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A NOTE ON VALUATIONS, p-PRIMES AND THE HOLOMORPHY SUBRING OF A COMMUTATIVE RING

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Introduction. In this note we use a recent result from the theory of valuations to identify p-primes in a special class of commutative rings. Our identification process will also provide the associated field with p-prime in each case. We refer the reader to [**5**], for all definitions pertaining to the p-prime invariants which, due to their length, will not be included here. At the conclusion of this paper we will apply our new information on the structure of all 0-primes to obtain a result on holomorphy subrings which is closely related to some earlier work by the author (see [**4**]).

We take this opportunity to introduce some notation which we will employ throughout this paper. Firstly, all rings will be understood to be commutative and unitary. For an arbitrary subset, S, of a ring R, set

$$I(R,S)=\{r\in R|rS\subseteq S\}$$

and

$$(R:S)=\{r\in R|rR\subseteq S\}.$$

We let $U(R)$ denote the group of units of R, and, in the case R is a domain, $\mathrm{qf}(R)$ will represent the field of fractions of R. For I a prime ideal of R, let R_I be the localization of R at I. If R is assumed to be a local ring with maximal ideal M, then $k(R)$ and π will be used, respectively, to signify the residue class field R/M and the natural projection from R onto this field. Lastly, for $B\subseteq A$ we write $A\backslash B$ for $\{a\in A|a\notin B\}$.

1. Let R be a commutative ring. By a *valuation* of R we will mean a map $v:R\to G^*$, where G^* is a totally ordered abelian (possibly trivial) group which has been extended by ∞, satisfying

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(1) $v(xy) = v(x) + v(y)$,

(2) $v(x + y) \geq \min\{v(x), v(y)\}$,

(3) $v(1) = 0$ and $v(0) = \infty$.

We will, without loss, also assume that G is generated by $\mathrm{im}(v)\backslash\{\infty\}$. If G is $\mathrm{im}(v)\backslash\{\infty\}$, we then say that v is a *Manis valuation*, or simply an *M-valuation*.

Let $v : R \to G^*$ and $w : R \to H^*$ be valuations of R. We say that v and w are equivalent if there exists an order isomorphism $\theta : G \to H$ such that $w = \theta^* \circ v$. Let $\mathrm{Val}(R)$ (respectively $\mathrm{MVal}(R)$) denote the set of all (equivalence classes of) valuations (respectively M-valuations) of R.

For $v \in \mathrm{Val}(R)$ we know that $A(v) := \{r \in R | v(r) \geq 0\}$ is a subring of R and that $p(v) := \{s \in R | v(s) > 0\}$ is a prime ideal of $A(v)$. We shall refer to these as the *valuation subring* and *valuation prime* associated to v. Lastly we have the *infinite ideal* associated to v, namely $I(v) := \{t \in R | v(t) = \infty\}$.

We note that if v is an M-valuation, then it is completely determined by the pair $(A(v), p(v))$. For subrings A and B of R with prime ideals p and q (respectively) we say (B, q) *dominates* (A, p) if $A \subseteq B$ and $p = A \cap q$. The following result can be found in [**3**].

Proposition 1.1 (Manis). *The following are equivalent*:

(1) $(B, q) = (A(v), p(v))$ *for some* M-*valuation* v,

(2) $r \in R\backslash B \Rightarrow \exists x \in q$ *with* $rx \in B\backslash q$, *and*

(3) (B, q) *is maximal with respect to domination*.

Proposition 1.2. *Given* $v : R \to G^*$ *in* $\mathrm{Val}(R)$, *there is a valuation* $\hat{v} : F(v) \to G^*$, *where* $F(v)$ *denotes the field of fractions of* $R/I(v)$, *with*

$$\hat{v}(r + I(v)/s + I(v)) = v(r) - v(s)$$

for all $r \in R$ *and* $s \in R\backslash I(v)$.

Proof. One checks or sees [**1**].

Let R possess a large Jacobson radical, $J(R)$. That is, for every $r \in R$, there exists an $s \in R$ with $r + s \in U(R)$ and $rs \in J(R)$. In [**2**], J. Grater proved

PROPOSITION 1.3. *Let R be as above and A be a valuation subring of R. Then A is the valuation subring for some* M-*valuation.*

Further, in [**6**] the author points out that the M-valuation guaranteed us by this proposition is nothing more than a slight modification of the original (possibly non-Manis) valuation. Specifically, we have

PROPOSITION 1.4. *For R as above and $v : R \to G^*$ a valuation of R, the mapping $v' : R \to G^*$ defined by*

$$v'(r) = \left\{ \begin{array}{ll} v(r) & \text{if } r \notin (R : Pv) \\ \infty & \text{if } r \in (R : Pv) \end{array} \right\}$$

is an M-*valuation of R.*

Clearly the construction described in this proposition gives $A(v') = A(v)$ and $p(v') = p(v)$.

2. We now proceed to determine the set of p-primes in a special class of commutative rings.

LEMMA 2.1. *Let F be a field and R a valuation subring of F with $m(R)$ its maximal ideal. If $(B(v), q(v))$ is an* M-*valuation pair of R, then there exists a valuation subring A of F with $B(v) = R \cap A$ and $q(v) = R \cap m(A)$.*

PROOF. Since R is a valuation subring of F, there exists a unique valuation subring C of F with $R \subseteq C$ and $m(C) = I(v)$. One now checks that we may naturally identify $\mathrm{qf}(R/I(v))$ with $k(C)$. With this, there exists a valuation subring A of F with $A \subseteq C$ and $(B(\hat{v}), q(\hat{v})) = (\overline{A}, m(\overline{A}))$ (where $\overline{}$ denotes the image in $k(C)$). But the diagram

$$\begin{tikzcd} C \arrow[r, two heads] & k(C) = \mathrm{qf}(R/I(v)) \\ R \arrow[u, hook] \arrow[ur, "\phi(I)"'] & \end{tikzcd}$$

commutes, where $\phi(I)(r) = \overline{r}/\overline{1}$ for all $r \in R$. So

$$\begin{aligned}(B(v), q(v)) &= (\phi^{-1}(B(\hat{v})), \phi^{-1}(q(\hat{v}))) \\ &= (R \cap A, R \cap m(A)). \qquad \square\end{aligned}$$

For the remainder of this section, R will be a Prüfer domain with large Jacobson radical and F its quotient field.

LEMMA 2.2. *Let R be as above. For $(A, m(A)) \in \mathrm{Val}(F)$, the assignment*

$$(A, m(A)) \mapsto (R \cap A, R \cap m(A))$$

provides a well-defined map, ρ, from $\mathrm{Val}(F)$ to $\mathrm{MVal}(R)$. Further, this map is a bijection. We also have

$$(\widehat{R \cap A}, \widehat{R \cap m}(A)) = (\overline{A}, m(\overline{A})),$$

where $\overline{}$ denotes the image under the projection

$$\pi_I : R \twoheadrightarrow k(R_I) \quad \textit{and} \quad I = (R : m(A)).$$

PROOF. For $(A, m(A)) \in \mathrm{Val}(F)$, let $v(A)$ be the valuation which is determined by A. Setting $v = v(A)|_R$, we have v is a valuation of R and, by Proposition 1.4, $B(v) = R \cap A$ and $q(v) = R \cap m(A)$ form an M-valuation pair of R. Thus our mapping is well-defined.

Continuing with the notation above, we employ a bit of abuse and let v denote the M-valuation determined by $(R \cap A, R \cap m(A))$. Recall we know that

$$v(r) = \begin{cases} v(A)(r), & \text{for } r \notin (R : m(A)), \\ \infty, & \text{for } r \in (R : m(A)). \end{cases}$$

Now, letting $I = I(v) = (R : m(A)), R_I$ is a valuation subring of F and $k(R_I)$ is naturally isomorphic to $F(v)$. With $\pi_I : R \twoheadrightarrow k(R_I)$ the natural projection, $\pi_I^{-1}(B(\hat{v}))$ is a valuation subring of F with $\pi_I^{-1}(B(\hat{v})) \subseteq R_I$ and $\pi_I^{-1}(q(\hat{v}))$ its maximal ideal. Also, one can easily verify that

$$R_I = \{r/u | r \in R \text{ and } u \in (R \cap A) \backslash (R \cap m(A))\}.$$

Claim 1. $A = \pi_I^{-1}(B(\hat{v}))$.

Reason. Let $r/u \in \pi_I^{-1}(B(\hat{v}))$. If $r \in I$, then $r \in (R : m(A)) \subseteq A$. If this is not the case, then $\infty \neq \hat{v}(\pi_I(r/u)) = v(A)(r) - v(A)(u) \geq 0$ and again $r \in A$. Thus $\pi_I^{-1}(B(\hat{v})) \subseteq A$.

Now $\pi_I^{-1}(q(\hat{v})) \subseteq \pi_I^{-1}(B(\hat{v})) \cap m(A)$. Let $s/w \in \pi_I^{-1}(B(\hat{v})) \cap m(A)$ where $s \in R$ and $w \in U(A) \cap R$. Then $s \in m(A)$ and hence $s/w \in \pi_I^{-1}(q(\hat{v}))$. Thus $(A, m(A))$ dominates $(\pi_I^{-1}(B(\hat{v})), \pi_I^{-1}(q(\hat{v})))$ and, since the former is a valuation pair, they must in fact be equal. This establishes the claim. Note that, at this point, we have proved that

$$\widehat{R \cap A} = \overline{A} \quad \text{and} \quad \widehat{R \cap m}(A) = m(\overline{A}).$$

We now wish to show that our map is surjective. To this end let $(B(w), q(w)) \in \mathrm{MVal}(R)$. Again we know that $I(w) = (R : q(w))$ and (with w denoting the valuation associated to this pair) w determines an M-valuation $\hat{w}$ of the valuation subring $R_{I(w)}$ of F. Now, using Lemma 2.1, our assertion quickly follows.

Let $A(i)$ be valuation subrings of F with maximal ideals $m(i)$ for $i = 1, 2$. Assume that $R \cap A(1) = R \cap A(2)$ and $R \cap m(1) = R \cap m(2)$. Then $I(1) = (R : m(1)) = (R : m(2)) = I(2)$. Further, by our assumption,

$$(\widehat{R \cap A}(1), \widehat{R \cap m}(1)) = (\widehat{R \cap A}(2), \widehat{R \cap m}(2)).$$

Thus, by the above, $(\overline{A}(1), \overline{m}(1)) = (\overline{A}(2), \overline{m}(2)) \subseteq k(R_{I(1)})$. Hence $(A(1), m(1)) = (A(2), m(2))$, and our proof is complete. □

THEOREM 2.3. *Let R be a Prüfer domain with large Jacobson radical. Then there is a well-defined map $\rho_0 : P_0(F) \to P_0(R)$ with*

$$\rho_0(A, P) = (R \cap A, R \cap P).$$

Further, this mapping provides a bijective correspondence between $P_0(F)$ *and* $P_0(R)$.

Proof. Using Lemma 2.2, we immediately see that $(R \cap A, R \cap P)$ is a 0-prime of R for any 0-prime (A, P) of F. Let $(\widehat{R \cap A}, \widehat{R \cap P})$ denote the associated 0-prime of $k(R_I)$ where $I = (R : m(A))$. From the above (and continuing with the notation introduced there) we also know that $\widehat{R \cap A} = \overline{A}$ and $\widehat{R \cap m}(A) = m(\overline{A})$. Moreover we have

$$\widehat{R \cap P} = \{\overline{x/u} \,|\, x \in R \cap P,\ u \in (R \cap P)\backslash(R \cap m(A))\}.$$

Now, with our identifications, $\widehat{R \cap P} = \overline{P}$.

Let (B, T) be a 0-prime of R with $B = (B, q)$ and let $(A, m(A))$ be the unique valuation of F with $B = R \cap A$ as well as $q = R \cap m(A)$. By definition we may think of T as an ordering of A having $\mathrm{Supp}(T) := T \cap (-T) = q$. Thus T induces an ordering, T', of $\mathrm{qf}(B/q)$. Now, via the natural isomorphism from $\mathrm{qf}(B/q)$ to $k(A)$, the set

$$P := \{a \in A \,|\, a + m(A) \in T'\}$$

is an ordering of A with $\mathrm{Supp}(P) = m(A)$. Thus (A, P) is a 0-prime of F and one checks that $R \cap P = T$.

Let $(A(i), P(i))$ be 0-primes of F for $i = 1, 2$ with $\rho_0(A(1), P(1)) = \rho_0(A(2), P(2))$. By Lemma 2.2 we already know that $(A(1), m(1)) = (A(2), m(2)) = (A, m(A))$. Further, by our assumption and the above remarks, $(P(1))' = (P(2))' \subseteq k(A)$. But this gives $P(1) = P(2)$, and our map is bijective. $\square$

We continue with the notation as above while letting p denote a rational prime. For $(A, P) \in P_p(F)$ we shall include it in the subset $P_p^*(F)$ if the following condition is met:

$$(*) \qquad r \in R \quad \text{and} \quad r(R \cap A) \subseteq P \to r \in m(A).$$

For convenience, set $B = R \cap A$, $q = R \cap m(A)$ and $Q = R \cap P$. Also, for specificity in the work that follows, we will employ a new bit of notation. Namely, if (D, n) is an M-valuation pair of a commutative ring S, we set $F(S,D,n)$ to be $\mathrm{qf}(S/(S:n))$. As above $(\hat{D},\hat{n})$ will continue to denote the induced valuation pair of $F(S,D,n)$.

From the definition of a p-prime we know that $I = I(A,P)$ and P form an M-valuation pair of A with $m(A) \subseteq P$. Thus, by Proposition 0.4 of [**5**], I is a valuation subring of F with P its maximal ideal. We have also seen that $F(A,I,P)$ may be identified with $k(A)$ and that, via this identification, $\hat{I} = \pi_A(I)$ and $\hat{P} = \pi_A(P)$. Since R possesses a large Jacobson radical, both (B,q) and $(I \cap R, Q)$ are M-valuation pairs of R and $I \cap R \subseteq B$. Hence the latter pair may also be considered as an M-valuation pair of B. One can check that $J := I(B,Q)$ coincides with $I \cap R$.

By $(^*)$, we have $(B:Q) = q$, and with this $F(B,J,Q) \cong k(A)$ via an isomorphism satisfying $(b+q)/(u+q) \mapsto bu^{-1} + m(A)$ for all $b \in B$ and $u \in B \backslash q$. Hence, up to identification, $\hat{J} = \pi_A(I), \hat{Q} = \pi_A(P)$ and $\dim(\hat{J}/p\hat{J}) < \infty$ over $\mathbf{Z}/p\mathbf{Z}$.

Now, setting $L = (R : m(A))$, we have R_L is a valuation subring of F and $A \subseteq R_L$. Let $\overline{}$ denote class modulo the maximal ideal of R_L in $k(R_L)$. Further, we consider three subsets of $F(R,B,q)$:

$$\tilde{B} := \{(b+L)/(u+L) \,|\, b \in B \text{ and } u \in J\backslash Q\},$$

$$\tilde{q} := \{(x+L)/(u+L) \,|\, x \in q \text{ and } u \in J\backslash Q\}$$

and

$$\tilde{Q} := \{(y+L)/(u+L) \,|\, y \in Q \text{ and } u \in J\backslash Q\}.$$

We have $\tilde{B} = \overline{A}$ and $\tilde{q} = m(\overline{A})$ up to the identification of $F(R,B,q)$ with $k(R_L)$.

We may arrange the information thus far in the following diagram:

$$
\begin{tikzcd}
R & F & & \\
& R_L \arrow[u, hook] \arrow[r, two heads, "\pi_L"] & k(R_L) \cong F(R,B,q) & \\
& & (\overline{A}, m(\overline{A})) \arrow[u, hook] \arrow[dr, two heads, "\pi_{\bar{A}}"] & \\
(B,q) \arrow[uuu, hook] & (A, m(A)) \arrow[l, "\rho"'] \arrow[uu, hook] \arrow[ur, "\pi_L"] \arrow[rr, two heads] & & k(A) \cong k(\overline{A}) \\
& & (\overline{I}, \overline{P}) \arrow[uu, hook] \arrow[dr, "\pi_{\bar{A}}"] & \\
(J,Q) \arrow[uu, hook] & (I,P) \arrow[l, "\rho"'] \arrow[uu, hook] \arrow[ur, "\pi_L"] \arrow[rr, "\pi_{\bar{A}}"'] & & (\pi_A(I), \pi_A(P)) \arrow[uu, hook]
\end{tikzcd}
$$

Lemma 2.4. $\tilde{Q} = \overline{P}$.

Proof. Let $(y+L)/(u+L) \in \tilde{Q}$. But $u \in J \backslash Q \subseteq I \backslash P$ which gives $u^{-1} \in I \backslash P$. Hence $yu^{-1} + L \in \overline{P}$. With $\tilde{J}$ defined in a manner which is analogous to the above, we have $\tilde{J} = I(\tilde{B}, \tilde{Q}), (\tilde{J}, \tilde{Q})$ and $(\overline{I}, \overline{P})$ are both valuations of $k(R_L)$ and $\tilde{J} \subseteq \overline{I}$. Clearly $\tilde{Q} \subseteq \tilde{J} \cap \overline{P}$. Also, given $(j+L)/(u+L) \in \overline{P}$, we see $ju^{-1} \in P$ but $u^{-1} \notin P$. By the definition of a p-prime it follows that $j \in P$. Thus $j \in Q, \tilde{Q} = \tilde{J} \cap P$, and, by maximality, $(\tilde{J}, \tilde{Q}) = (\overline{I}, \overline{P})$. $\square$

The valuation $(\overline{I}, \overline{P}) \subseteq (\overline{A}, m(\overline{A}))$ determines a valuation $(\pi_{\bar{A}}(\overline{I}), \pi_{\bar{A}}(\overline{P}))$ of $k(\overline{A})$. But we may also identify $k(\overline{A})$ with $k(A)$ via $(a+L)+m(\overline{A}) \mapsto a+m(A)$, and using this we see that $(\pi_{\bar{A}}(\overline{I}), \pi_{\bar{A}}(\overline{P})) = (\pi_A(I), \pi_A(P))$ as valuations of $k(A)$. Letting $G(I)$ denote the (additive) value group associated to this valuation, we have

$$\mathrm{REL}(P,n) = G(I)/nG(I) = \mathrm{REL}(\overline{P},n) = \mathrm{REL}(\tilde{Q},n)$$

for any prime number n (see [**5**, Proposition 1.5]). Further $|\mathrm{REL}(\tilde{Q}, n)| = |\mathrm{REL}(Q, n)|$ (see [**5**, Claim 3 of Theorem 2.3]), and, since (A, P) is a p-prime of F, we may conclude that $|\mathrm{REL}(Q, n)| = n$ for any prime number n. Thus we have proved

PROPOSITION 2.5. *By defining* $\rho_p(A, P) = (R \cap A, R \cap P)$, *where* $R \cap A = (R \cap A, R \cap m(A))$, *a well-defined mapping,* $\rho_p \colon P_p^*(F) \to P_p(R)$, *results.*

THEOREM 2.6. *The map* $\rho_p : P_p^*(F) \to P_p(R)$ *is a bijection.*

PROOF. Fix $(B, Q) \in P_p(R)$ with $B = (B, q)$. By Lemma 2.2, there exists a valuation subring A of F with $B = R \cap A$ and $q = R \cap m(A)$. Also, $J = I(B, Q)$, together with Q, is an M-valuation pair of B, and $(B : Q) = q$. As above, $F(B, J, Q) \cong k(A)$ so that we may view $(\hat{J}, \hat{Q})$ as a valuation of $k(A)$. Hence, there exists a unique valuation subring C of F with $C \subseteq A$, $\hat{J} = \pi_A(C)$ and $\hat{Q} = \pi_A(m(C))$.

Now $(R \cap C, R \cap m(C))$ is an M-valuation pair of B. Let $j \in J$. Then $j + m(A) \in \hat{J} = \pi_A(C)$. So $j \in C$ and $J \subseteq R \cap C$. Similarly we have $Q = J \cap m(C)$, and, by maximality, $(J, Q) = (R \cap C, R \cap m(C))$.

We consider $(A, m(C))$. It follows that $I(A, m(C)) = C$, $\hat{C} = \pi_A(C)$ and $\widehat{m(C)} = \pi_A(m(C))$. So $\dim(\hat{C}/p\hat{C}) = \dim(\hat{J}/p\hat{J}) < \infty$ over $\mathbf{Z}/p\mathbf{Z}$. Also, following the proof above, $|\mathrm{REL}(Q, n)| = |\mathrm{REL}(m(C), n)| = n$ for any prime number, n. Thus $(A, m(C))$ is a p-prime of F which restricts to (B, Q) in R. By construction we see that $(A, m(C))$ satisfies $(^*)$ and our map is surjective.

To see that ρ_p is injective we let $(A(i), P(i))$ be in $P_p^*(F)$, for $i = 1, 2$, such that $\rho_p(A(1), P(2)) = \rho_p(A(2), P(2))$. We know immediately that $A(1) = A(2) = A$ and, letting $J = I(R \cap A, R \cap P(1))$ and $I(i) = I(A, P(i))$, for $i = 1, 2$, $R \cap I(1) = J = R \cap I(2)$. Also, via our past work, $\pi_A(I(1)) = \hat{J} = \pi_A(I(2))$ (as valuation subrings of $k(A)$), and, consequently, $P(1) = P(2)$. This completes the proof. □

3. We take this opportunity to return to concepts which were introduced in [**4**]. Specifically we are interested in the holomorphy subring,

$H(S)$, of a semi-real ring S (see §5 of [**4** or **6**] for appropriate definitions and notation). From the definition, one can immediately verify that $H(S) \subseteq S \cap H(\mathrm{qf}(S))$ for any domain S. We are particularly interested in identifying those domains for which the inclusion is equality. In any such case it easily follows that $H(S)$ would coincide with subring $A(T)$ for the preordering $T = S \cap \sigma(\mathrm{qf}(S))$ of S.

Assume that R is a Prüfer domain with large Jacobson radical. Here we have that $H(R) = H_M(R)$. Further, by Theorem 2.3, we know that every 0-prime of R is uniquely determined by a 0-prime of $\mathrm{qf}(R)$ via restriction. Thus it is not surprising that we have (cf. Theorem 5.2 (b))

PROPOSITION 3.1. *For R as above and $F = \mathrm{qf}(R)$, $H(R) = R \cap H(F)$.*

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