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ESSENTIAL NORMS OF WEIGHTED COMPOSITION OPERATORS BETWEEN BLOCH-TYPE SPACES

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ABSTRACT. We compute the essential norm of a weighted composition operator uC_φ acting from an analytic Lipschitz space into a weighted Bloch-type space on the disk, and give estimates for the essential norm of uC_φ when it maps the standard Bloch space into a weighted Bloch-type space. We also study boundedness and compactness of weighted composition operators on analytic Lipschitz spaces from a geometric perspective.

1. Introduction. Let D be the open unit disk in the complex plane. Let u be a fixed analytic function on D and φ an analytic self-map of D. We can define a linear operator uC_φ on the space of analytic functions on D, called a *weighted composition operator*, by

$$uC_\varphi f = u(f \circ \varphi).$$

It is easy to see that an operator defined in this manner is linear. We can regard this operator as a generalization of a multiplication operator and a composition operator.

Our main interest here is in determining the essential norm of a weighted composition operator acting from one weighted Bloch space, defined below, to another. In the case that $u(z) \equiv 1$ or $\varphi(z) = z$, our formulas give the essential norm of the multiplication operator M_u or the composition operator C_φ, respectively.

Essential norm formulas for composition operators are known in various settings. When C_φ acts from the Hardy space $H^2(D)$ to itself, Shapiro [**9**] gives a formula for $\|C_\varphi\|_e$, the essential norm of C_φ, in terms of the Nevanlinna counting function for φ. A similar formula, using a generalized Nevanlinna counting function, for the essential norm of C_φ acting on the Bergman space $A^2(D)$ is given in [**6**]. In [**3**], Donaway

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gives upper and lower estimates for $\|C_\varphi\|_e$ when C_φ maps the Bloch, Dirichlet, or a Besov-p space to itself. In general he obtains upper estimates which are (fixed) constant multiples of his lower estimates. In the case of the Bloch space, Montes-Rodriguez [**7**] gives an exact formula, namely,

$$\|C_\varphi\|_e = \lim_{s\to 1} \sup_{|\varphi(z)|>s} \frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)|.$$

The spaces we consider here are weighted Bloch-type spaces defined as follows. Let $0<\alpha<\infty$. A function f analytic in D is said to belong to the α-Bloch space B^α if

$$\sup_{z\in D} |f'(z)|(1-|z|^2)^\alpha < \infty$$

and to the little α-Bloch space B_0^α if

$$\lim_{|z|\to 1} |f'(z)|(1-|z|^2)^\alpha = 0.$$

It is known that, under the norm

$$\|f\|_{B^\alpha} = |f(0)| + \sup_{z\in D} |f'(z)|(1-|z|^2)^\alpha,$$

B^α becomes a Banach space and B_0^α is a closed subspace of B^α. When no confusion can occur, we write $\|f\|_\alpha$ for $\|f\|_{B^\alpha}$. When $\alpha=1$, we have the standard Bloch space and we write B for B^1. For $0<\beta<\infty$, let H_β^∞ be the weighted Banach space of analytic functions f on D satisfying

$$\|f\|_{H_\beta^\infty} = \sup_{z\in D} |f(z)|(1-|z|^2)^\beta < \infty.$$

Correspondingly, let $H_{\beta,0}^\infty$ be the space of analytic functions f on D satisfying

$$\lim_{|z|\to 1} |f(z)|(1-|z|^2)^\beta = 0.$$

The essential norms of weighted composition operators between these weighted Banach spaces of analytic functions were obtained recently by Montes-Rodriguez in [**8**] and by Contreras and Hernandez-Diaz in [**1**]. It is known that, for $\alpha>1$, the space B^α coincides with the space

$H^\infty_{\alpha-1}$ and B^α_0 coincides with the space $H^\infty_{\alpha-1,0}$. Further, the norms $\|f\|_{B^\alpha}$ and $\|f\|_{H^\infty_{\alpha-1}}$ are comparable. Thus the work in [**1**] and [**8**] gives information about the essential norm of uC_φ on α-Bloch spaces when $\alpha > 1$. So in this paper we focus our attention on the case $\alpha \le 1$.

For $\alpha \le 1$, we collect some basic properties of functions in the α-Bloch space and the little α-Bloch space here. Recall that

$$\beta(z,w) = \frac{1}{2}\log\frac{1+\rho(z,w)}{1-\rho(z,w)}$$

is the Bergman metric on D, where $\rho(z,w) = |z-w|/|1-\bar{z}w|$ and $z, w \in D$. It is well known (see, for example, [**11**, Theorem 5.1.6]) that the following hold:

$$|f(z) - f(w)| \le \|f\|_B \beta(z,w) \quad \text{for } f \in B. \tag{1}$$

From equation (1) we can show that, for $f \in B$,

$$|f(z)| \le \frac{1}{\log 2}\|f\|_B \log\frac{2}{1-|z|^2}. \tag{2}$$

For $\alpha < 1$, the space B^α coincides with the Lipschitz space $\mathrm{Lip}_{1-\alpha}$ [**4**, p. 74], consisting of all analytic functions f on D, satisfying

$$|f(z) - f(w)| \le C|z-w|^{1-\alpha}$$

for some finite positive constant C and all $z, w \in D$. It is easy to see that C can be taken to be a multiple of the norm of f in B^α. Boundedness and compactness of weighted composition operators between Bloch-type spaces are characterized in [**5**]. The parts of this work which are relevant here are given in the following two theorems.

Theorem 1 [**5**]. *When $0 < \alpha < 1$ and $\beta > 0$ the weighted composition operator uC_φ maps B^α boundedly into B^β if and only if $u \in B^\beta$ and*

$$\sup_{z\in D} |u(z)|\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}|\varphi'(z)| < \infty.$$

The operator is compact if and only if it is bounded and

$$\lim_{s\to 1}\sup_{|\varphi(z)|>s} |u(z)|\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}|\varphi'(z)| = 0.$$

Theorem 2 [**5**]. *The operator* uC_φ *is bounded from* B *to* B^β *if and only if* $u \in B^\beta$ *and*

(a) $\sup_{z\in D} |u'(z)|(1-|z|^2)^\beta \log \frac{1}{1-|\varphi(z)|^2} < \infty$, *and*

(b) $\sup_{z\in D} |u(z)|\frac{(1-|z|^2)^\beta}{1-|\varphi(z)|^2}|\varphi'(z)| < \infty$.

Furthermore, uC_φ *is compact from* B *to* B^β *if and only if it is bounded and the* $\limsup_{|\varphi(z)|\to 1^-}$ *of these same expressions is* 0.

On the basis of the compactness criterion, it is reasonable to expect that the essential norm of uC_φ acting from B^α, $0<\alpha<1$ to B^β should be given by the related "lim sup" expression. That is the content of our first main theorem.

Theorem 3. *Suppose* $0<\alpha<1$ *and* $0<\beta<\infty$ *and suppose the weighted composition operator* uC_φ *is bounded from* B^α *to* B^β*. Then*

$$\|uC_\varphi\|_e = \lim_{s\to 1} \sup_{|\varphi(z)|>s} |u(z)||\varphi'(z)| \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}.$$

We remark that a similar result can be obtained for the essential norm of uC_φ acting boundedly from B_0^α to B_0^β, $0<\alpha<1$, where in the essential norm formula of Theorem 3 we may replace "$\lim_{s\to 1}\sup_{|\varphi(z)|>s}$" by "$\limsup_{|z|\to 1}$." This equivalence can be shown by an argument like that in Proposition 2.2 of [**7**] or Theorem 2.2 of [**8**], using the characterization of boundedness of uC_φ from B_0^α to B_0^β given in [**5**, Theorem 4.1].

In order to simplify the notation in the statement of the next result, we write

$$A(u,\varphi,\beta) = \lim_{s\to 1} \sup_{|\varphi(z)|>s} |u(z)||\varphi'(z)| \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)} \tag{3}$$

and

$$B(u,\varphi,\beta) = \lim_{s\to 1} \sup_{|\varphi(z)|>s} |u'(z)|(1-|z|^2)^\beta \log \frac{1}{(1-|\varphi(z)|^2)}. \tag{4}$$

Our second main result is

Theorem 4. *Suppose $0 < \beta < \infty$ and suppose the weighted composition operator uC_φ is bounded from B to B^β. Then*

$$\max\left\{A(u,\varphi,\beta), \frac{1}{6}\,B(u,\varphi,\beta)\right\} \le \|uC_\varphi\|_e \le A(u,\varphi,\beta) + B(u,\varphi,\beta).$$

When $u \equiv 1$, Theorems 3 and 4 recover the essential norm formulas obtained by Montes-Rodriguez in [**8**, Theorem 2.3].

The proofs of these two theorems will be given in Sections 2 and 3, with lower estimates obtained in Section 2 and upper estimates in Section 3. In Section 4 we return to the question of compactness and boundedness of weighted composition operators on B^α and obtain some results relating the boundary values of the multiplier u and the radial limits and angular derivative of the symbol φ.

2. The lower estimates. We first give the following lower estimate.

Proposition 1. *Let $0 < \alpha \le 1$ and $0 < \beta < \infty$, and let uC_φ be bounded from B^α into B^β. Then*

$$\|uC_\varphi\|_e \ge \lim_{s\to 1} \sup_{|\varphi(z)|>s} |u(z)||\varphi'(z)| \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}.$$

Proof. Since uC_φ is bounded from B^α into B^β we have by Theorems 1 and 2, $u \in B^\beta$ and for $\alpha = 1$,

$$\sup_{z\in D} |u'(z)|(1-|z|^2)^\beta \log \frac{1}{1-|\varphi(z)|^2} < \infty. \tag{5}$$

Note that for $n \ge 1$ and $0 < \alpha \le 1$

$$\begin{aligned}\|z^n\|_{B^\alpha} &= \max_{z\in D} n|z|^{n-1}(1-|z|^2)^\alpha \\ &= n\left(\frac{2\alpha}{n-1+2\alpha}\right)^\alpha \left(\frac{n-1}{n-1+2\alpha}\right)^{(n-1)/2},\end{aligned}$$

where the maximum is attained at any point on the circle with radius

$$r_n = \left(\frac{n-1}{n-1+2\alpha}\right)^{1/2}.$$

Let $f_n = z^n/\|z^n\|_{B^\alpha}$. Then $\|f_n\|_{B^\alpha} = 1$ and f_n converges to 0 weakly in B^α. This follows since a bounded sequence contained in B_0^α which tends to 0 uniformly on compact subsets of D converges weakly to 0 in B^α. In particular, if K is any compact operator from B^α to B^β, then $\lim_{n\to\infty} \|Kf_n\|_\beta = 0$.

Let $A_n = \{z \in D : r_n \le |z| \le r_{n+1}\}$. Then

$$\min_{z\in A_n} |f_n'(z)|(1-|z|^2)^\alpha = \left(\frac{n-1+2\alpha}{n+2\alpha}\right)^\alpha \left(\frac{n^2+(2\alpha-1)n}{n^2+(2\alpha-1)n-2\alpha}\right)^{(n-1)/2}.$$

Note that this minimum tends to 1 as $n \to \infty$. Take any compact operator K from B^α to B^β. We have

$$\|uC_\varphi - K\| \ge \limsup_{n\to\infty} \|(uC_\varphi - K)f_n\|_{B^\beta} \ge \limsup_{n\to\infty} \|uC_\varphi f_n\|_{B^\beta}.$$

Thus for uC_φ acting from B^α to B^β, $0 < \alpha \le 1$ and $0 < \beta < \infty$

$$\begin{aligned}
\|uC_\varphi\|_e &\ge \limsup_{n\to\infty} \|uC_\varphi f_n\|_{B^\beta} \\
&\ge \limsup_{n\to\infty} \sup_{z\in D} |(uC_\varphi f_n)'(z)|(1-|z|^2)^\beta \\
&\ge \lim_{n\to\infty} \sup_{\varphi(z)\in A_n} |(uC_\varphi f_n)'(z)|(1-|z|^2)^\beta \\
&\ge \lim_{n\to\infty} \sup_{\varphi(z)\in A_n} \{|u(z)||\varphi'(z)||f_n'(\varphi(z))| \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}(1-|\varphi(z)|^2)^\alpha \\
&\quad - |u'(z)f_n(\varphi(z))|(1-|z|^2)^\beta\}.
\end{aligned}$$

When $0 < \alpha < 1$ we know that $u \in B^\beta$ so that

$$\begin{aligned}
\lim_{n\to\infty} \sup_{\varphi(z)\in A_n} &|u'(z)|\,|f(n)(\varphi(z))|(1-|z|^2)^\beta \\
&\le \|u\|_{B^\beta} \lim_{n\to\infty} \sup_{\varphi(z)\in A_n} |f_n(\varphi(z))| \\
&= \|u\|_{B^\beta} \lim_{n\to\infty} \frac{(\frac{n}{n+2\alpha})^{n/2}}{n(\frac{2\alpha}{n-1+2\alpha})^\alpha(\frac{n-1}{n-1+2\alpha})^{(n-1)/2}} \\
&= 0.
\end{aligned}$$

When $\alpha = 1$ we have, by (5),

$$\lim_{n\to\infty} \sup_{\varphi(z)\in A_n} |u'(z)|\,|f_n(\varphi(z))|(1-|z|^2)^\beta \leq C \lim_{n\to\infty} \sup_{\varphi(z)\in A_n} |f_n(\varphi(z))| \bigg(\log \frac{1}{1-|\varphi(z)|^2} \bigg)^{-1}.$$

Since

$$\lim_{n\to\infty} \sup_{\varphi(z)\in A_n} |f_n(\varphi(z))| = \lim_{n\to\infty} \frac{(\frac{n}{n+2})^{n/2}}{n(\frac{2}{n+1})(\frac{n-1}{n+1})^{(n-1)/2}} = \frac{1}{2},$$

we get that

$$\lim_{n\to\infty} \sup_{\varphi(z)\in A_n} |u'(z)|\,|f_n(\varphi(z))|(1-|z|^2)^\beta \leq \frac{C}{2} \lim_{n\to\infty} \sup_{\varphi(z)\in A_n} \bigg(\log \frac{1}{1-|\varphi(z)|^2} \bigg)^{-1} = 0.$$

Therefore, for $0 < \alpha \leq 1$,

$$\begin{aligned} \|uC_\varphi\|_e &\geq \lim_{n\to\infty} \sup_{\varphi(z)\in A_n} |u(z)|\,|\varphi'(z)| \\ &\quad \times \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} \min_{\varphi(z)\in A_n} |f_n'(\varphi(z))|(1-|\varphi(z)|^2)^\alpha \\ &\geq \lim_{n\to\infty} \sup_{\varphi(z)\in A_n} |u(z)|\,|\varphi'(z)| \\ &\quad \times \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} \min_{w\in A_n} |f_n'(w)|(1-|w|^2)^\alpha, \end{aligned}$$

where the minimum is attained at any point on the circle with radius r_{n+1}. Because

$$\limsup_{n\to\infty} \min_{w\in A_n} |f_n'(w)|(1-|w|^2)^\alpha = 1,$$

we get

$$\|uC_\varphi\|_e \geq \lim_{n\to\infty} \sup_{\varphi(z)\in A_n} |u(z)|\,|\varphi'(z)| \frac{(1-|z|^2)^\beta}{(1-\varphi(z)|^2)^\alpha},$$

as desired. □

In Theorem 2.1 of [**7**], a similar argument is used to obtain a lower estimate on $\|C_\varphi\|_e$ for C_φ acting on B or B_0.

Proposition 2. *Let $0 < \beta < \infty$, and let uC_φ be bounded from B into B^β. Then*

$$\|uC_\varphi\|_e \geq \max\left\{A(u,\varphi,\beta), \frac{1}{6}\,B(u,\varphi,\beta)\right\},$$

where $A(u,\varphi,\beta)$ and $B(u,\varphi,\beta)$ are given by (3), (4).

Proof. The inequality

$$\|uC_\varphi\|_e \geq A(u,\varphi,\beta) \tag{6}$$

was already given in Proposition 1. So we need only prove

$$\|uC_\varphi\|_e \geq \frac{1}{6}\,B(u,\varphi,\beta). \tag{7}$$

There is nothing to prove if $B(u,\varphi,\beta)=0$ so we may assume $\|\varphi\|_\infty = 1$ and take a sequence of points $\{a_n\}$ in D such that $|\varphi(a_n)| \to 1$ and so that

$$\lim_{n\to\infty} |u'(a_n)|(1-|a_n|^2)^\beta \log\frac{1}{1-|\varphi(a_n)|^2} = B(u,\varphi,\beta).$$

Let $\lambda_n(z) = -\log(1-\overline{\varphi(a_n)}z)$ and

$$f_n(z) = \frac{\lambda_n^2(z)}{2\lambda_n(\varphi(a_n))}.$$

Then we have $f_n(0)=0$,

$$\begin{aligned} f_n(\varphi(a_n)) &= \frac{1}{2}\log\frac{1}{1-|\varphi(a_n)|^2}; \\ f_n'(z) &= \frac{\lambda_n(z)}{\lambda_n(\varphi(a_n))}\frac{\overline{\varphi(a_n)}}{1-\overline{\varphi(a_n)}z}; \\ f_n'(\varphi(a_n)) &= \frac{\overline{\varphi(a_n)}}{1-|\varphi(a_n)|^2}; \end{aligned}$$

and $\{f_n\}$ is a bounded sequence in B such that $f_n(z) \to 0$ uniformly on compact subsets of D. We estimate $\|f_n\|_B$ next.

$$\begin{aligned}\|f_n\|_B &= \sup_{z\in D} |f_n'(z)|(1-|z|^2)\\ &= \sup_{z\in D} \Big|\frac{\lambda_n(z)}{\lambda_n(\varphi(a_n))}\Big| \frac{|\varphi(a_n)|}{|1-\overline{\varphi(a_n)}z|}(1-|z|^2)\\ &\le 2\sup_{z\in D}\Big|\frac{\lambda_n(z)}{\lambda_n(\varphi(a_n))}\Big|\\ &\le 2\bigg(\frac{\log\frac{1}{1-|\varphi(a_n)|}+2\pi}{\log\frac{1}{1-|\varphi(a_n)|}+\log\frac{1}{1+|\varphi(a_n)|}}\bigg).\end{aligned}$$

Thus, $\lim_{n\to\infty}\|f_n\|_B \le 2$. Let $g_n = f_n/\|f_n\|_B$. Then $\|g_n\|_B = 1$ and $g_n \to 0$ uniformly on compact subsets of D. Since $g_n \in B_0$, this ensures that g_n tends to 0 weakly in B, and thus for any compact operator $K: B \to B^\beta$, $\lim_{n\to\infty}\|Kg_n\|_B = 0$. Therefore

$$\begin{aligned}\|uC_\varphi - K\| &= \sup_{\|f\|_B\le 1}\|(uC_\varphi - K)f\|_{B_\beta}\\ &\ge \lim_{n\to\infty}(uC_\varphi - K)g_n\|_{B^\beta}\\ &\ge \lim_{n\to\infty}\|uC_\varphi g_n\|_{B^\beta}.\end{aligned}$$

Hence

$$\begin{aligned}\|uC_\varphi\|_e &= \inf_{K\text{ compact}}\|uC_\varphi - K\|\\ &\ge \lim_{n\to\infty}\|uC_\varphi g_n\|_{B^\beta}\\ &= \limsup_{n\to\infty}\sup_{z\in D}|u(z)g_n'(\varphi(z))\varphi'(z)+u'(z)g_n(\varphi(z))|(1-|z|^2)^\beta\\ &\ge \limsup_{n\to\infty}\frac{1}{\|f_n\|_B}\big[|u'(a_n)f_n(\varphi(a_n))|\\ &\quad -|u(a_n)f_n'(\varphi(a_n))\varphi'(a_n)|\big](1-|a_n|^2)^\beta\\ &\ge \frac{1}{2}\limsup_{n\to\infty}\bigg[\frac{1}{2}|u'(a_n)|(1-|a_n|^2)^\beta\log\frac{1}{1-|\varphi(a_n)|^2}\\ &\quad -|u(a_n)||\varphi'(a_n)||\varphi(a_n)|\frac{(1-|a_n|^2)^\beta}{1-|\varphi(a_n)|^2}\bigg].\end{aligned}$$

By equation (6),

$$\|uC_\varphi\|_e \geq \limsup_{n\to\infty} |u(a_n)|\,|\varphi'(a_n)|\,|\varphi(a_n)|\frac{(1-|a_n|^2)^\beta}{1-|\varphi(a_n)|^2}.$$

Thus,

$$\|uC_\varphi\|_e \geq \frac{1}{4}\lim_{n\to\infty} |u'(a_n)|(1-|a_n|^2)^\beta \log\frac{1}{1-|\varphi(a_n)|^2} - \frac{1}{2}\|uC_\varphi\|_e.$$

Hence

$$\frac{3}{2}\|uC_\varphi\|_e \geq \frac{1}{4}\lim_{n\to\infty} |u'(a_n)|(1-|a_n|^2)^\beta \log\frac{1}{1-|\varphi(a_n)|^2}.$$

So

$$\|uC_\varphi\|_e \geq \frac{1}{6}\lim_{n\to\infty} |u'(a_n)|(1-|a_n|^2)^\beta \log\frac{1}{1-|\varphi(a_n)|^2}.$$

The proof is complete. □

3. The upper estimates.

We begin with two lemmas. For $r \in (0,1)$, let $K_r f(z) = f(rz)$. Then K_r is a compact operator on the space B^α or B_0^α for any positive number α, with $\|K_r\| \leq 1$.

Lemma 1. *Let $0 < \alpha < 1$. Then there is a sequence $\{r_k\}$, $0 < r_k < 1$, tending to 1, such that the compact operator $L_n = (1/n)\sum_{k=1}^n K_{r_k}$ on B_0^α satisfies*

(i) *For any $t \in [0,1)$, $\lim_{n\to\infty} \sup_{\|f\|_{B^\alpha}\leq 1} \sup_{|z|\leq t} |((I-L_n)f)'(z)| = 0$*

(ii) $\lim_{n\to\infty} \sup_{\|f\|_{B^\alpha}\leq 1} \sup_{z\in D} |(I-L_n)f(z)| = 0$.

(iii) $\limsup_{n\to\infty} \|I-L_n\| \leq 1$.

Furthermore, these statements hold as well for the sequence of biadjoints L_n^{**} on B^α.

Proof. The argument is much like that given in the proof of Proposition 2.1 of [**8**], where a similar result is obtained for the operators L_n acting on the weighted Banach spaces H_β^∞, so we only highlight the new ideas needed in the Lipschitz space setting. It is enough to prove that, for any t, $0 < t < 1$, and any $\varepsilon > 0$, there is an $N > 0$ such that, for any $n > N$,

$$\sup_{\|f\|_{B^\alpha}\le 1}\sup_{|z|\le t}|((I-L_n)f)'(z)| < \varepsilon; \tag{8}$$

$$\sup_{\|f\|_{B^\alpha}\le 1}\sup_{z\in D}|(I-L_n)f(z)| < \varepsilon; \tag{9}$$

and

$$\|I-L_n\| < 1+2\varepsilon. \tag{10}$$

We show that if the r_k's used to define L_n are chosen to satisfy

$$r_k > 1-\frac{1}{k^{2/(1-\alpha)}} \tag{11}$$

then (9) will be satisfied for all n sufficiently large. This follows from the fact that, for $0 < \alpha < 1$, $B^\alpha = \mathrm{Lip}_{1-\alpha}$, with comparable norms. Thus, for all n,

$$\begin{aligned}\sup_{\|f\|_{B^\alpha}\le 1}\sup_{z\in D}|(I-L_n)f(z)| &\le \frac{1}{n}\sup_{\|f\|_{B^\alpha}\le 1}\sup_{z\in D}\sum_{k=1}^{n}|f(z)-f(r_kz)|\\ &\le \frac{1}{n}\sup_{z\in D}C\sum_{k=1}^{n}|z-r_kz|^{1-\alpha}\\ &\le \frac{C}{n}\sum_{k=1}^{n}(1-r_k)^{1-\alpha}\\ &\le \frac{C}{n}\sum_{k=1}^{n}\frac{1}{k^2}=\frac{C\pi^2}{6n}.\end{aligned}$$

We take $N > (C\pi^2/6\varepsilon)$. Then, for $n > N$,

$$\sup_{\|f\|_{B^\alpha}\le 1}\sup_{z\in D}|(I-L_n)f(z)| < \varepsilon$$

and (9) is satisfied. Note that, since $t < 1$, (8) follows immediately from (9) by Cauchy's derivative formula.

Property (10) follows exactly as in the proof of Proposition 2.1 of [**8**], so we omit the details here. This proves the statements for B_0^α.

The proof of the corresponding statements for the biadjoints L_n^{**} on B^α follows similarly to the corresponding argument in [**8**, Proposition 2.1]. For $0 < \alpha < 1$, the dual of B_0^α can be identified with the Bergman space A^1 of analytic functions in $L^1(D, dA)$ [**12**] and the dual of A^1 is identified with B^α, under the pairing $(\sum a_n z^n, \sum b_n z^n) = \lim_{t\to 1} \sum \Gamma(\alpha) n!/\Gamma(\alpha+n+1) a_n \overline{b_n} t^{2n}$. The operators K_r^* and K_r^{**} assign to each function g its dilate g_r, hence we may write $L_n^{**} = L_n$. Then property (iii) follows from $\|I - L_n^{**}\| = \|I - L_n\|$. Finally note that the proofs of (i) and (ii) are unchanged when B_0^α is replaced by B^α, since that part of the argument only depended on the identification of B^α as $\mathrm{Lip}_{1-\alpha}$ and the restriction $r_k > 1 - k^{2/(\alpha-1)}$. □

The analogue of Lemma 1 for the case $\alpha = 1$ is next.

Lemma 2. *There is a sequence* $\{r_k\}$, $0 < r_k < 1$, *tending to* 1, *such that the compact operator* $L_n = (1/n)\sum_{k=1}^n K_{r_k}$ *acting on* B_0 *satisfies*

(i) *For any* $t \in [0,1)$, $\lim_{n\to\infty} \sup_{\|f\|_B \le 1} \sup_{|z|\le t} |((I - L_n)f)'(z)| = 0$

(iia) $\limsup_{n\to\infty} \sup_{\|f\|_B\le 1} \sup_{|z|>s} |(I - L_n)f(z)| \left(-\log(1-|z|^2)\right)^{-1} \le 1$ *for* s *sufficiently close to* 1 *and*

(iib) $\lim_{n\to\infty} \sup_{\|f\|_B\le 1} \sup_{|z|\le s} |(I - L_n)f(z)| = 0$, *for the above* s.

(iii) $\limsup_{n\to\infty} \|I - L_n\| \le 1$.

Furthermore, the same is true for the sequence of biadjoints L_n^{**} *on* B.

Proof. The proof is similar to the proof of Lemma 1, except that in considering $|(I - L_n)f(z)|$ we choose our increasing sequence $\{r_k\}$ tending to 1 such that

$$r_k \ge \frac{(1+s) - (1-s)e^{1/k^2}}{[(1+s) + (1-s)e^{1/k^2}]s}, \tag{12}$$

where $\frac{e-1}{e+1} < s < 1$, which ensures that $r_k \in (0,1)$ for any integer $k \ge 1$. For $\|f\|_B \le 1$, we use the Bloch space estimate from equation (1) to see that

$$\begin{aligned}|(I-L_n)f(z)|&\left(\log\frac{1}{1-|z|^2}\right)^{-1}\\&\le\frac{1}{n}\sum_{k=1}^{n}|f(z)-f(r_kz)|\left(\log\frac{1}{1-|z|^2}\right)^{-1}\\&\le\frac{1}{n}\sum_{k=1}^{n}\frac{1}{2}\log\left(\frac{1+|z|}{1-|z|}\cdot\frac{1-r_k|z|}{1+r_k|z|}\right)\left(\log\frac{1}{1-|z|^2}\right)^{-1}\\&\le\frac{1}{n}\sum_{k=1}^{n}\frac{1}{2}\log\left(\frac{2}{1-|z|}\right)\left(\log\frac{1}{1-|z|^2}\right)^{-1}\le 1\end{aligned}$$

provided $7/8\le|z|<1$.

Thus for $\max\{\frac{e-1}{e+1},\frac{7}{8}\}<s<1$,

$$\limsup_{n\to\infty}\sup_{\|f\|_B\le 1}\sup_{|z|>s}|(I-L_n)f(z)|\left(\log\frac{1}{1-|z|^2}\right)^{-1}\le 1, \tag{13}$$

which is (iia).

For $|z|\le s$ we see also by equation (1) that, if $\|f\|_B\le 1$,

$$\begin{aligned}|(I-L_n)f(z)|&\le\frac{1}{n}\sum_{k=1}^{n}|f(z)-f(r_kz)|\\&\le\frac{1}{n}\sum_{k=1}^{n}\frac{1}{2}\log\left(\frac{1+s}{1-s}\cdot\frac{1-r_ks}{1+r_ks}\right)\\&\le\frac{1}{2n}\sum_{k=1}^{n}\frac{1}{k^2}\le\frac{\pi^2}{12n}.\end{aligned}$$

For the penultimate inequality we have used the assumption (12). This gives (iib).

Parts (i) and (iii) are obtained exactly as in Lemma 1; we leave the details to the reader. Similarly, the statement about the biadjoints proceeds along the same lines as in Lemma 1, since $B_0^*=A^1$ and $(A^1)^*=B$, where A^1 is the unweighted Bergman space of analytic functions in $L^1(dA)$.

Proposition 3. *Suppose* $uC_\varphi : B^\alpha \to B^\beta$ *is bounded, where* $0 < \alpha < 1$ *and* $\beta > 0$. *Then*

$$\|uC_\varphi\|_e \le \lim_{s\to 1} \sup_{|\varphi(z)|>s} |u(z)|\,|\varphi'(z)| \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}. \tag{14}$$

Proof. Let $\{L_n\}$ be the sequence of operators given in Lemma 1. Since each L_n is compact as an operator from B^α to B^α, so is $uC_\varphi L_n$ and we have

$$\begin{aligned}\|uC_\varphi\|_e &\le \|uC_\varphi - uC_\varphi L_n\| = \|uC_\varphi(I-L_n)\| \\ &= \sup_{\|f\|_{B^\alpha}\le 1} \|uC_\varphi(I-L_n)f\|_{B^\beta}.\end{aligned}$$

We bound this last expression from above by

$$\sup_{\|f\|_{B^\alpha}\le 1} |u(0)|\,|(I-L_n)f(\varphi(0))| \tag{15}$$

$$+ \sup_{\|f\|_{B^\alpha}} \sup_{z\in D} |u(z)|\,|((I-L_n)f)'(\varphi(z))|\,|\varphi'(z)|(1-|z|^2)^\beta \tag{16}$$

$$+ \sup_{\|f\|_{B^\alpha}} \sup_{z\in D} |u'(z)|\,|(I-L_n)f(\varphi(z))|(1-|z|^2)^\beta. \tag{17}$$

Lemma 1 (ii) guarantees that the supremum in (15) can be made arbitrarily small as $n\to\infty$. Since $u \in B^\beta$ and thus $\sup_{z\in D}|u'(z)|(1-|z|^2)^\beta < \infty$, Lemma 1(ii) also ensures that the supremum in (17) tends to 0 as $n\to\infty$.

Now we need only consider the term

$$\sup_{\|f\|_{B^\alpha}\le 1} \sup_{z\in D} |u(z)|\,|((I-L_n)f)'(\varphi(z))|\,|\varphi'(z)|(1-|z|^2)^\beta.$$

For arbitrary $0 < s < 1$, consider

$$\sup_{\|f\|_{B^\alpha}\le 1} \sup_{|\varphi(z)|\le s} |u(z)|\,|((I-L_n)f)'(\varphi(z))|\,|\varphi'(z)|(1-|z|^2)^\beta \tag{18}$$

and

$$\sup_{\|f\|_{B^\alpha}\le 1} \sup_{|\varphi(z)|>s} |u(z)|\,|((I-L_n)f)'(\varphi(z))|\,|\varphi'(z)|(1-|z|^2)^\beta. \tag{19}$$

Since uC_φ is bounded from B^α into B^β, by Theorem 1 we have

$$\sup_{z\in D}|u(z)|\,|\varphi'(z)|\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}<\infty.$$

Thus

$$\sup_{|\varphi(z)|\le s}|u(z)|\,|\varphi'(z)|(1-|z|^2)^\beta<\infty.$$

Thus from (i) of Lemma 1, we see that

(20)

$$\lim_{n\to\infty}\sup_{\|f\|_{B^\alpha}\le 1}\sup_{|\varphi(z)|\le s}|u(z)|\,|(I-L_n)f)'(\varphi(z))|\,|\varphi'(z)|(1-|z|^2)^\beta=0.$$

We write the expression in (19) as

$$\sup_{\|f\|_{B^\alpha}\le 1}\sup_{|\varphi(z)|>s}\frac{|u(z)|\,|\varphi'(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}(1-|\varphi(z)|^2)^\alpha|((I-L_n)f)'(\varphi(z))|$$

and observe that this is bounded above by

$$\|I-L_n\|\sup_{|\varphi(z)|>s}|u(z)|\,|\varphi'(z)|\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}.$$

Thus by (iii) of Lemma 1,

$$\limsup_{n\to\infty}\sup_{\|f\|_{B^\alpha}\le 1}\sup_{|\varphi(z)|>s}|u(z)|\,|((I-L_n)f)'(\varphi(z))|\,|\varphi'(z)|(1-|z|^2)^\beta$$

is bounded by

$$\sup_{|\varphi(z)|>s}|u(z)|\,|\varphi'(z)|\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}.$$

Combining the estimates for (15), (16) and (17) as $n\to\infty$, we get

$$\|uC_\varphi\|_e\le\sup_{|\varphi(z)|>s}|u(z)|\,|\varphi'(z)|\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}.$$

Since s was arbitrary, we see that (14) holds, and the proof is completed.
□

Proposition 4. *Suppose $uC_\varphi : B \to B^\beta$ is bounded. Then*

$$\|uC_\varphi\|_e \le A(u,\varphi,\beta) + B(u,\varphi,\beta),$$

where $A(u,\varphi,\beta)$ and $B(u,\varphi,\beta)$ are given by (3) *and* (4).

Proof. The proof is similar to that of Proposition 3, so we outline the argument and leave the details to the interested reader. Estimate $\|uC_\varphi\|_e$ from above by

$$\|uC_\varphi\|_e \le \sup_{\|f\|_B \le 1} \|uC_\varphi(I-L_n)f\|_{B^\beta},$$

where $\{L_n\}$ is the sequence of operators given in Lemma 2. This expression is then bounded by the sum of the expressions in (15), (16) and (17), where the supremums are now over all $f \in B$, $\|f\| \le 1$. The first supremum

$$\sup_{\|f\|_B \le 1} |u(0)|\,|(I-L_n)f(\varphi(0))|$$

tends to 0 as $n \to \infty$ by Lemma 2(iib). For $\|f\|_B \le 1$ we estimate

$$\sup_{z \in D} |u(z)|\,|(I-L_n)f)'(\varphi(z))|\,|\varphi'(z)|(1-|z|^2)^\beta$$

by the sum

$$\sup_{|\varphi(z)|>s} \frac{|u(z)|\,|\varphi'(z)|(1-|z|^2)^\beta}{1-|\varphi(z)|^2} |((I-L_n)f)'(\varphi(z))|(1-|\varphi(z)|^2)$$

and the supremum over $\{z : |\varphi(z)| \le s\}$ of the same expression, where s is as in Lemma 2. We use the boundedness of the first factor in D, Theorem 2, and Lemma 2(i) to see that

$$\lim_{n\to\infty} \sup_{|\varphi(z)|\le s} \frac{|u(z)|\,|\varphi'(z)|(1-|z|^2)^\beta}{1-|\varphi(z)|^2} |((I-L_n)f)'(\varphi(z))|(1-|\varphi(z)|^2) = 0.$$

For the supremum over $\{|\varphi(z)| > s\}$, note that

$$|((I-L_n)f)'(\varphi(z))|(1-|\varphi(z)|^2) \le \|(I-L_n)f\|_B$$

and apply Lemma 2(iii) to see that

(21)

$$\limsup_{n\to\infty} \sup_{|\varphi(z)|>s} \frac{|u(z)|\,|\varphi'(z)|(1-|z|^2)^\beta}{1-|\varphi(z)|^2} |((I-L_n)f)'(\varphi(z))|(1-|\varphi(z)|^2)$$

is bounded by

$$\sup_{|\varphi(z)|>s} \frac{|u(z)|\,|\varphi'(z)|(1-|z|^2)^\beta}{1-|\varphi(z)|^2}. \tag{22}$$

Finally we estimate, for $\|f\|_B \leq 1$,

$$\sup_{z\in D} |u'(z)|\,|(I-L_n)f(\varphi(z))|(1-|z|^2)^\beta$$

by

$$\sup_{|\varphi(z)|\leq s} |u'(z)|\,|(I-L_n)f(\varphi(z))|(1-|z|^2)^\beta \\ + \sup_{|\varphi(z)|>s} |u'(z)|\,|(I-L_n)f(\varphi(z))|(1-|z|^2)^\beta.$$

Since uC_φ is bounded from B to B^β, $u \in B^\beta$. This observation, together with Lemma 2(iib) give

$$\lim_{n\to\infty} \sup_{\|f\|_B\leq 1} \sup_{|\varphi(z)|\leq s} |u'(z)|\,|(I-L_n)f(\varphi(z))|(1-|z|^2)^\beta = 0.$$

Also, the boundedness of uC_φ from B to B^β guarantees, by Theorem 2, that

$$\sup_{z\in D} |u'(z)|(1-|z|^2)^\beta \log \frac{1}{(1-|\varphi(z)|^2)} < \infty. \tag{23}$$

Using (23) and Lemma 2(iia) we see that

$$\limsup_{n\to\infty} \sup_{\|f\|_B\leq 1} \sup_{|\varphi(z)|>s} |u'(z)|\,|(I-L_n)f(\varphi(z))|(1-|z|^2)^\beta$$

is bounded above by

$$\sup_{|\varphi(z)|>s} |u'(z)|(1-|z|^2)^\beta \log \frac{1}{1-|\varphi(z)|^2}. \tag{24}$$

Taking the limit as $s \to 1$ in (22) and (24) gives the desired conclusion.

4. Compact weighted composition operators.

For (unweighted) composition operators $C_\varphi : B^\alpha \to B^\alpha$, $0<\alpha<1$, it is known that compactness of C_φ implies $\|\varphi\|_\infty < 1$, [**2, 10**]. This does not seem to be an obvious consequence of the characterization of compactness of C_φ on B^α, $0<\alpha<1$, given by the case $u \equiv 1$ of Theorem 1. The next result gives a generalization of this result to weighted composition operators. Recall that boundedness of uC_φ on B^α implies that $u \in B^\alpha$ so that u extends continuously to the closed disk when $\alpha < 1$.

Theorem 5. *If $uC_\varphi : B^\alpha \to B^\alpha$, $0<\alpha<1$, is compact, then $u(\zeta)=0$ whenever $\lim_{r\to 1^-} \varphi(r\zeta)$ exists and has modulus 1.*

Proof. Suppose $\lim_{r\to 1^-}\varphi(r\zeta) = \eta \in \partial D$. Set $\psi(z) = (\zeta+z)/2$, so that ψ is a self-map of D with $\psi(\zeta)=\zeta$ and $|\psi(\omega)|<1$ if $\omega \neq \zeta \in \partial D$. If $\tau = \varphi\circ\psi$, then $\lim_{r\to 1^-}\tau(r\zeta)$ has modulus 1, so, C_τ is not compact on B^α by the above remarks. By the compactness criteria in Theorem 1, this says either

$$\limsup_{|\tau(w)|\to 1} \left(\frac{1-|w|^2}{1-|\tau(w)|^2}\right)^\alpha |\tau'(w)| = d > 0, \text{ or} \tag{25}$$

$$C_\tau : B^\alpha \longrightarrow B^\alpha \text{ is not bounded.} \tag{26}$$

If (25) holds, then the rest of the argument proceeds simply. Take a sequence w_n with $|\tau(w_n)| \to 1$ and

$$\left(\frac{1-|w_n|^2}{1-|\tau(w_n)|^2}\right)^\alpha |\tau'(w_n)| \longrightarrow d > 0.$$

(Note that necessarily $w_n \to \zeta$.) Since C_ψ is bounded on B^α, the weighted composition operator $(u\circ\psi)C_\tau = C_\psi(uC_\varphi)$ is compact on B^α and hence

$$\lim_{|\tau(w)|\to 1} |u(\psi(w))| \left(\frac{1-|w|^2}{1-|\tau(w)|^2}\right)^\alpha |\tau'(w)| = 0.$$

Considering the sequence w_n as above, we see that $u(\psi(w_n)) \to 0$; by continuity of u on $\overline{D}$, this says $u(\zeta) = 0$.

This leaves consideration of the case where (26) holds but (25) does not. In this case, again by Theorem 1, we find $v_n \in D$ with

$$\left(\frac{1-|v_n|^2}{1-|\tau(v_n)|^2}\right)^{\alpha} |\tau'(v_n)| \longrightarrow \infty. \tag{27}$$

Without loss of generality, we may assume $v_n \to v_0 \in \overline{D}$; necessarily, $v_0 \in \partial D$ if (27) holds. If $v_0 \neq \zeta$, then $\psi(v_0) \in D$ and $\lim_{n\to\infty} |\tau(v_n)| < 1$, so (27) cannot hold since

$$(1-|v_n|^2)^{\alpha}|\tau'(v_n)| = \frac{1}{2}(1-|v_n|^2)^{\alpha}|\varphi'(\psi(v_n))|$$

has finite limit as $n \to \infty$ since $\psi(v_n) \to \psi(v_0) \in D$. Thus, $v_n \to \zeta$. If this convergence is nontangential, then the definition of ψ shows that $\psi(v_n) \to \zeta$ nontangentially and $\lim_{n\to\infty} \tau(v_n) = \eta$. Thus we are in fact in the case (25), which has already been dispensed with.

It remains only to consider the case that $v_n \to \zeta$ in such a way that some subsequence does not converge nontangentially. Passing to this subsequence, but not relabeling it, we have

$$1 - |v_n|^2 < |\zeta - v_n|. \tag{28}$$

For simplicity of notation from this point on, we take $\zeta = 1$. Write $v_n = r_n e^{i\theta_n}$ so that equation (28) says

$$3 - 4r_n^2 + r_n^4 < 3 - r_n^2 - 2r_n\cos(\theta_n).$$

Setting $\tilde{v}_n = \psi(v_n) = (1+v_n)/2$ one sees by this estimate that

$$\frac{1-|v_n|^2}{1-|\tilde{v}_n|^2} \leq 2.$$

Now consider uC_φ on B^α. Since this is bounded,

$$\sup_n |u(\tilde{v}_n)| \left(\frac{1-|\tilde{v}_n|^2}{1-|\varphi(\tilde{v}_n)|^2}\right)^{\alpha} |\varphi'(\tilde{v}_n)| < \infty. \tag{29}$$

If $|\tau(v_n)| = |\varphi(\tilde{v}_n)| \to 1$, then equation (27) implies that condition (25) holds and there is nothing further to do. Thus, without loss of generality, we may assume that $\sup_n |\varphi(\tilde{v}_n)| < 1$. In this case we have

$$(1-|v_n|^2)^\alpha |\tau'(v_n)| \longrightarrow \infty$$

which means

$$(1-|\tilde{v}_n|^2)^\alpha \left(\frac{1-|v_n|^2}{1-|\tilde{v}_n|^2}\right)^\alpha |\varphi'(\tilde{v}_n)| \longrightarrow \infty.$$

Our estimates show that the second factor on the left is bounded by 2^α so that

$$(1-|\tilde{v}_n|^2)^\alpha |\varphi'(\tilde{v}_n)| \longrightarrow \infty.$$

This fact, together with (29), forces $\lim_n u(\tilde{v}_n) = 0$; since $\tilde{v}_n \to 1$ and u is continuous on the closed disk, $u(1) = 0$ as desired. □

For unweighted composition operators C_φ acting boundedly from B^α to itself, $0 < \alpha < 1$, it is known [**2**, Corollary 4.10] that φ has finite angular derivative at every $\zeta \in \partial D$ at which φ has radial limit of modulus 1. The next result gives a generalization of this result to weighted composition operators.

Theorem 6. *Suppose uC_φ is bounded from B^α to itself, where $0 < \alpha < 1$. Then $u(\zeta) = 0$ whenever $\lim_{r\to 1^-} \varphi(r\zeta)$ exists and has modulus 1 and φ has infinite angular derivative at ζ.*

Proof. First suppose $\|\varphi\|_\infty = 1$ and φ has infinite angular derivative at every point of ∂D. Then

$$\limsup_{|z|\to 1} \frac{1-|z|}{1-|\varphi(z)|} = 0.$$

Boundedness of uC_φ on B^α implies by Theorem 1 that

$$\sup_{z\in D} |u(z)| \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^\alpha |\varphi'(z)| < \infty.$$

Fix $\alpha < \alpha' < 1$. Then we write

$$\limsup_{|\varphi(z)|\to 1} |u(z)| \left(\frac{1-|z|^2}{1-|\varphi(z)|^2} \right)^{\alpha'} |\varphi'(z)|$$

as

$$\limsup_{|\varphi(z)|\to 1} |u(z)| \left(\frac{1-|z|^2}{1-|\varphi(z)|^2} \right)^{\alpha} |\varphi'(z)| \left(\frac{1-|z|^2}{1-|\varphi(z)|^2} \right)^{\alpha'-\alpha} = 0.$$

Since uC_φ is bounded on B^α and $\alpha' > \alpha$ it is easy to see that uC_φ is bounded on $B^{\alpha'}$. Thus, by Theorem 1, uC_φ is compact from $B^{\alpha'}$ to itself. By Theorem 5, $u(\zeta) = 0$ whenever $\lim_{r\to 1} \varphi(r\zeta)$ exists and has modulus 1. This gives the desired result in the case that φ has no finite angular derivative at any point.

Now suppose φ has radial limit of modulus 1 at ζ_0 and further assume that φ has infinite angular derivative at ζ_0. Set $\psi(z) = (\zeta_0 + z)/2$, so that $\psi(\zeta_0) = \zeta_0$ and $|\psi(z)| < 1$ if $|z| = 1$ and $z \neq \zeta_0$. Now $\tau = \varphi \circ \psi$ has radial limit of modulus one at ζ_0, and infinite angular derivative there and at every other point of ∂D as well. Moreover, $(u \circ \psi)C_\tau$ is bounded on B^α since $(u \circ \psi)C_\tau = C_\psi(uC_\varphi)$. By the first part of the proof, $u \circ \psi(\zeta_0) = 0$ and thus $u(\zeta_0) = 0$ as desired. $\square$

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