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CONVOLUTION TRANSFORM FOR BOEHMIANS

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ABSTRACT. The theory of Boehmians was initiated by J. Mikusinski and P. Mikusinski in 1981 and later, several applications of Boehmians were discovered by P. Mikusinski, D. Nemzer and others. The main aim of this paper is to study certain properties of integral transform, which carries $f(t)$ into $F(x)$ as a convolution, through a kernel $G(x-y)$, given by the map

$$f(t) \longrightarrow F(x) = \int_{\mathbf{R}} f(t)G(x-t)\,dt.$$

We treat the convolution transform as a continuous linear operator on a suitably defined Boehmian space. In this paper, we construct a suitable Boehmian space on which the convolution transform can be defined and the generalized function space $\mathcal{L}'_{c,d}$ can be imbedded. In addition to this, our definition extends the convolution transform to more general spaces and that the definition remains consistent for $\mathcal{L}'_{c,d}$ elements under a suitable condition on c and d. We also discuss the operational properties of the convolution transform on Boehmians and finally end with an example of a Boehmian which is not in any $\mathcal{L}'_{c,d}$ but is convolution transformable.

1. Introduction. Boehmian spaces, defined first in [**8**] is a generalization of Boehme's regular operators [**1**] and Schwartz's distributions [**21**]. The general construction of Boehmians is algebraic in nature. The construction applied to various function spaces yields various spaces of Boehmians [**2, 6, 9, 11, 13, 16**–**17**] and the spaces obtained contain the generalized function spaces defined as dual spaces. Hence, the theory of integral transforms has a natural extension to Boehmian spaces (see, for example, [**4, 5, 7, 10, 12**–**15, 18**]).

The conventional convolution of two suitably restricted functions f and G is given by

$$F(x) = \int_{\mathbf{R}} f(t)G(x-t)\,dt.$$

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This definition can be viewed as an integral transformation that transforms $f(t)$ into $F(x)$ with $G(x-t)$ as its kernel. In this set up, the transform is called the "convolution transform" with kernel G. The theory of convolution transform was first studied by Hirschman and Widder [**3**]. For certain choices of the kernel G, the convolution transform, through appropriate change of variable, reduces to the Laplace, $G(t)=\exp(t)\exp(-\exp(t))$, the Stieltjes, $G(t)=(1/2)\text{sech}\,(t/2)$, the Hilbert $G(t)=1/t$, the Weierstrass, $G(t)=\exp(-t^2/4)$, and the K-transforms (see [**22**, Chapter 6]), respectively. In [**22**], Zemanian extended the convolution transform to the generalized function space $\mathcal{L}'_{c,d}$ under a suitable restriction on c and d.

In [**4, 7**], the authors have discussed the Hilbert transform and the Weierstrass transform on Boehmian spaces. Therefore, it would be of interest to see whether the convolution transform can be extended to the spaces of Boehmians. The main objective of this paper is to define the convolution transform on Boehmians and to discuss certain properties of it. To carry out this task, we construct a suitable Boehmian space on which the convolution transform can be defined and the generalized function space $\mathcal{L}'_{c,d}$ can be imbedded. Moreover, this definition extends the convolution transform to more general spaces than $\mathcal{L}'_{c,d}$ and preserves the transform for $\mathcal{L}'_{c,d}$ elements under suitable conditions on c and d. As in the papers [**4, 5, 7**], we define the convolution transform as a continuous linear map between two Boehmian spaces. Moreover, here the transform becomes an operator on the Boehmian space $\mathcal{B}(T,\Delta)$.

The organization of the material is as follows. In Section 2 we include basic definitions and certain results that are required for our investigation. In Section 3 we discuss the construction of the required Boehmian space. In Section 4 the convolution transform on Boehmians is defined whereas, in Section 5, the operational properties of the convolution transform on Boehmians are studied. Our final section ends with an example of a Boehmian which is not in any of the $\mathcal{L}'_{c,d}$ spaces.

2. Preliminaries. Throughout the paper, $K_{c,d}(x)$ denotes a function of real variable x defined by

$$K_{c,d}(x) = \begin{cases} e^{cx} & x \geq 0, \\ e^{dx} & x < 0, \end{cases} \tag{2.1}$$

where $c, d \in \mathbf{R}$. Then it can be easily shown that

$$\sup_{\substack{x\in\mathbf{R} \\ |t|\leq m}} \frac{K_{c,d}(x-t)}{K_{c,d}(x)} \leq e^{\max\{|c|,|d|\}m}. \tag{2.2}$$

Indeed, this inequality follows from the definition of $K_{c,d}(x)$, if we write the expressions for $K_{c,d}(x-t)$ and $K_{c,d}(x)$ explicitly for various cases such as $x \geq 0$, $x-t<0$, and so on.

As usual, $\mathbf{C}^\infty = \mathbf{C}^\infty(\mathbf{R})$ denotes the space of all complex-valued, infinitely differentiable functions on $\mathbf{R}$ and the members of $\mathbf{C}^\infty$ are called smooth functions or simply the $\mathbf{C}^\infty$-functions on $\mathbf{R}$. Let $\mathcal{D} = \mathcal{D}(\mathbf{R})$ denote the Schwartz testing function space, i.e., $\mathcal{D}$ consists of functions $f \in \mathbf{C}^\infty$ for which the support, denoted by $\operatorname{supp} f$, is compact. Now we recall the definition of the testing function space $\mathcal{L}_{c,d}$ and its dual $\mathcal{L}'_{c,d}$.

The space $\mathcal{L}_{c,d}$ consists of smooth functions f for which the functionals γ_p defined by

$$\gamma_p(f) = \gamma_{c,d,p}(f) = \sup_{x\in\mathbf{R}} |K_{c,d}(x)D^p f(x)|$$

assume finite values. It is easy to see that the testing function space $\mathcal{L}_{c,d}$ is linear under the pointwise addition and the scalar multiplication of its members. In [**22**, p. 49] it is proved that $\{\gamma_p\}_{p=0}^\infty$ is a multinorm on $\mathcal{L}_{c,d}$. Further, by assigning the topology generated by $\{\gamma_p\}$ to $\mathcal{L}_{c,d}$, it can be shown that the space $\mathcal{L}_{c,d}$ is complete and, therefore, it is a Fréchet space (see [**22**, p. 49]). The dual of $\mathcal{L}_{c,d}$ is denoted by $\mathcal{L}'_{c,d}$ and it consists of the continuous linear functionals on $\mathcal{L}_{c,d}$. By assigning the weak topology to $\mathcal{L}'_{c,d}$, it is proved in [**22**, p. 49] that $\mathcal{L}'_{c,d}$ is a complete linear space. Since $\mathcal{L}_{c,d}$ is a testing function space, the dual $\mathcal{L}'_{c,d}$ is a space of generalized functions.

We will require the following well-known result for dual space [**22**, p. 52]. For each $f \in \mathcal{L}'_{c,d}$, a positive constant C and a nonnegative integer r exist such that, for all $\phi \in \mathcal{L}_{c,d}$,

$$|\langle f, \phi \rangle| \leq C \max_{0 \leq p \leq r} \gamma_{c,d,p}(\phi). \tag{2.3}$$

Moreover, from [**22**, p. 53], if $f(t)$ is a locally integrable function such that $f(t)/K_{c,d}(t)$ is absolutely integrable on $-\infty < t < \infty$, then $f(t)$ generates a member of $\mathcal{L}'_{c,d}$ through the representation

$$\langle f, \phi \rangle = \int_{\mathbf{R}} f(t)\phi(t)\, dt, \quad \phi \in \mathcal{L}_{c,d}. \tag{2.4}$$

In this case f is called a *regular element* of $\mathcal{L}'_{c,d}$. We require a few more definitions for our present investigation.

2.5 Definition. Let $f \in \mathcal{L}'_{c,d}$ and $\psi \in \mathcal{L}_{c,d}$. The convolution $f * \psi$ is defined as a $\mathbf{C}^\infty$-function through

$$(f * \psi)(x) = \langle f, \tau_x \check{\psi} \rangle, \quad \text{for all } x \in \mathbf{R},$$

where $\check{\psi}(t) = \psi(-t)$ and $\tau_x \psi(t) = \psi(t - x)$.

We remark that $f * \psi$ can also be written as $(f * \psi)(x) = \langle f(t), \psi(x - t) \rangle$.

2.6 Definition. The convolution $f * g$ of two generalized functions f and g in $\mathcal{L}'_{c,d}$ is defined as an element of $\mathcal{L}'_{c,d}$ through

$$\langle f * g, \phi \rangle = \langle f(t), \langle g(y), \phi(t + y) \rangle \rangle, \quad \phi \in \mathcal{L}_{c,d}.$$

Now we recall the following basic result on convolutions from [**22**, p. 76].

2.7 Theorem. *If $f, g \in \mathcal{L}'_{c,d}$, $c \leq d$, then $f * g \in \mathcal{L}'_{c,d}$. In particular, if f and g are regular elements of $\mathcal{L}'_{c,d}$, then so is the convolution $f * g$.*

A simple application of Theorem 2.7 is described in

2.8 Theorem. *If* $f \in \mathcal{L}'_{c,d}$, $\phi, \psi \in \mathcal{D}$, *then* $(f*\phi)*\psi = f*(\phi*\psi) = (f*\psi)*\phi$.

Proof. As $\mathcal{D} \subseteq \mathcal{L}_{c,d}$, by Definition 2.5, $f*\phi \in \mathbf{C}^\infty$. Since $f*\phi \in \mathbf{C}^\infty$ and $\psi \in \mathcal{D}$, it can be easily seen that (see also Theorem 6.30(b) in [**19**, p. 156]), $(f*\phi)*\psi \in \mathbf{C}^\infty$. Similarly, we find that $(f*\psi)*\phi \in \mathbf{C}^\infty$. Now $\phi, \psi \in \mathcal{D}$ implies that $\phi*\psi \in \mathcal{D} \subseteq \mathcal{L}_{c,d}$, and therefore $f*(\phi*\psi) \in \mathbf{C}^\infty$. These observations imply that all the terms in the required equalities are $\mathbf{C}^\infty$ functions.

We also observe that, since $f \in \mathcal{L}'_{c,d}$ and $\phi \in \mathcal{D} \subseteq \mathcal{L}_{c,d} \subseteq \mathcal{L}'_{c,d}$, $f*\phi \in \mathcal{L}'_{c,d}$, by Theorem 2.7. Thus, in view of Definitions 2.5 and 2.6, we can write

$$\begin{aligned}((f*\phi)*\psi)(x) &= \langle f*\phi, \tau_x\check{\psi}\rangle \\ &= \langle f(t), \langle \phi(y), (\tau_x\check{\psi})(t+y)\rangle\rangle \\ &= \langle f(t), (\phi*\psi)(x-t)\rangle \\ &= (f*(\phi*\psi))(x)\end{aligned}$$

which shows that

$$((f*\phi)*\psi)(x) = (f*(\phi*\psi))(x).$$

Similarly, we can prove that

$$((f*\psi)*\phi)(x) = (f*(\psi*\phi))(x) = (f*(\phi*\psi))(x),$$

and the proof is complete. □

From now on, G denotes the convolution kernel and D stands for the conventional differentiation. We assume that, to every permissible kernel G, at least one sequence of operators $\{P_n(D)\}_{n=0}^\infty$ exists, where $P_0(D)$ is the identity operator and

$$P_n(D) = e^{B_nD}Q_n(D), \quad n = 1, 2, 3, \dots, \tag{2.9}$$

where B_n is a real constant and Q_n is a polynomial with real coefficients. Recall that, for each differentiable function $\phi(t)$,

$$e^{aD}\phi(t) = \left(\sum_{k=0}^\infty \frac{a^k}{k!}D^k\right)\phi(t) = \sum_{k=0}^\infty \frac{a^k\phi^{(k)}(t)}{k!} = \phi(t+a) \tag{2.10}$$

and, in particular, $e^{B_n D}\phi(t) = \phi(B_n + t)$. Let $G_n(t) = P_n(D)G(t)$. For our discussion on generalized convolution transformation, we require the following conditions on G [**22**, p. 231]:

A1. $G(t)$ is smooth (and hence $G_n(t)$ is smooth on $\mathbf{R}$).

A2. $G_n(t) \geq 0$, $n = 0, 1, 2, \dots$.

A3. $\displaystyle\int_{-\infty}^{\infty} G_n(t)\, dt = 1$, $n = 0, 1, 2, \dots$.

A4. $\lim_{n\to\infty} G_n(t) = 0$, $0 < |t| < \infty$.

A5. For every $\delta > 0$, $\lim_{n\to\infty} \displaystyle\int_{|t|>\delta} G_n(t) = 0$.

A6. Given any real number c and δ with $\delta > 0$, a positive integer $N(c,\delta)$ exists such that for each $n > N(c,\delta)$, $e^{-ct}G_n(t)$ is monotonic increasing on $-\infty < t < \delta$ and monotonic decreasing on $\delta < t < \infty$.

A7. $G(t)$ has the following asymptotic relation as $t \to \pm\infty$:

(i) At least one real negative number β exists such that, for each $k = 0, 1, 2, \dots,$

$$D^k G(t) = O(e^{\beta t}), \quad t \to \infty.$$

(ii) At least one real positive number α exists such that, for each $k = 0, 1, 2, \dots,$

$$D^k G(t) = O(e^{\alpha t}), \quad t \to -\infty.$$

Let α_1 be the infimum of all β for which (i) holds, and α_2, the supremum of all α for which (ii) holds. Here we also allow $\alpha_1 = -\infty$ and $\alpha_2 = +\infty$. Examples of G satisfying all the above conditions are given by Hirschman and Widder [**3**] (see also [**22**, p. 232]).

We recall the following important results about convolution kernel from [**22**, p. 234].

2.11 Theorem. *Let $c < \alpha_2$ and $d > \alpha_1$. Then, for any fixed real number x, the kernel $G(x - t)$ as a function of t is in $\mathcal{L}_{c,d}$.*

In view of Theorem 2.11, we frame the following definition for the convolution transform on $\mathcal{L}'_{c,d}$, and we use this definition for our present investigation.

2.12 Definition. For $c < \alpha_2$, $d > \alpha_1$ and $f \in \mathcal{L}'_{c,d}$, we define the convolution transform F of f as $F(x) = \langle f(t), G(x-t)\rangle$ for $x \in \mathbf{R}$.

2.13 Theorem. *Let $c < \alpha_2$, $d > \alpha_1$ and $f \in \mathcal{L}'_{c,d}$. Then, we have*

(i) *The convolution transform F of f is a smooth function on $\mathbf{R}$ and*

$$F^{(k)}(x) = \langle f(t), G^{(k)}(x-t)\rangle, \quad k = 1, 2, 3\ldots,$$

where $F^{(k)}(x)$ denotes the kth derivative of F with respect to x.

(ii) *The convolution transform F of f is in $\mathcal{L}_{a,b}$ whenever a and b are such that $a < \min\{-\alpha_1, -c\}$ and $b > \max\{-\alpha_2, -d\}$.*

2.14 Theorem. *For a given kernel G, let α_1 and α_2 be defined as in assumption* A7*, and let $P_n(D)$, $n = 0, 1, 2, \ldots$, be defined by* (2.9)*. If F is the convolution transform of $f \in \mathcal{L}'_{c,d}$ on $\mathbf{R}$, $c < \alpha_2$, $d > \alpha_1$, then $\lim_{n\to\infty} P_n(D)F = f$ in the sense of convergence in $\mathcal{D}'$. That is, for every $\psi \in \mathcal{D}$,*

$$\lim_{n\to\infty} \langle P_n(D)F, \psi\rangle = \langle f, \psi\rangle.$$

3. Boehmians. Now we discuss the construction of a Boehmian space in which the dual space $\mathcal{L}'_{c,d}$ can be imbedded. For our construction, we need some preparation. Let G be a convolution kernel, and let α_1, α_2 be defined as in assumption (A7). For $a, b \in \mathbf{R}$, let the space $T_{a,b}$ consist of functions $f \in \mathbf{C}^\infty$ such that

$$\|f\|_{a,b,p} = \sup_{x\in\mathbf{R}} \left|\frac{D^p f(x)}{K_{a,b}(x)}\right| < \infty, \quad p = 0, 1, 2, \ldots.$$

Let $\{a_n\}$ be an increasing sequence of positive real numbers converging to α_2 and $\{b_n\}$ a decreasing sequence of negative real numbers converging to α_1. Define the space T by

$$T = \bigcup_{n=1}^{\infty} T_{a_n,b_n}.$$

It is easy to show that $T_{a_n,b_n} \subseteq T_{a',b'}$, whenever $a_n \le a' \le \alpha_2$ and $\alpha_1 \le b' \le b_n$. In particular, $T_{a_n,b_n} \subseteq T_{a_{n+1},b_{n+1}}$. Therefore, if $f \in T$, then $f \in T_{a_n,b_n}$, and hence belongs to T_{a_m,b_m} whenever $m \geq n$. For the sake of simplicity, we write $\|f\|_{n,p} := \|f\|_{a_n,b_n,p}$. It can be easily proved that, with the family of semi-norms given by $\{\|\cdot\|_{n,p}\}_{p\geq 0}$, each T_{a_n,b_n}, $n \in \mathbf{N}$, is a Fréchet space. Moreover, T is a complete countable union space and, therefore, T is a testing function space with $\mathcal{D} \subseteq T$. For a general discussion on *testing function space*, we refer to [**22**, p. 38].

A sequence $\{f_m\}$ in T is said to converge to f in T if and only if all the f_m and f are contained in one of T_{a_n,b_n} and, for each $p \in \mathbf{N} \cup \{0\}$,

$$\|f_m - f\|_{n,p} \longrightarrow 0 \quad \text{as } m \to \infty.$$

Now we produce a simple example of a function in $T_{a,b}$. Consider the $\mathbf{C}^\infty$-function defined by $f(x) = e^{-|x|}$ for $|x| > 1$. Then $f^{(p)}(x) = (-1)^p e^{-|x|}$, for each $p \in \mathbf{N} \cup \{0\}$. Now

$$\frac{D^p f(x)}{K_{a,b}(x)} = \begin{cases} e^{-x(a+1)} & x \geq 0, \\ e^{-x(b-1)} & x < 0, \end{cases}$$

and therefore, for each $a \in [-1, \infty)$ and $b \in (-\infty, 1]$, we have $\|f\|_{a,b,p} \leq 1$. Thus, $f \in T_{a,b}$ whenever $a \in [-1, \infty)$ and $b \in (-\infty, 1]$.

We start proving our first result which implies that the convolution '$*$' can be thought of as a map from $T \times \mathcal{D}$ to T, where $T \times \mathcal{D} = \{f * \phi : f \in T, \phi \in \mathcal{D}\}$.

3.1 Theorem. *If $f \in T$ and $\phi \in \mathcal{D}$, then $f * \phi \in T$.*

Proof. Since $f \in T$, a $k \in \mathbf{N}$ exists such that $f \in T_{a_k,b_k}$. Therefore $f \in \mathbf{C}^\infty$ and, for each $p \in \mathbf{N} \cup \{0\}$,

$$\|f\|_{k,p} = \sup_{x\in\mathbf{R}} \left| \frac{D^p f(x)}{K_{a_k,b_k}(x)} \right| < \infty. \tag{3.2}$$

As $\phi \in \mathcal{D}$, $f * \phi$ is in $\mathbf{C}^\infty$ (see also Theorem 6.31 in [**19**, p. 156]). We shall now proceed to obtain the estimates for $f * \phi$. Since $\phi \in \mathcal{D}$, we can take $\operatorname{supp} \phi \subseteq [-m, m]$. Therefore,

$$D^p(f * \phi)(x) = \int_{\mathbf{R}} D^p f(x - y)\phi(y)\, dy = \int_{-m}^{m} D^p f(x - y)\phi(y)\, dy$$

so that

$$
\begin{aligned}
|D^p(f*\phi)(x)| &\le \int_{-m}^{m} |D^p f(x-y)||\phi(y)|\,dy \\
&\le \int_{-m}^{m} \|f\|_{k,p} K_{a_k,b_k}(x-y)|\phi(y)|\,dy \quad \text{by (3.2)}, \\
&= \|f\|_{k,p} \int_{-m}^{m} \frac{K_{a_k,b_k}(x-y)}{K_{a_k,b_k}(x)} K_{a_k,b_k}(x)|\phi(y)|\,dy.
\end{aligned} \tag{3.3}
$$

By (2.2), $\sup_{|y|\le m}(K_{a_k,b_k}(x-y)/K_{a_k,b_K}(x)) \le e^{\max\{a_k,|b_k|\}m} < \infty$ and, since $\phi \in \mathcal{D}$, we obtain that $\int_{-m}^{m} |\phi(y)|\,dy < \infty$. In view of these observations, we can write (3.3) as

$$
\frac{|D^p(f*\phi)(x)|}{K_{a_k,b_k}(x)} < \infty \quad \text{for all } x \in \mathbf{R},
$$

from which we get that, for each $p \in \mathbf{N} \cup \{0\}$,

$$
\sup_{x\in\mathbf{R}} \left| \frac{D^p(f*\phi)(x)}{K_{a_k,b_k}(x)} \right| < \infty.
$$

This observation shows that $f*\phi \in T_{a_k,b_k}$ and hence $f*\phi \in T$. □

3.4 Definition. Let Δ be a class of sequences from $\mathcal{D}$. The elements of Δ are called "Delta sequences" if the following conditions are satisfied [**9**, p. 160]:

1. If $f, g \in T$, $\{\phi_n\} \in \Delta$ and $f*\phi_n = g*\phi_n$ for all $n \in \mathbf{N}$, then $f = g$ in T.

2. If $\{\phi_n\}, \{\psi_n\} \in \Delta$, then $\{\phi_n * \psi_n\} \in \Delta$.

We aim to construct a class Δ satisfying the conditions of *Delta sequences*. For this we consider the class Δ consisting of sequences $\{\phi_n\}$ from $\mathcal{D}$ satisfying the conditions

(Δ1) $\int_{\mathbf{R}} \phi_n(x)\,dx = 1,$

(Δ2) $\int_{\mathbf{R}} |\phi_n(x)|\,dx \le M,$

(Δ3) $\operatorname{supp}\phi_n \to 0$ as $n \to \infty$.

3.5 Theorem. *Let $f \in T$ and $\{\phi_n\} \in \Delta$. Then $f * \phi_n \to f$ as $n \to \infty$ in T.*

Proof. As $f \in T$, a $k \in \mathbf{N}$ exists such that $f \in T_{a_k, b_k}$. Let $c = \max\{a_k, |b_k|\}$. Choose $m \in \mathbf{N}$ such that $\alpha_1 \le b_m \le b_k$, $a_k \le a_m \le \alpha_2$ and

$$b_k - b_m = a_m - a_k = \eta, \tag{3.6}$$

where the equalities are assumed whenever a_k and b_k are constant sequences. As $f \in T_{a_k, b_k}$, $f \in T_{a_m, b_m}$ and, by Theorem 3.1, $f * \phi_n \in T_{a_m, b_m}$ for each $n \in \mathbf{N}$. Let us now show that $f * \phi_n \to f$ as $n \to \infty$ in T_{a_m, b_m}. As $f \in \mathbf{C}^\infty$, $f(x)$ is uniformly continuous on $[-2q, 2q]$ for any $q \in \mathbf{R}$ and, therefore, given $\varepsilon > 0$, a $\delta > 0$ exists such that for $|x| \le q$,

$$|f(x-t) - f(x)| < \frac{\varepsilon}{M} \quad \text{whenever } |t| < \delta. \tag{3.7}$$

Now choose $q > 0$ large enough such that, for $|x| > q$,

$$e^{-\eta|x|} < \frac{\varepsilon}{2M\|f\|_{k,p}} e^{-c\delta}, \tag{3.8}$$

where M is given by $(\Delta 2)$. By $(\Delta 1)$, we can write

$$D^p(f * \phi_n)(x) - D^p f(x) = \int_{\mathbf{R}} D^p(f(x-t) - f(x))\phi_n(t)\, dt.$$

By $(\Delta 3)$ we can choose n large so that $\operatorname{supp} \phi_n \subseteq [-\delta, \delta]$ and thus, we have

$$D^p(f * \phi_n)(x) - D^p f(x) = \int_{-\delta}^{\delta} D^p(f(x-t) - f(x))\phi_n(t)\, dt$$

so that

$$|D^p((f * \phi_n)(x) - f(x))| \le \int_{-\delta}^{\delta} |D^p(f(x-t) - f(x))||\phi_n(t)|\, dt.$$

In view of (3.7) and $(\Delta 2)$, for $|x| \le q$, we can write the above inequality as

$$|D^p(f * \phi_n)(x) - D^p f(x)| \le \frac{\varepsilon}{M} \int_{-\delta}^{\delta} |\phi_n(t)|\, dt \le \varepsilon.$$

From (2.1), one has

$$\frac{1}{K_{a_m,b_m}(x)} = \begin{cases} e^{-a_m x} & x \geq 0 \\ e^{-b_m x} & x < 0. \end{cases}$$

Since $a_m > 0$ and $b_m < 0$, the last equality implies that $\sup_{|x|\leq q}(1/K_{b_m,a_m}(x)) \leq 1$. Thus

$$\sup_{|x|\leq q} \frac{|D^p((f*\phi_n - f))(x)|}{K_{b_m,a_m}(x)} \leq \varepsilon. \tag{3.9}$$

Let us now proceed to obtain an estimate for $\sup_{|x|>q}|D^p((f*\phi_n - f))(x)|/K_{b_m,a_m}(x)$. As $f \in T_{a_k,b_k}$, for each $p \in \mathbf{N} \cup \{0\}$, $\|f\|_{k,p} < \infty$ and

$$\begin{aligned} |D^p f(x-t) - D^p f(x)| &\leq \|f\|_{k,p}\{|K_{a_k,b_k}(x-t)| + |K_{a_k,b_k}(x)|\} \\ &\leq \|f\|_{k,p} K_{a_k,b_k}(x)\left\{\left|\frac{K_{a_k,b_k}(x-t)}{K_{a_k,b_k}(x)}\right| + 1\right\} \\ &\leq \|f\|_{k,p} K_{a_k,b_k}(x)(1+e^{c\delta}), \quad \text{by (2.2)}, \\ &\leq 2\|f\|_{k,p} e^{c\delta} K_{a_k,b_k}(x) \end{aligned}$$

which gives the estimate

$$\frac{|D^p(f(x-t) - f(x))|}{K_{a_m,b_m}(x)} \leq 2\|f\|_{k,p} e^{c\delta} \frac{K_{a_k,b_k}(x)}{K_{a_m,b_m}(x)}.$$

Using (2.1) and (3.6), we can write

$$\frac{K_{a_k,b_k}(x)}{K_{a_m,b_m}(x)} = K_{-\eta,\eta}(x) = e^{-\eta|x|}$$

and, therefore, for each $p \in \mathbf{N} \cup \{0\}$ and $|x| > q$,

$$\frac{|D^p(f(x-t) - f(x))|}{K_{b_m,a_m}(x)} \leq 2\|f\|_{k,p} e^{\delta k_1} e^{-\eta|x|} < \frac{\varepsilon}{M},$$

by (3.8). Thus, in view of ($\Delta 2$), we have

$$\begin{aligned} \sup_{|x|>q} & \frac{|D^p((f*\phi_n - f)(x))|}{K_{b_m,a_m}(x)} \\ &\leq \int_{-\delta}^{\delta} \frac{|D^p(f(x-t) - f(x))|}{K_{b_m,a_m}(x)} |\phi_n(t)|\, dt < \varepsilon. \end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10), we get for each $p \in \mathbf{N} \cup \{0\}$,

$$\sup_{x\in\mathbf{R}} \frac{|D^p((f*\phi_n - f)(x))|}{K_{b_m,a_m}(x)} < 2\varepsilon$$

which proves that $(f*\phi_n - f) \to 0$ as $n \to \infty$ in T_{a_m,b_m}. Thus, $f*\phi_n \to f$ as $n \to \infty$ in T and we complete the proof. □

The following result is an immediate consequence of Theorem 3.5.

3.11 Corollary. *Let $f, g \in T$ and $\{\phi_n\} \in \Delta$. If $f*\phi_n = g*\phi_n$ for each $n \in \mathbf{N}$, then $f = g$ in T.*

3.12 Lemma. *If two sequences $\{\phi_n\}$ and $\{\psi_n\}$ belong to Δ, then so does the sequence $\{\phi_n*\psi_n\}$.*

Proof. The proof follows by routine calculations and the conditions $(\Delta 1)$–$(\Delta 3)$. □

Thus, in view of Corollary 3.11 and Lemma 3.12, we observe that the sequences in the class Δ are delta sequences. By $T^{\mathbf{N}}$ we mean the class of sequences from T. Let $\mathcal{A}$ be the set defined as

$$\mathcal{A} = \{(\{f_n\}, \{\phi_n\}) : \{f_n\} \in T^{\mathbf{N}}, \{\phi_n\} \in \Delta\}.$$

A pair of sequences $(\{f_n\}, \{\phi_n\}) \in \mathcal{A}$ is called a quotient of sequences and is denoted by f_n/ϕ_n if and only if

$$f_m*\phi_n = f_n*\phi_m \quad \text{for all } m, n \in \mathbf{N}.$$

Two quotients of sequences f_n/ϕ_n and g_n/ψ_n are said to be equivalent, denoted by $f_n/\phi_n \sim g_n/\psi_n$, if and only if

$$f_m*\psi_n = g_n*\phi_m \quad \text{for all } m, n \in \mathbf{N}.$$

It can be verified that $\sim$ is an equivalence relation on $\mathcal{A}$ and, hence, it splits $\mathcal{A}$ into equivalence classes. The equivalence classes are called "Boehmians." A Boehmian is represented by $[f_n/\phi_n]$ and the Boehmian space by $\mathcal{B}(T, \Delta)$.

A function $f \in T$ is identified in the Boehmian space $\mathcal{B}(T,\Delta)$ as $[f * \delta_n/\delta_n]$ where $\{\delta_n\} \in \Delta$ and the identification is independent of the delta sequences. We define addition, scalar multiplication and differentiation in $\mathcal{B}(T,\Delta)$ as

$$\left[\frac{f_n}{\phi_n}\right] + \left[\frac{g_n}{\psi_n}\right] = \left[\frac{(f_n * \psi_n) + (g_n * \phi_n)}{\phi_n * \psi_n}\right]$$
$$\left[\frac{\alpha f_n}{\phi_n}\right] = \left[\frac{\alpha f_n}{\phi_n}\right], \quad \alpha \in \mathbf{C}$$
$$D^k\left[\frac{f_n}{\phi_n}\right] = \left[\frac{D^k f_n}{\phi_n}\right],$$

where D^k denotes the kth derivative. The convolution of Boehmians (whenever it makes sense) is defined as

$$\left[\frac{f_n}{\phi_n}\right] * \left[\frac{g_n}{\psi_n}\right] = \left[\frac{f_n * g_n}{\phi_n * \psi_n}\right].$$

3.13 Definition. Let $\mathcal{B}(T,\Delta)$ and $\mathcal{B}(G,\Delta)$ be two Boehmian spaces.

(i) A sequence $\{x_n\}$ in $\mathcal{B}(T,\Delta)$ is said to be δ-convergent to x in $\mathcal{B}(T,\Delta)$ and is denoted by $x_n \xrightarrow{\delta} x$, if a sequence $\{\phi_k\} \in \Delta$ exists such that $x_n * \phi_k$ and $x * \phi_k$ are in T for each $k \in \mathbf{N}$ and that $x_n * \phi_k$ converges to $x * \phi_k$ as $n \to \infty$ in T for each $k \in \mathbf{N}$.

(ii) A map $L : \mathcal{B}(T,\Delta) \to \mathcal{B}(G,\Delta)$ is said to be continuous, if $x_n \xrightarrow{\delta} x$ in $\mathcal{B}(T,\Delta)$ implies $Lx_n \xrightarrow{\delta} Lx$ in $\mathcal{B}(G,\Delta)$.

3.14 Theorem. *If $f \in \mathcal{L}'_{c,d}$, for $c < \alpha_2$ and $d > \alpha_1$, and $\phi \in \mathcal{D}$, then $f * \phi \in T$.*

Proof. As $\mathcal{D} \subseteq \mathcal{L}_{c,d}$, we find that $\phi \in \mathcal{L}_{c,d}$ and, therefore, by Definition 2.5, $f * \phi \in \mathbf{C}^\infty$. Since $c < \alpha_2$ and $\{a_n\}$ is an increasing sequence of positive real numbers, we can find an $a_k > 0$ such that $c < a_k \le \alpha_2$. Similarly, we can find a $b_k < 0$ such that $\alpha_1 \le b_k < d$. Let us now show that $f * \phi \in T_{a_k,b_k}$. That is, we have to show that, for each $p \in \mathbf{N} \cup \{0\}$,

$$\sup_{x \in \mathbf{R}} \left| \frac{D^p(f * \phi)(x)}{K_{a_k,b_k}(x)} \right| < \infty.$$

We note that $\phi \in \mathcal{D}$ implies that $D^p\phi \in \mathcal{D}$, and $D^p(f * \phi) = f * D^p\phi$. Therefore, by Definition 2.5, we have

$$D^p(f * \phi)(x) = (f * D^p\phi)(x) = \langle f(t), D^p\phi(x-t)\rangle$$

so that

$$|D^p(f*\phi)(x)| = |\langle f(t), D^p\phi(x-t)\rangle| \leq C \max_{0\leq q\leq r} \gamma_{c,d,q} D^p\phi(x-t), \tag{3.15}$$

by (2.3). Recall that

$$\gamma_{c,d,q}(D^p\phi(x-t)) = \sup_{t\in\mathbf{R}} |K_{c,d}(t) D_t^{p+q}\phi(x-t)|.$$

Here the differential operator D_t^{p+q} refers to the $(p+q)$ times differentiation with respect to t. If $A = \operatorname{supp} D^{p+q}\phi$, then $D^{p+q}\phi(x-t) = 0$ whenever $x - t \notin A$. Putting $x - t = y$, we see that

$$\begin{aligned}
\gamma_{c,d,q}(D^p\phi(x-t)) &= \sup_{y\in A} |K_{c,d}(x-y) D^{p+q}\phi(y)| \\
&= \sup_{y\in A} \left| \frac{K_{c,d}(x-y)}{K_{c,d}(x)} K_{c,d}(x) D^{p+q}\phi(y)\right| \\
&\leq K_{c,d}(x) e^{\max\{|c|,|d|\}|y|} \sup_{y\in A} |D^{p+q}\phi(y)|,
\end{aligned} \tag{3.16}$$

by (2.2). If $M_{p,q}(c,d) = e^{\max\{|c|,|d|\}|y|} \sup_{y\in A} |D^{p+q}\phi(y)|$, then $M_{p,q}(c,d) < \infty$, as A is compact. Therefore, we can write (3.16) as

$$\gamma_{c,d,q}(D^p\phi(x-t)) \leq M_{p,q}(c,d) K_{c,d}(x),$$

and in view of this inequality, (3.15) reduces to

$$|D^p(f * \phi)(x)| \leq C \max_{0\leq q\leq r} M_{p,q}(c,d) K_{c,d}(x) \leq M_p(c,d) K_{c,d}(x), \tag{3.17}$$

where $M_p(c,d) = C \max_{0\leq q\leq r} M_{p,q}(c,d)$. Therefore,

$$\left| \frac{D^p(f * \phi)(x)}{K_{a_k,b_k}(x)}\right| \leq M_p(c,d) \frac{K_{c,d}(x)}{K_{a_k,b_k}(x)}.$$

Now,

$$\frac{K_{c,d}(x)}{K_{a_k,b_k}(x)} = \begin{cases} e^{(c-a_k)x} & x \geq 0 \\ e^{(d-b_k)x} & x < 0, \end{cases}$$

which is bounded by unity by the choices of a_k and b_k. Thus, for each $p \in \mathbf{N} \cup \{0\}$, we have

$$\sup_{x\in\mathbf{R}} \left| \frac{D^p(f * \phi)(x)}{K_{c,d}(x)} \right| < \infty$$

from which we conclude that $f * \phi \in T_{a_k,b_k}$ and, hence, $f * \phi \in T$. $\square$

3.18 Theorem. *The dual space $\mathcal{L}'_{c,d}$, $c < \alpha_2$, $d > \alpha_1$, can be imbedded in $\mathcal{B}(T, \Delta)$.*

Proof. Let $f \in \mathcal{L}'_{c,d}$ and $\{\phi_n\} \in \Delta$ be a delta sequence. By Theorem 3.14, we see that $f * \phi_n \in T$ for each $n \in \mathbf{N}$. Let $f_n = f * \phi_n$ for each $n \in \mathbf{N}$. In view of Theorem 2.8, it follows that f_n/ϕ_n is a quotient, and hence, $[f_n/\phi_n] \in \mathcal{B}(T, \Delta)$. We shall now prove that the map is one-to-one. By letting $[(f * \phi_n)/\phi_n] = 0$ in $\mathcal{B}(T, \Delta)$, we deduce that $f * \phi_n = 0$ in T for each $n \in \mathbf{N}$ from which it can be proved that $f = 0$ in $\mathcal{L}'_{c,d}$. Hence, the given mapping is one-to-one and we complete the proof. $\square$

4. Convolution transform on Boehmian space. In this section we extend the convolution transform to the Boehmian space $\mathcal{B}(T, \Delta)$. Here it turns out that the convolution transform is an operator on $\mathcal{B}(T, \Delta)$. We also show that the convolution transform on $\mathcal{B}(T, \Delta)$ remains consistent for the elements of $\mathcal{L}'_{c,d}$ with $c < \alpha_2$ and $d > \alpha_1$.

If $f \in T$, then $f \in T_{a_k,b_k}$ for some $0 < a_k \leq \alpha_2$ and $\alpha_1 \leq b_k < 0$. Therefore, in view of (2.4), f is a regular element of $\mathcal{L}'_{a_k,b_k}$. Further, by Theorem 2.11, the convolution kernel G is in $\mathcal{L}_{c,d}$ whenever $c < \alpha_2$ and $d > \alpha_1$. Hence by Definition 2.5, the convolution transform $f * G$ of $f \in T$ is defined which, in view of (2.4), gives

$$(f * G)(x) = \langle f(t), (\tau_x \check{G})(t) \rangle = \int_{\mathbf{R}} f(t)G(x - t)\, dt, \quad -\infty < x < \infty.$$

4.1 Lemma. *Let G be a convolution kernel, and let α_1 and α_2 be as defined in* A7. *Let $c < \alpha_2$ and $d > \alpha_1$ be any two real numbers such* *that $c \le a \le \alpha_2$ and $\alpha_1 \le d \le b$, for some $a, b \in \mathbf{R}$. Then for each $p \in \mathbf{N} \cup \{0\}$, the function $K(x, y)$ defined by*

$$K(x,y) = \frac{K_{c,d}(y)}{K_{a,b}(x)} |D^p G(x-y)|$$

satisfies the inequality

$$K(x,y) \le Be^{-\gamma|y|} \quad \text{for all } (x,y) \in \mathbf{R}^2, \tag{4.2}$$

for some absolute constants B and γ. In particular, $K(x, y)$ is bounded on $\mathbf{R}^2$.

Proof. Let G be a convolution kernel. Then, by property (A7), at least one negative real number β with $\alpha_1 < \beta < d$ and a positive real number α with $a < \alpha < \alpha_2$ exist such that, for each $p \in \mathbf{N} \cup \{0\}$,

$$D^p G(x-y) \le \begin{cases} Be^{\beta(x-y)} & \text{if } x \ge y, \\ Be^{\alpha(x-y)} & \text{if } x < y, \end{cases}$$

for some constant B. To complete the proof we need to consider six cases. We provide a detailed proof of the case $x \ge y \ge 0$ and the remaining cases follow in a similar manner. If $x \ge y \ge 0$, then

$$\begin{aligned} K(x,y) &\le Be^{cy}e^{-ax}e^{\beta(x-y)} \\ &= Be^{(c-\beta)y}e^{(\beta-a)x} \\ &\le Be^{(c-\beta)y}e^{(\beta-a)y} \quad \text{as } \beta - a < 0 \\ &= Be^{(c-a)y}. \end{aligned}$$

Indeed, all six cases can be arranged as

$$K(x,y) \le \begin{cases} Be^{(c-\beta)y}e^{(\beta-a)x} \le Be^{(c-a)y} & \text{for } 0 \le y \le x \\ Be^{(c-\alpha)y}e^{(\alpha-a)x} \le Be^{(c-a)y} & \text{for } 0 \le x \le y \\ Be^{(d-\beta)y}e^{(\beta-b)x} \le Be^{(d-b)y} & \text{for } y \le x \le 0 \\ Be^{(d-\alpha)y}e^{(\alpha-b)x} \le Be^{(d-b)y} & \text{for } x \le y \le 0 \\ Be^{(d-\beta)y}e^{(\beta-a)x} \le Be^{(d-\beta)y} & \text{for } y \le 0 \le x \\ Be^{(c-\alpha)y}e^{(\alpha-b)x} \le Be^{(c-\alpha)y} & \text{for } x \le 0 \le y. \end{cases}$$

The last inequalities and the choices of a, b, α, β show that a positive real number γ exists such that $K(x,y) \le Be^{-\gamma|y|}$, for all $(x,y) \in \mathbf{R}^2$. Thus, we conclude that $K(x,y)$ is bounded in $\mathbf{R}^2$. □

4.3 Theorem. *If $f \in T$ and if G is a convolution kernel, then $f * G \in T$.*

Proof. As $f \in T$, a $k \in \mathbf{N}$ exists such that $f \in T_{a_k,b_k}$. Therefore, f is a regular element of $\mathcal{L}'_{a_k,b_k}$ and, in view of Theorem 2.13, it follows that $f*G \in \mathbf{C}^\infty$. Choose $m \in \mathbf{N}$ such that $a_k \le a_m \le \alpha_2$ and $\alpha_1 \le b_m \le b_k$. Let us now show that $f * G$ is in T_{a_m,b_m} which in turn implies that $f * G \in T$. That is, we have to show that for each $p \in \mathbf{N} \cup \{0\}$, $\sup_{x\in\mathbf{R}} |D^p(f*G)(x)/K_{a_m,b_m}(x)|$ is finite. Now by Theorem 2.13 (i) we have,

$$D^p(f*G)(x) = (f * D^pG)(x) = \int_{\mathbf{R}} f(y) D^pG(x-y)\,dy$$

so that

$$\begin{aligned}|D^p(f*G)(x)| &\le \int_{\mathbf{R}} |f(y)||D^pG(x-y)|\,dy \\ &\le \|f\|_{k,0} \int_{\mathbf{R}} K_{a_k,b_k}(y)|D^pG(x-y)|\,dy,\end{aligned}$$

where $\|f\|_{k,0}$ denotes the zeroth semi-norm of f in T_{a_k,b_k}. Now

$$\frac{|D^p(f*G)(x)|}{K_{a_m,b_m}(x)} \le \|f\|_{k,0} \int_{\mathbf{R}} \frac{K_{a_k,b_k}(y)}{K_{a_m,b_m}(x)} |D^pG(x-y)|\,dy. \tag{4.4}$$

In view of Lemma 4.1,

$$\frac{K_{a_k,b_k}(y)}{K_{a_m,b_m}(x)} |D^pG(x-y)| \le Be^{-\gamma}|y|$$

for all $(x,y) \in \mathbf{R}^2$. Therefore, (4.4) can be written as

$$\frac{|D^p(f*G)(x)|}{K_{a_m,b_m}(x)} \le B\|f\|_{k,0} \int_{\mathbf{R}} e^{-\gamma|y|}\,dy$$

which is finite for each $p \in \mathbf{N} \cup \{0\}$. Thus we conclude that $f * G \in T_{a_m,b_m} \subseteq T$ and complete the proof. $\square$

4.5 Theorem. *If* $f \in \mathcal{L}'_{c,d}$, $c < \alpha_2$, $d > \alpha_1$, *and if* G *is a convolution kernel, then* $f * G \in T$.

Proof. Since $c < \alpha_2$ and $\{a_n\}$ is a sequence of positive real numbers converging to α_2, we can find an $a_k > 0$ such that $c < a_k \leq \alpha_2$. Similarly, we can find a $b_k < 0$ such that $\alpha_1 \leq b_k < d$. By Theorem 2.13, $f * G \in \mathbf{C}^\infty$. We shall now show that, for each $p \in \mathbf{N} \cup \{0\}$, $\sup_{x\in\mathbf{R}} |D^p(f * G)(x)/K_{a_k,b_k}(x)|$ is finite, which in turn implies that $f * G \in T_{a_k,b_k}$ and, hence, $f * G \in T$. In view of Theorem 2.13 (i),

$$D^p(f * G)(x) = (f * D^p G)(x)$$

and, by Definition 2.5,

$$(f * D^p G)(x) = \langle f, \tau_x (D^p G)^V \rangle.$$

Therefore, by (2.3),

$$\begin{aligned} \left| \frac{D^p(f * G)(x)}{K_{a_k,b_k}(x)} \right| &= \left| \left\langle f, \frac{\tau_x (D^p G)^V}{K_{a_k,b_k}(x)} \right\rangle \right| \\ &\leq C \max_{0\leq q\leq r} \gamma_{c,d,q} \left(\frac{\tau_x (D^p G)^V}{K_{a_k,b_k}(x)} \right) \end{aligned} \tag{4.6}$$

for some nonnegative integer r. Now consider

$$\gamma_{c,d,q} \left(\frac{\tau_x (D^p G)^V}{K_{a_k,b_k}(x)} \right) = \sup_{y\in\mathbf{R}} \left| \frac{K_{c,d}(y)}{K_{a_k,b_k}(x)} D_y^{p+q} G(x-y) \right|. \tag{4.7}$$

By Lemma 4.1, $|(K_{c,d}(y)/K_{a_k,b_k}(x)) D_y^{p+q} G(x-y)| < e^{-\gamma|y|}$ for all $(x, y) \in \mathbf{R}^2$, which, in view of (4.7), proves that $\gamma_{c,d,q}(\frac{\tau_x (D^p G)^V}{K_{a_k,b_k}(x)})$ is bounded for all $x \in \mathbf{R}$. Finally, from (4.6) we obtain that, for each $p \in \mathbf{N} \cup \{0\}$,

$$\sup_{x\in\mathbf{R}} \frac{|D^p(f * G)(x)|}{K_{a_k,b_k}(x)} < \infty,$$

so that $f * G \in T_{a_k,b_k}$, and we complete the proof. $\square$

4.8 *Remark.* Since the elements of T are regular elements of $\mathcal{L}'_{c,d}$ for $0 < c < \alpha_2$ and $\alpha_1 < d < 0$, Theorem 4.3 can be deduced from Theorem 4.5.

4.9 Theorem. *Let $f_n \to f$ as $n \to \infty$ in T and G be a convolution kernel. Then $f_n * G$ converges to $f * G$ as $n \to \infty$ in T.*

Proof. Let $f_n \to f$ as $n \to \infty$ in T. Then a $k \in \mathbf{N}$ exists such that $f, f_n \in T_{a_k,b_k}$ for large n and, for each $p \in \mathbf{N} \cup \{0\}$,

$$\sup_{x \in \mathbf{R}} \left| \frac{D^p(f_n - f)(x)}{K_{a_k,b_k}(x)} \right| = \|f_n - f\|_{k,p} \longrightarrow 0 \quad \text{as } n \to \infty. \tag{4.10}$$

Choose $m \in \mathbf{N}$ such that $a_k \le a_m \le \alpha_2$ and $\alpha_1 \le b_m \le b_k$. Clearly, by Theorem 4.3, we find that $f_n * G$ and $f * G$ are in T_{a_m,b_m}. Now our aim is to show that $f_n * G \to f * G$ as $n \to \infty$ in T_{a_m,b_m}. For this, we consider

$$D^p((f_n - f) * G)(x) = ((f_n - f) * D^p G)(x) = \int_{\mathbf{R}} (f_n - f)(y) D^p G(x - y)\, dy$$

so that

$$\begin{aligned} |D^p((f_n - f) * G)(x)| &\le \int_{\mathbf{R}} |(f_n - f)(y)||D^p G(x - y)|\, dy \\ &\le \|f_n - f\|_{k,0} \int_{\mathbf{R}} K_{a_k,b_k}(y) |D^p G(x - y)|\, dy. \end{aligned}$$

The last inequality is equivalent to

$$\begin{aligned} &\frac{|D^p((f_n - f) * G)(x)|}{K_{a_m,b_m}(x)} \\ &\qquad \le \|f_n - f\|_{k,0} \int_{\mathbf{R}} \frac{K_{a_k,b_k}(y)}{K_{a_m,b_m}(x)} |D^p G(x - y)|\, dy. \end{aligned} \tag{4.11}$$

By Lemma 4.1, a $\gamma > 0$ exists such that

$$\frac{K_{a_k,b_k}(x)}{K_{a_m,b_m}(x)} |D^p G(x - y)| \le B e^{-\gamma |y|}$$

for all $(x, y) \in \mathbf{R}^2$. This observation shows that, a positive constant C exists such that

$$\int_{\mathbf{R}} \frac{K_{a_k,b_k}(y)}{K_{a_m,b_m}(x)} |D^p G(x-y)| \, dy \le B \int_{\mathbf{R}} e^{-\gamma|y|} \, dy \le C$$

for all $x \in \mathbf{R}$. Applying this inequality in (4.11), we get

$$\left| \frac{D^p((f_n - f) * G)(x)}{K_{a_m,b_m}(x)} \right| \le C \|f_n - f\|_{k,0}$$

which, by (4.10) tends to 0 as $n \to \infty$. Thus we conclude that $f_n * G \to f * G$ as $n \to \infty$ in T_{a_m,b_m} and complete the proof. □

The proof of the following theorem is similar to that of Theorem 2.8 and therefore we omit the details.

4.12 Theorem. *Let $f \in \mathcal{L}'_{c,d}$, $\phi \in \mathcal{D}$ and G be a convolution kernel. If $c < \alpha_2$ and $d > \alpha_1$, then the equality $(f * \phi) * G = (f * G) * \phi$ holds.*

4.13 Corollary. *If $f \in T$, $\phi \in \mathcal{D}$ and G is a convolution kernel, then we have $(f * \phi) * G = (f * G) * \phi$.*

Proof. Since the elements of T are regular elements of $\mathcal{L}'_{c,d}$, the result follows from Theorem 4.12. □

Let f_n/ϕ_n be a quotient in $\mathcal{B}(T, \Delta)$. Since $f_n \in T$, by Theorem 4.3 we see that $f_n * G \in T$ for each $n \in \mathbf{N}$. As f_n/ϕ_n is a quotient in $\mathcal{B}(T, \Delta)$, we have

$$f_m * \phi_n = f_n * \phi_m \quad \text{for each } m, n \in \mathbf{N}.$$

Now it follows that

$$(f_m * \phi_n) * G = (f_n * \phi_m) * G \quad \text{for each } m, n \in \mathbf{N},$$

which by Corollary 4.13 gives

$$(f_m * G) * \phi_n = (f_n * G) * \phi_m \quad \text{for each } m, n \in \mathbf{N}.$$

Therefore, by the definition of the quotient of sequences, we conclude that $(f_n{*}G)/\phi_n$ is a quotient in $\mathcal{B}(T,\Delta)$. In view of this information, we can define the convolution transform of elements in $\mathcal{B}(T,\Delta)$ as follows.

4.14 Definition. Let $X = [f_n/\phi_n] \in \mathcal{B}(T,\Delta)$ and G be a convolution kernel. The convolution transform $\hat{X}$ of X is defined in $\mathcal{B}(T,\Delta)$ as $\hat{X} = [(f_n * G)/\phi_n]$.

Clearly, this definition is well defined. Indeed, if $X = [f_n/\phi_n]$ and $Y = [g_n/\psi_n]$ are in $\mathcal{B}(T,\Delta)$ such that $X = Y$ in $\mathcal{B}(T,\Delta)$, then $\hat{X} = [(f_n * G)/\phi_n]$ and $\hat{Y} = [(g_n * G)/\psi_n]$ are in $\mathcal{B}(T,\Delta)$. Further,

$$\begin{aligned} X = Y &\Longrightarrow \frac{f_n}{\phi_n} \sim \frac{g_n}{\psi_n} \\ &\Longrightarrow f_m * \psi_n = g_n * \phi_m, \quad \forall\, m,n \in \mathbf{N} \\ &\Longrightarrow (f_m * \psi_n) * G = (g_n * \phi_m) * G, \quad \forall\, m,n \in \mathbf{N} \\ &\Longrightarrow (f_m * G) * \psi_n = (g_n * G) * \phi_m, \quad \forall\, m,n \in \mathbf{N} \\ &\Longrightarrow \frac{f_n * G}{\phi_n} \sim \frac{g_n * G}{\psi_n} \\ &\Longrightarrow \hat{X} = \hat{Y} \quad \text{in } \mathcal{B}(T,\Delta), \end{aligned}$$

which proves that Definition 4.14 is well defined.

4.15 Theorem. *The convolution transform on $\mathcal{B}(T,\Delta)$ remains consistent for the elements of $\mathcal{L}'_{c,d}$, $c < \alpha_2$, $d > \alpha_1$.*

Proof. Let $f \in \mathcal{L}'_{c,d}$ for $c < \alpha_2$ and $d > \alpha_1$. Then by Theorem 4.5, the convolution transform $f * G$ of f is in T. By Theorem 3.18, f can be represented as $X = [(f * \phi_n)/\phi_n]$ in $\mathcal{B}(T,\Delta)$. Moreover, by Definition 4.14, the convolution transform $\hat{X}$ of X is given by $\hat{X} = [((f{*}\phi_n){*}G)/\phi_n]$ in $\mathcal{B}(T,\Delta)$. Now by Theorem 4.12, $(f{*}\phi_n){*}G = (f * G) * \phi_n$ for each $n \in \mathbf{N}$. Therefore,

$$\hat{X} = \left[\frac{(f * G) * \phi_n}{\phi_n}\right] = f * G$$

in $\mathcal{B}(T,\Delta)$. Thus, the convolution transform of a Boehmian representing $f \in \mathcal{L}'_{c,d}$ coincides with the Boehmian representing $\hat{f}$, and we complete the proof. $\square$

5. Operational properties. In this section, we first prove that the convolution transform is a continuous linear operator on $\mathcal{B}(T,\Delta)$ and then obtain the inversion theorem.

5.1 Theorem. *The convolution transform is a linear operator on $\mathcal{B}(T,\Delta)$.*

Proof. Let $X=[f_n/\phi_n]$ and $Y=[g_n/\psi_n]$ be in $\mathcal{B}(T,\Delta)$. Suppose that $\hat{X}=[(f_n*G)/\phi_n]$ and $\hat{Y}=[(g_n*G)/\psi_n]$ are the convolution transforms of X and Y, respectively. Then

$$X+Y=\left[\frac{(f_n*\psi_n)+(g_n*\phi_n)}{\phi_n*\psi_n}\right]$$

and the convolution transform of $X+Y$ is given by

$$(X+Y)^\wedge=\left[\frac{((f_n*\psi_n)+(g_n*\phi_n))*G}{\phi_n*\psi_n}\right].$$

Using Corollary 4.13, the above equality can be written as

$$\begin{aligned}(X+Y)^\wedge&=\left[\frac{((f_n*G)*\psi_n)+((g_n*G)*\phi_n))}{\phi_n*\psi_n}\right]\\&=\left[\frac{f_n*G}{\phi_n}\right]+\left[\frac{g_n*G}{\psi_n}\right]=\hat{X}+\hat{Y}.\end{aligned}$$

If $\alpha\in\mathbf{C}$, $\alpha X=[\alpha f_n/\phi_n]$ and its convolution transform is given by

$$(\alpha X)^\wedge=\left[\frac{(\alpha f_n)*G}{\phi_n}\right]=\left[\frac{\alpha(f_n*G)}{\phi_n}\right]=\alpha\left[\frac{f_n*G}{\phi_n}\right]=\alpha\hat{X},$$

and we are done. □

5.2 Theorem. *If $X\in\mathcal{B}(T,\Delta)$ and $\psi\in\mathcal{D}$, then $X*\psi\in\mathcal{B}(T,\Delta)$ and $(X*\psi)^\wedge=\hat{X}*\psi$.*

Proof. Let $X=[f_n/\phi_n]\in\mathcal{B}(T,\Delta)$. Using the definition of convolution of Boehmians, we get $X*\psi=[(f_n*\psi)/\phi_n]$. Clearly

$X * \psi \in \mathcal{B}(T, \Delta)$ and the convolution transform of $X * \psi$ is given by $(X * \psi)^\wedge = [((f_n * \psi) * G)/\phi_n]$. Using Corollary 4.13, we can write

$$(X * \psi)^\wedge = \left[\frac{(f_n * \psi) * G}{\phi_n}\right] = \left[\frac{(f_n * G) * \psi}{\phi_n}\right] = \left[\frac{f_n * G}{\phi_n}\right] * \psi = \hat{X} * \psi,$$

and the proof follows. □

5.3 Theorem. *If $\{X_n\}$ is a sequence of Boehmians such that $X_n \xrightarrow{\delta} X$ as $n \to \infty$ in $\mathcal{B}(T, \Delta)$, then $\hat{X}_n \xrightarrow{\delta} \hat{X}$ in $\mathcal{B}(T, \Delta)$.*

Proof. Now $X_n \xrightarrow{\delta} X$ in $\mathcal{B}(T, \Delta)$ implies that, a delta sequence $\{\phi_k\} \in \Delta$ exists such that $f_{n,k} = X_n * \phi_k$, $f_k = X * \phi_k$ are in T for each $k \in \mathbf{N}$ and $f_{n,k} \to f_k$ as $n \to \infty$ in T for each $k \in \mathbf{N}$. Therefore, in view of Theorem 4.9, we have for each $k \in \mathbf{N}$,

$$\hat{f}_{n,k} = f_n * G \longrightarrow f * G = \hat{f}_k \quad \text{as } n \to \infty \text{ in } T. \tag{5.4}$$

Now, $\hat{f}_{n,k} = (X_n * \phi_k)^\wedge = \hat{X}_n * \phi_k$ by Theorem 5.2. Similarly, we can write $\hat{f}_k = \hat{X} * \phi_k$. In view of the last two equalities, for each $k \in \mathbf{N}$, (5.4) can equivalently be written as

$$\hat{X}_n * \phi_k \longrightarrow \hat{X} * \phi_k \quad \text{as } n \to \infty \text{ in } T.$$

That is, $\hat{X}_n \xrightarrow{\delta} \hat{X}$ as $n \to \infty$ in $\mathcal{B}(T, \Delta)$, and we complete the proof. □

5.5 Theorem (Inversion theorem). *Let $Y = [g_m/\phi_m] \in \mathcal{B}(T, \Delta)$ be such that $Y = \hat{X}$ for some $X \in \mathcal{B}(T, \Delta)$. For each $m, n \in \mathbf{N}$, let $f_{n,m} = P_n(D)g_m$, where $P_n(D)$ is defined by (2.9). If $f_{n,m} \to f_m$ as $n \to \infty$ in $\mathcal{D}'$, then $X = [(f_m * \phi_m)/(\phi_m * \phi_m)] \in \mathcal{B}(T, \Delta)$ and $\hat{X} = Y$.*

Proof. By hypothesis, $g_m \in T$ and, therefore, $g_m \in \mathbf{C}^\infty$. Now

$$P_n(D)g_m(t) = Q_n(D_t)g_m(t + B_n) \in \mathbf{C}^\infty.$$

Indeed, if $Q_n(D) = \sum_{k=0}^{n} a_k D^k$ with real coefficients a_k, then

$$\begin{aligned} P_n(D)g_m(t) &= e^{B_n D} Q_n(D) g_m(t) \\ &= e^{B_n D}\left(\sum_{k=0}^{n} a_k D^k g_m(t)\right) \\ &= \sum_{k=0}^{n} (e^{B_n D} a_k D^k g_m(t)) \\ &= \sum_{k=0}^{n} a_k (D^k g_m(t+B_n)), \qquad \text{by (2.10)}, \\ &= Q_n(D_t) g_m(t+B_n), \end{aligned}$$

where D_t is the differential operator with respect to the variable t. That is, $f_{n,m} \in \mathbf{C}^\infty \subset \mathcal{D}'$ for each $m, n \in \mathbf{N}$. In view of Theorem 2.14, we deduce that g_m is the convolution transform of f_m and $f_m \in \mathcal{L}'_{c,d}$, $c < \alpha_2$, $d > \alpha_1$. That is,

$$g_m = \hat{f}_m, \quad \text{where } f_m \in \mathcal{L}'_{c,d}. \tag{5.6}$$

Since $f_m \in \mathcal{L}'_{c,d}$, by Theorem 3.14, $f_m * \phi_m \in T$ for each $m \in \mathbf{N}$. It is easy to check that $(f_m * \phi_m)/(\phi_m * \phi_m)$ is a quotient and, therefore, $X = [(f_m * \phi_m)/(\phi_m * \phi_m)] \in \mathcal{B}(T, \Delta)$. The convolution transform of X is given by

$$\hat{X} = \left[\frac{(f_m * \phi_m) * G}{\phi_m * \phi_m}\right] = \left[\frac{(f_m * G) * \phi_m}{\phi_m * \phi_m}\right] = \left[\frac{g_m * \phi_m}{\phi_m * \phi_m}\right] = \left[\frac{g_m}{\phi_m}\right] = Y,$$

and we are done. □

6. Example. Finally we give an example of a Boehmian which is convolution transformable but is not an element of $\mathcal{L}'_{c,d}$ for any values of c and d. This shows that the convolution transform defined on Boehmian spaces actually extends the transform to a more general set up than $\mathcal{L}'_{c,d}$ spaces.

The class $C\{M_k\}$ where $M_k = \{k!\}^{3/2}$ is not quasi-analytic in view of Theorem 19.11 (e) in [**20**, p. 380]. Therefore, by Theorem 19.10 [**20**, p. 379], a sequence $(\phi_n) \in \Delta$ exists (see also [**1**]) such that

(i) $\operatorname{supp}\phi_n \subseteq [-1/n, 1/n]$,

(ii) $\displaystyle\sum_{k=1}^{\infty} \frac{\phi_n^{(k)}}{(2k)!}$ is uniformly convergent on $\mathbf{R}$.

Let

$$f_n = \sum_{k=1}^{\infty} \frac{\phi_n^{(k)}}{(2k)!} \tag{6.1}$$

so that $\operatorname{supp} f_n \subseteq [-1/n, 1/n]$ for all n. Further, f_n, being the uniform limit of a sequence of $\mathbf{C}^\infty$ functions is itself a $\mathbf{C}^\infty$-function. Since $f_n \in \mathcal{D}$, $f_n \in T$ for each $n \in \mathbf{N}$. It can easily be shown that f_n/ϕ_n is a quotient and, hence, $X = [f_n/\phi_n] \in \mathcal{B}(T, \Delta)$. Therefore, it is convolution transformable and $\hat{X}$ is given by

$$\hat{X} = \left[\frac{f_n * G}{\phi_n}\right] \in \mathcal{B}(T, \Delta).$$

It can be shown that X does not represent any element of $\mathcal{D}'$ and hence cannot represent any element of $\mathcal{L}'_{c,d}$.

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