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ON PRIME SUBMODULES

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Throughout this paper R will denote a commutative ring with identity and M a unital module. Several authors have extended the notion of prime ideal to modules, see, for example [**1, 2**]. In this paper, we continue these investigations.

A proper submodule N of M is prime if for any $r \in R$ and $m \in M$ such that $rm \in N$, either $rM \subseteq N$ or $m \in N$. It is easy to show that if N is a prime submodule of M then the annihilator P of the module M/N is a prime ideal of R. Also it is not difficult to see that N is a prime submodule of M if and only if $(N : K) = (N : M)$ for all submodules K of M properly containing N.

It is well known that a submodule N of M is prime if and only if $P = (N : M)$ is a prime ideal of R and the (R/P)-module M/N is fully faithful. For a prime ideal P of R, McCasland and Smith [**8**] defined the set $M(P)$ and asked the question: When does $M = M(P)$? In this paper we give an answer to this question and also describe the interrelation between the attached primes and prime submodules of an Artinian R-module.

Let N be a proper submodule of an R-module M. The radical of N in M, denoted by $\mathrm{rad}_M N$, is defined to be the intersection of all prime submodules of M containing N. Should there be no prime submodule of M containing N, then we put $\mathrm{rad}_M N = M$. On the other hand, $\mathrm{rad}\, R$ denotes the intersection of all prime ideals of R. Let I be an ideal of R. Then it is well known that $\sqrt{I} = \{r \in R : r^n \in I \text{ for some } n \in \mathbf{N}\}$. The envelope submodule $RE_M(N)$ of N in M is a submodule of M generated by the set $E_M(N) = \{rm : r \in R \text{ and } m \in M \text{ such that } r^n m \in N \text{ for some } n \in \mathbf{N}\}$.

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We will call N a *McCasland submodule in* M if it satisfies the radical formula, that is, if $\mathrm{rad}_M N = RE_M(N)$. Likewise, M will be called a McCasland module if every submodule of M is a McCasland submodule. A ring R is said to satisfy the radical formula if every R-module M is a McCasland module, equivalently, if $\mathrm{rad}_M 0 = RE_M(0)$. The question as to what kinds of module are McCasland modules has been considered in [**4, 5, 7, 8, 10**]. In this paper we continue the investigation begun in [**7**] into conditions under which a submodule satisfies the radical formula. In the first section we deal with the question as to when a representable module is a McCasland module.

Recall that M is called a *multiplication module* provided that for each submodule N of M there exists an ideal I of R such that $N = IM$. It is also known that $\mathrm{rad}RM \subseteq RE_M(0) \subseteq \mathrm{rad}_M 0$. Example 2.4 shows that they are not equal in general, but the equality holds if M is a multiplication R-module. We also prove, in Section 2, see Theorem 2.6, that the equality is true if M is a projective module. We also characterize the radical of a submodule N of an R-module M with M/N a projective R-module, as $\mathrm{rad}_M N = \mathrm{rad}RM + N = RE_M(N)$. We show that if the ring R has $R/\mathrm{rad}\, R$ semi simple and N is a submodule of an R-module M, then $\mathrm{rad}_M N = \mathrm{rad}RM + N = RE_M(N) = \sqrt{(N : M)}M + N$.

In [**5**], Leung and Man proved that any Artinian ring satisfies the radical formula. Also it is well known that for any Artinian ring R, $R/\mathrm{rad}\, R$ is semi simple. On the other hand, there are many examples showing that the converse is not true in general. We prove in the last section that if $R/\mathrm{rad}R$ is semi simple for any ring R, then R satisfies the radical formula.

In [**7**], McCasland and Moore proved that if N is a submodule of a finitely generated multiplication R-module M, then $\mathrm{rad}_M N = \sqrt{(N : M)}M$. They concluded their paper by mentioning that for any R-module M and a submodule N of M one has in general $\sqrt{(N : M)}M \subseteq RE_M(N) \subseteq \mathrm{rad}_M N$ and asking when equality holds. At the end of this note we also give necessary and sufficient conditions for this equality to hold.

1. Secondary modules. Let P be a prime ideal of a ring R. We recall from [**8**] the subset $M(P)$ of M defined by

$$M(P) = \{m \in M \mid Bm \subseteq PM \text{ for some ideal } B \nsubseteq P\}.$$

We will need the following lemma from [**8**].

Lemma 1.1, (1) *Let I be an ideal of R. Then there exists a proper submodule N of M such that $I = (N : M)$ if and only if $IM \neq M$ and $I = (IM : M)$.*

(2) *For a prime ideal P of R let $N = M(P)$. Then $N = M$ or N is a prime submodule of M such that $P = (N : M)$.*

Let us recall from [**11**] what it means for M to have a secondary representation.

Definition 1.2. A nonzero R-module M is said to be *secondary* if for all $x \in R$, either $xM = M$ or there exists $n \in \mathbf{N}$ such that $x^n M = 0$. If M is a secondary R-module then $\sqrt{0 : M} = P$ is a prime ideal of R and M is then called *P-secondary.*

Definition 1.3. A *secondary representation* for an R-module M is an expression of the form $M = M_1 + \cdots + M_r$, $r \geq 0$, where M_i is a secondary submodule of M for all $i = 1, \ldots, r$. We say that the secondary representation is *minimal* if

(i) *For $P_i = \sqrt{0 : M_i}$, $i = 1, \ldots, r$, the $P_1, \ldots, P_r$ are all distinct, and*

(ii) *No term in the sum is redundant.*

The set $\{P_1, \ldots, P_r\}$ of prime ideals of R is independent of the choice of minimal secondary representation for M and is called the *set of attached primes of M*, denoted by $\mathrm{Att}\,(M)$. In this case M is said to be a *representable module.*

In this section, we study the relation between $\mathrm{Att}\,(M)$ and the prime submodule of M. We also give a condition for a representable module to be a McCasland module.

Let N be a submodule of an R-module M such that $(N : M)$ is a prime ideal in R. Then N need not be a prime submodule of M and also for any prime ideal P of R there may be no prime submodule N such that $P = (N : M)$. Now we give the following:

Theorem 1.4. *Let M be an Artinian R-module and $M = M_1 + \cdots + M_r$ a minimal secondary representation with $\sqrt{0 : M_i} = P_i$ for all $i = 1, \ldots, r$. Also suppose that M/P_iM is finitely generated for some i, $1 \leq i \leq r$. Then M has a prime submodule N such that $P_i = (N : M)$.*

Proof. Since M/P_iM is finitely generated we have $M/P_iM = R\bar{x}_1 + \cdots + R\bar{x}_n$, where $x_i \in M$ for all $i = 1, \ldots, n$. Then we get $M = Rx_1 + Rx_2 + \cdots + Rx_n + P_iM$. Since $P_i \in \operatorname{Att}(M)$, by [**11**, Corollary 2.6], M has a nonzero homomorphic image with annihilator P_i. Thus M has a proper submodule N such that $P_i = (N : M)$ and so we obtain $P_i = (P_iM : M)$. Now we claim that $M \neq M(P)$. Otherwise, for each i there exists an ideal B_i with $B_i \nsubseteq P_i$ such that $B_ix_i \subseteq P_iM$. Let $B = \cap_{i=1}^{n} B_i$. Then $BM \subseteq P_iM$, which is a contradiction. The result now follows from Lemma 1.1. □

Let M be a nonzero Artinian module. Then for the reverse relationship between the attached primes of M and the prime submodule of M, we suppose that N is a prime submodule of the Artinian R-module M. Then $P = (N : M)$ is a prime ideal of R and so by [**11**, Corollary 2.6], P belongs to $\operatorname{Att}(M)$.

Now we show that the condition in Theorem 1.4, that M/P_iM is a finitely generated R-module for some $P_i \in \operatorname{Att}(M)$, is necessary. Let $M = \mathbf{Z}(p^\infty)$ be an Artinian $\mathbf{Z}$-module, whence M has a minimal secondary representation. If M/q_iM were a finitely generated $\mathbf{Z}$-module for some $q_i \in \operatorname{Att}(M)$, then by Theorem 1.4, $M(q_i)$ would be a prime submodule of M. But this is impossible, as we show in the following example.

Example 1.5. Let $M = \mathbf{Z}(p^\infty)$ be an Artinian $\mathbf{Z}$-module. Then we claim that for any prime ideal q in $\mathbf{Z}$, $M(q) = M$.

Let $r/p^n + \mathbf{Z} \in \mathbf{Z}(p^\infty)$ for some $r \in \mathbf{Z}$, $n \in \mathbf{N}$. If $r \in q$, then it is clear that $r/p^n + \mathbf{Z} \in M(q)$. If $r \notin q$, then take $A = (r)$ and so $A \nsubseteq q$. There exist elements u and s in $\mathbf{Z}$ such that $qu + sp^n = 1$ and so $r = rqu + rsp^n$. Let $rt \in A$. Then $rt((r/p^n) + \mathbf{Z}) = (tr^2u)/p^n + \mathbf{Z} \in q\mathbf{Z}(p^\infty)$, and so we have $M(q) = M$.

This example also gives a partial answer to the question raised in [**8**, Proposition 1.7], namely when does $M = M(P)$?

Let $\mathrm{Spec}_{P}(M)$ denote the collection of all prime submodules K of M such that $P = (K : M)$, together with the module M. Let M be an Artinian R-module. Suppose that $M = M_1 + \cdots + M_r$ is a minimal secondary representation for M with $\sqrt{0 : M_i} = P_i$ for all $i = 1, \dots, r$. Then by Theorem 1.4 and [**14**], all prime submodules of M can be classified as the set $\{\mathrm{Spec}_{P_i}(M) : P_i \in \mathrm{Att}\,(M)\}$.

Recall from [**10**] that an R-module M is called *special* if, for each $m \in M$ and each element a of any maximal ideal $\mathcal{M}$, there exists $n \in \mathbf{N}$ and $c \in R \backslash \mathcal{M}$ such that $ca^n m = 0$. Also a module M is called *semi-artinian* if every homomorphic image of M has a nonzero socle. In [**10**], Pusat and Smith proved that every semi-artinian module is special. They also proved that any special module is a McCasland module. This gives us that any Artinian module is a McCasland module. The class of representable R-modules is, in general, larger than the class of Artinian R-modules. Hence we investigate when a representable R-module M is a McCasland module. First we prove that, if M is Noetherian representable over a one dimensional domain R, then M is a McCasland module.

It is well known that if M is a McCasland module then so is any homomorphic image of M. Although the proof of the following lemma is very similar to the proof of [**10**, Theorem 2.2], it is given for completeness.

Lemma 1.6. *Let R be a domain and $M = M_1 + M_2$ an R-module. If M_1 is a McCasland module and M_2 a divisible module, then M is a McCasland module.*

Proof. The mapping α from M_1 to M/M_2 defined by $\alpha(s_1) = s_1 + M_2$ is an epimorphism and so M/M_2 is a McCasland module. Let N be a submodule of M and $m \in \mathrm{rad}_M N$. Then $m = s_1 + s_2$, whence

$$\begin{aligned} m + M_2 \in (\mathrm{rad}_M N + M_2)/M_2 &= \mathrm{rad}_{M/M_2}(N + M_2/M_2) \\ &= RE_{M/M_2}(N + M_2/M_2) \end{aligned}$$

and so

$$s_1 + M_2 = r_1(k_1 + M_2) + \cdots + r_n(k_n + M_2),$$

where $r_i^{t_i}(k_i + M_2) \in N + M_2/M_2$, and so $r_i^{t_i} k_i \in N + M_2$. Then there exist $n_i \in N$, $d_i \in M_2$ such that $r_i^{t_i} k_i = n_i + d_i$ for $t_i \in \mathbf{N}$. Since M_2 is divisible, $d_i = r_i^{t_i} c_i$ for some $c_i \in M_2$ for all i, and so $r_i^{t_i}(k_i - c_i) \in N$, $1 \le i \le n$. Therefore, we have

$$s_1 + s_2 = r_1(k_1 - c_1) + \cdots + r_n(k_n - c_n) + x$$

for some $x \in M_2$. It follows that $x \in \mathrm{rad}_M N$. There exist a nonzero $c \in R$ and $y \in M_2$ such that $cx \in N$ and $x = cy$. Hence it follows that $c^2 y \in N$ and so $x = cy \in RE_M(N)$. Therefore $\mathrm{rad}_M N = RE_M(N)$. $\square$

Let T be a multiplicatively closed subset of R, and let S be a $\mathcal{P}$-secondary R-module. If $\mathcal{P} \cap T \neq \varnothing$ then clearly $T^{-1}S = 0$. Otherwise, $T^{-1}S$ is a $T^{-1}\mathcal{P}$-secondary $T^{-1}R$-module. By Lemma 1.6 any divisible R-module over a domain is a McCasland module and by [**10**, Theorem 4.8], any special R-module over a domain is a McCasland module. Therefore, if R is a local domain with $\dim R = 1$ then any secondary R-module is a McCasland module. Hence we have the following.

Theorem 1.7. *Let R be a domain with $\dim R = 1$. If M is a Noetherian representable R-module, then M is a McCasland module.*

Proof. Let $M = M_1 + \cdots + M_n$ be the minimal secondary representation with $\sqrt{(0 : M_i)} = \mathcal{P}_i$ for $i = 1, \dots, n$. Let $\mathcal{M}$ be a maximal ideal of R. Then $M_{\mathcal{M}} = M_{1\mathcal{M}} + \cdots + M_{n\mathcal{M}}$. Assume that $\mathcal{P}_k = 0$ for at least for one k, $1 \le k \le n$. Without loss of generality let $k = 1$. In this case M_1 is a divisible R-module. Now we have the following two cases:

(i) $\mathcal{M} = \mathcal{P}_j$ for some j. Then $M_{i\mathcal{M}} = 0$ for all $i \neq j$, $2 \leq i \leq n$ and so we have $M_{\mathcal{M}} = M_{1_{\mathcal{M}}} + M_{j_{\mathcal{M}}}$. Hence $M_{\mathcal{M}}$ is a McCasland module.

(ii) Let $\mathcal{M} \neq \mathcal{P}_i$ for all $i = 2, \dots, n$. In this case $M_{i_{\mathcal{M}}} = 0$ and so $M_{\mathcal{M}} = M_{1_{\mathcal{M}}}$ is again a McCasland module.

If $\mathcal{P}_i \neq 0$ for all i, then $M_{\mathcal{P}_i}$ is a McCasland module since $\dim R = 1$. Therefore M is a McCasland module is all cases. □

Now we continue our investigation of the conditions under which a representable module is a McCasland module.

Lemma 1.8. *Let R be a domain and $M = M_1 + M_2$ an R-module with representable submodule M_2. Let N be a submodule of M. If $r^t k + d \in N$, where $r \in R$, $k \in M_1$, $d \in M_2$ and $t \in \mathbf{N}$, then $r(k + c) \in RE_M(N)$ for some $c \in M_2$.*

Proof. Assume that $M_2 = L_1 + \cdots + L_n$ is a minimal secondary representation with $\sqrt{0 : L_i} = P_i$ for all $i = 1, \dots, n$. Then d can be written as $d = x_{i_1} + \cdots + x_{i_t}$ for $x_{i_j} \in L_{i_j}$, $1 \leq j \leq t$. Now we use induction on t. Let $t = 1$.

(a) If $rL_1 = L_1$, then we have $d = r^t c$ for some $c \in M_2$ and so $r^t(k + c) \in N$. Thus $r(k + c) \in RE_M(N)$.

(b) If $r^l L_1 = 0$ for some $l \in \mathbf{N}$, then $r^l(r^t k + d) = r^{t+l} k \in N$ and so $rk \in RE_M(N)$.

Suppose now that $t > 1$. We will divide the rest of the proof into two parts:

1. Assume first that for at least one i_j we have $l \in \mathbf{N}$ such that $r^l x_{i_j} = 0$. Without loss of generality we may assume $i_j = i_t$. Then

$$r^l(r^t k + d) = r^{t+l} k + (r^l x_{i_1} + \cdots + r^l x_{i_{t-1}}) \in N$$

and, by hypothesis, $r(k + c) \in RE_M(N)$ for some $c \in M_2$.

2. Now assume that $r^l x_i \neq 0$ for all i, $1 \leq i \leq t$, and for all l in $\mathbf{N}$. Then $r^l L_i = L_i$ and so there exists $c_{i_j} \in L_{i_j}$ such that $x_{i_j} = r^l c_{i_j}$ for all i, $1 \leq i \leq t$. It follows that $r^t(k + c_{i_1} + \cdots + c_{i_t}) \in N$ and so $r(k + c_{i_1} + \cdots + c_{i_t}) \in RE_M(N)$. □

Let N be a submodule of an R-module M. We say that N *satisfies* $(*)$ if for $x \in \mathrm{rad}_M N$ and $c \in R$ such that $cx \in E_M(0) \cap N$ implies $x \in RE_M(N)$. M is said to *satisfy* $(*)$ if every submodule of M satisfies $(*)$. Clearly any torsion free R-module over a domain satisfies $(*)$. By using the same argument as in the proof of Lemma 1.8, we have the following lemma:

Lemma 1.9. *Let $M = M_1 + M_2$ be an R-module over a domain satisfying $(*)$ and M_2 a representable submodule of M. Let N be a submodule of M. If $c \in R$ and $x \in \mathrm{rad}_M N \cap M_2$ are such that $cx \in N$ then $x \in RE_M(N)$.*

Theorem 1.10. *Let $M = M_1 + M_2$ be an R-module over a domain satisfying $(*)$. If M_1 is a McCasland module and M_2 a representable submodule of M, then M is a McCasland module.*

Proof. Let N be a submodule of M. Take $m \in \mathrm{rad}_M N$. Then $m = m_1 + m_2$ where $m_1 \in M_1$ and $m_2 \in M_2$. As in the proof of Lemma 1.6, we have

$$m_1 + M_2 = r_1(k_1 + M_2) + \cdots + r_n(k_n + M_2)$$

for some $n \in \mathbf{N}$, $r_i \in R$, $k_i \in M$, $(1 \le i \le n)$, and there exist $t \in \mathbf{N}$, $u_i \in N$ and $v_i \in M_2$, $(1 \le i \le n)$, such that

$$r_i^{t_i} k_i = u_i + v_i, \quad 1 \le i \le n.$$

By Lemma 1.8, $r_i(k_i + c_i) \in RE_M(N)$ for some $c_i \in M_2$ and each $1 \le i \le n$. Thus

$$m = r_1(k_1 + c_1) + \cdots + r_n(k_n + c_n) + x$$

for some $x \in M_2$, whence there exists a $c \in R$ such that $cx \in N$. Therefore by Lemma 1.9 we get $x \in RE_M(N)$. This completes the proof. □

We do not know if Lemma 1.9 remains true when $M = M_1 + M_2$ is an arbitrary R-module. If so then Theorem 1.10 could be extended in the natural way.

2. The radicals of a submodule. In this section we characterize the radicals and envelopes for a certain class of submodules. Also we prove that a ring R for which with $R/\mathrm{rad}\, R$ is semi simple satisfies the radical formula. We begin this section with the following simple-known lemma.

Lemma 2.1. *Let N_1 and N_2 be submodules of an R-module M with $N_1 \subseteq N_2$. Then*

(i) $RE_{M/N_1}(N_2/N_1) = RE_M(N_2)/N_1$.

(ii) $\mathrm{rad}_{M/N_1}(N_2/N_1) = \mathrm{rad}_M(N_2)/N_1$.

In [**4**], James and Smith proved that if M is an R-module such that $\mathrm{rad}_M 0 = RE_M(0)$ then so is any direct sum of M. Now we will show that if M is a McCasland module then any direct summand N of M is a McCasland module. Let M be direct sum of the R-modules M_i, $i \in I$. Let $N = \oplus N_i$ be a submodule of M such that N_i is a submodule of M_i for all $i \in I$.

Lemma 2.2. *Let M and N be as above. Assume that P is a prime ideal of R. Then N is a P-prime submodule of M if and only if whenever $N_i \neq M_i$, N_i is a P-prime submodule of M_i for all $i \in I$.*

Proof. Let $N = \oplus N_i$, where N_i is a submodule of M_i, $i \in I$. Then N is a P-prime submodule of M if and only if $M/N = \oplus M_i / \oplus N_i \cong \oplus(M_i/N_i)$ is a torsion free (R/P)-module if and only if M_i/N_i is a torsion-free R/P-module for all $i \in I$ if and only if N_i is a P-prime submodule of M_i for all $i \in I$ such that $N_i \neq M_i$. $\square$

Now we show the condition in Lemma 2.2, that for all $i \in I$, N_i should be a P-prime submodule of M_i, is necessary: Let $R = \mathbf{Z}$ and assume that M is the R-module $\mathbf{Z} \oplus \mathbf{Z}$ and $N = 3\mathbf{Z} \oplus 2\mathbf{Z}$. Then it is easy to see that $(N : M) = 6\mathbf{Z}$ and so N is not a prime submodule of M.

Lemma 2.3. *Let M and N be as above. Then we have*

(i) $RE_M(N) = \oplus_{i \in I} RE_{M_i}(N_i)$.

(ii) $\mathrm{rad}_M N = \oplus \, \mathrm{rad}_{M_i} N_i$.

(iii) $\mathrm{rad}_{M_i} N_i = RE_{M_i}(N_i)$ *for all* $i \in I$ *if and only if* $\mathrm{rad}_M N = RE_M(N)$.

Proof. (i) Suppose that $rm \in RE_M(N)$, where $m = (m_i) \in \oplus M_i$ and $r \in R$. Then for some integer k, $r^k m \in N$ and so we have $r^k m = (r^k m_i) \in \oplus_{i \in I} N_i$. This means that $r^k m_i \in N_i$ and $rm_i \in RE_{M_i}(N_i)$ for all $i \in I$ and then $(rm_i) \in \oplus_{i \in I} RE_{M_i}(N_i)$. Therefore, $RE_M(N) = \oplus_{i \in I} RE_{M_i}(N_i)$.

(ii) Suppose that $m \in \mathrm{rad}_M N$ and $m \notin \oplus \, \mathrm{rad}_{M_i} N_i$. Let π_i denote the projection map from M to M_i. Then there exists $i \in I$ such that $\pi_i(m) \notin \mathrm{rad}_{M_i} N_i$. This means that there exists a prime submodule P_i of M_i such that $N_i \subseteq P_i$ but $\pi_i(m) \notin P_i$. Then $K = P_i \oplus (\oplus_{i \neq j} M_j)$ is a prime submodule of M such that $N \subseteq K$ and $m \notin K$. Thus $m \notin \mathrm{rad}_M N$, a contradiction. Hence, $\mathrm{rad}_M N \subseteq \oplus \, \mathrm{rad}_{M_i} N_i$.

(iii) This is clear from (i) and (ii). □

It is well known that $\mathrm{rad}\, RM \subseteq RE_M(0) \subseteq \mathrm{rad}_M 0$ for any R-module M. In general we do not have equality, as is seen from Example 2.4. However equality is known to hold for a multiplication module, and we will prove that it holds for a projective R-module also.

Example 2.4 [**13**]. Suppose that R denotes the polynomial ring $\mathbf{Z}[x]$, and let $M = R \oplus R$. Let N be the submodule $N = R(x,4) + R(0,x) + x^2 M$ of M. It is easy to check $RE_M(N) = N + xM = R(0,4) + xM$ and $\mathrm{rad}_M N = R(0,2) + xM$. Let $\mathcal{M} = M/N$. Then by Lemma 2.1, we have $\mathrm{rad}\, R\mathcal{M} = 0$, $RE_{\mathcal{M}}(0) = (R(x,4) + xM)/N$ and $\mathrm{rad}_{\mathcal{M}} 0 = (R(x,2) + xM)/N$.

Now we give the following simple lemma.

Lemma 2.5. *Let* M *and* N *be* R*-modules, and let* α *be an epimorphism from* M *to* N*. Then we have*

(i) *Let* P_i*,* $i \in I$*, be submodules of* M *satisfying* $\operatorname{Ker}\alpha \subseteq P_i$ *for all* $i \in I$*. Then* $\alpha(\cap P_i) = \cap \alpha(P_i)$*.*

(ii) $\alpha(\mathrm{rad}_M \operatorname{Ker}\alpha) = \mathrm{rad}_N 0$*. In particular,* $\alpha(\mathrm{rad}_M 0) \subseteq \mathrm{rad}_N 0$*.*

Proposition 2.6. *Let* M *be a projective* R*-module. Then* $\mathrm{rad}\, RM = RE_M(0) = \mathrm{rad}_M 0$*.*

Proof. Let M be a projective R-module. Then there exists a free R-module F and an R-module A such that $F = M \oplus A$.

First we prove that our claim is true for F. Let $\{x_i \mid i \in I\}$ be a basis for F. Then $F = \oplus Rx_i$ and so each $x \in F$ has a unique expansion $x = \sum r_i x_i$ where $r_i \in R$ and almost all $r_i = 0$. Define a homomorphism α_i from F to R by $\alpha_i(x) = r_i$. Then α_i is an epimorphism for all $i \in I$ and we obtain $x = \sum_{i\in I} \alpha_i(x) x_i$.

Let $u \in \mathrm{rad}_F 0$. Then $u = \sum r_i x_i = \sum \alpha_i(u) x_i$, where $r_i \in R$ and almost all $r_i = 0$. Hence, by Lemma 2.5 we have $u = \sum \alpha_i(u) x_i \in \mathrm{rad} F$. Now we have $\mathrm{rad}_F 0 \subseteq \mathrm{rad} F$ and so $\mathrm{rad}_F 0 = \mathrm{rad} F$.

Take $m \in rad_M 0$. By Lemma 2.3, it follows that $\mathrm{rad}_F 0 = \mathrm{rad}_M 0 \oplus \mathrm{rad}_A 0$ and so we have $m \in \mathrm{rad}_F 0 = \mathrm{rad}\, RF = \mathrm{rad}\, R(M \oplus A) = \mathrm{rad}\, RM \oplus \mathrm{rad} RA$. This implies that $m = \sum r_i m_i + \sum k_j a_j$, where $r_i,\ k_j \in \mathrm{rad}\, R$, $m_i \in M$ and $a_j \in A$. Therefore, $m = \sum r_i m_i \in \mathrm{rad}\, RM$. This completes the proof. □

The following theorem can be obtained using Lemma 2.1 and Proposition 2.6.

Theorem 2.7. *Let* N *be a submodule of an* R*-module* M *such that* M/N *is projective. Then* $\mathrm{rad}_M N = RE_M(\mathrm{rad}\, RM + N) = \mathrm{rad}\, RM + N$*.*

Let N be a prime submodule of an R-module M. Then $(N : M)$ is a prime ideal of R and $N = RE_M(N) = \mathrm{rad}_M N$. But the converse is not true in general. (Consider the $\mathbf{Z}$-module $\mathbf{Z} \oplus \mathbf{Z}$). Thus Theorem 2.7 has the following immediate consequences.

Corollary 2.8. *Let N be a submodule of an R-module M such that M/N is a projective R-module and* $\operatorname{rad} RM \subseteq N$. *Then* $\operatorname{rad}_M N = RE_M(N) = N$.

Denote $R/\operatorname{rad} R$ by $\overline{R}$, $M/\operatorname{rad} RM$ by $\overline{M}$ and consider a ring R such that $\overline{R}$ is semi simple.

Theorem 2.9. *Let R be a ring such that $\overline{R}$ is semi simple, and let N be a submodule of an R-module M. Then we have* $\operatorname{rad}_M N = RE_M(N) = \sqrt{(N:M)}\,M + N = \operatorname{rad} RM + N$.

Proof. First assume that $N = 0$. Then it will be enough to show that $\operatorname{rad}_M 0 = \operatorname{rad} RM$. Since $\overline{R}$ is semi simple, $\overline{M}$ is a semi simple R-module and so $\operatorname{rad} R\overline{M} = \operatorname{rad}_{\overline{M}} 0 = 0$. On the other hand, since $\operatorname{rad} RM \subseteq \operatorname{rad}_M 0$, we have

$$\operatorname{rad}_{\overline{M}} 0 = (\operatorname{rad}_M 0)/\operatorname{rad} RM = 0.$$

This means that $\operatorname{rad}_M 0 = \operatorname{rad} RM = \sqrt{(0:M)}\,M$.

Now let $N \neq 0$. Then we have

$$\operatorname{rad}_{M/N} 0 = \operatorname{rad} R(M/N) = \sqrt{(0:M/N)}\,M/N.$$

Therefore, $\operatorname{rad}_M N = \operatorname{rad} RM + N = \sqrt{(N:M)}\,M + N = RE_M(N)$. □

Corollary 2.10. *Let R be a ring such that $\overline{R}$ is semi simple. Then R satisfies the radical formula.*

Proof. Let M be any R-module. Then by Theorem 2.9, $\operatorname{rad}_M 0 = \sqrt{(0:M)}\,M = \operatorname{rad} RM$. As $\operatorname{rad} RM \subseteq RE_M(0) \subseteq \operatorname{rad}_M 0$, we get $\operatorname{rad} RM = RE_M(0) = \operatorname{rad}_M 0$. This means that M is a McCasland module, hence the result. □

We conclude this note by making the following observations. Let N be a submodule of an R-module M. It is easy to check that

$$\operatorname{rad} RM \subseteq \sqrt{(N:M)}\,M \subseteq RE_M(N) \subseteq \operatorname{rad}_M N$$

for any submodule N of an R-module M. In [**7**] McCasland and Moore ask when $\sqrt{(N:M)}M = RE_M(N) = \text{rad}_M(N)$. Now we can give an answer to their question. If the hypothesis of Theorem 2.7, or Theorem 2.9, are satisfied, then $N \subseteq \sqrt{(N:M)}\,M$ if and only if $\sqrt{(N:M)}\,M = RE_M(N) = \text{rad}_M N$. (In general $N \nsubseteq \sqrt{(N:M)}\,M$).

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REFERENCES

1. J. Dauns, *Prime modules*, J. Reine Angew. Math. **2** (1978), 156–181.

2. ———, *Prime modules and one-sided ideals*, in *Ring theory and algebra* III, Proc. of the Third Oklahoma Conf. (B.R. McDonald, ed.), Dekker, New York, 1980, pp. 301–344.

3. K.R. Goodearl and R.B Warfield, *An introduction to non commutative Noetherian rings*, London Math. Soc. Stud. Texts, vol. 16, Cambridge Univ. Press, Cambridge, 1989.

4. J. Jenkins and P.F. Smith, *On the prime radical of a module over a commutative ring*, Comm. Algebra **20** (1992), 3593–9602.

5. K.H. Leung and H.S. Man, *On Commutative Noetherian rings which satisfy the radical formula*, Glasgow Math. J. **39** (1997), 285–293.

6. C.P. Lu, *Prime submodules of modules*, Comm. Math. Univ. Sancti Pauli **33** (1984), 61–69.

7. R.L. McCasland and M.E. Moore, *On radicals of submodules*, Comm. Algebra **19** (1991), 1327–1341.

8. R.L. McCasland and P.F. Smith, *Prime submodules of Noetherian modules*, Rocky Mountain J. Math **23** (1993), 1041–1062.

9. J.C. McConnell and J.C. Robson, *Noncommutative Noetherian rings*, Wiley, Chichester, 1987.

10. D. Pusat-Yilmaz and P.F. Smith, *Modules which satisfy the radical formula*, Acta Math Hungar. **95** (2002), 155–167.

11. R.Y. Sharp, *A method for the study of Artinian modules with an application to asymptotic behaviour*, Math. Sci. Res. Inst. Publ., vol. 15, Springer, New York, 1989, pp. 443–465.

12. ———, *Steps in commutative algebra*, London Math. Soc. Stud. Texts, vol. 19, Cambridge Univ. Press, Cambridge, 1990.

13. P.F. Smith, *Primary modules over commutative rings*, Glasgow Math. J. **43** (2001), 103–111.

14. Y. Tıraş, A. Tercan and A.Harmancı, *Prime modules*, Honam Math. J. **18** (1996), 5–15.

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