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SOLUTION TO TWO PROBLEMS IN INVERSE INTERPOLATION

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ABSTRACT. Answering two problems raised by A.L. Horwitz and L.A. Rubel, we construct analytic functions f such that $L(L(f))$ is the set of all polynomials (here $L(f)$ denotes the set of all Lagrange interpolants of f on $[0,1]$).

Let $L(f)$ denote the set of all Lagrange interpolants based on knots in $[0,1]$ of the function f defined on $[0,1]$. A.L. Horwitz and L.A. Rubel [**1**] proved that if f and g are analytic on $[0,1]$ and $L(f)=L(g)$, then $f=g$; on the other hand, they constructed a large class of C^∞ -functions f for which $L(L(F))$ is the set of all polynomials. They asked

PROBLEM 1. *Is there a function f analytic on $[0,1]$ such that $L(L(f))$ is the set of all polynomials*?

and

PROBLEM 2. *If f and g are analytic on $[0,1]$ and $L(L(f))=L(L(g))$, then must $f=g$*?

In this paper we show that there are many analytic functions of the form

$$f(z)=\int_{\mathbf{R}}\frac{d\mu(t)}{1+tz},\quad z\in[0,1],$$

where μ is a finite signed measure, for which $L(L(f))$ is the set of all polynomials. Hence, the answer is positive for Problem 1 and it is negative for Problem 2.

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It is enough to construct functions f which are analytic on $[0,1]$ and which have the following property. Given arbitrary M and k, there are points $0<\xi_1<\xi_2<\cdots<\xi_k<1$ and an integer $n=n(k)$ such that if

$$T_{n-1}(f,x)=\sum_{\nu=0}^{n-1}\frac{f^{(\nu)}(0)}{\nu!}x^\nu$$

denotes the McLaurin polynomial of f of degree $(n-1)$, then

$$(-1)^s T_{n-1}(f,\xi_s)>M, \quad s=1,\dots,k.$$

In fact, in this case if P is any polynomial of degree l, say, and if

$$|P(x)|<M, \quad x\in[0,1],$$

then to this M and $k=l+2$ there are ξ_i's and an $n=n(k)$ with the above property. If $L(f,\{x_j\}_{j=1}^n;x)$ denotes the Lagrange interpolant of f based on the knots $\{x_j\}_{j=1}^n$, then

$$\lim_{t\to 0+0} L(f,\{tj/n\}_{j=1}^n;x)=T_{n-1}(f,x)$$

uniformly in $x\in[0,1]$, hence there is a $t>0$ such that

$$(-1)^s L(f,\{tj/n\}_{j=1}^n;\xi_s)>M, \quad s=1,2,\dots,k.$$

But then $L(f,\{tj/n\}_{j=1}^n;\chi)$ intersects P in at least $k-1=l+1$ points, hence $P\in L(L(f))$ as we claimed above.

We will use

LEMMA. *Let* $M>0, \quad 1/2>\varepsilon>0, \quad \eta>0, \quad k\in\mathbf{N}$,

$$2-\varepsilon<\alpha_1<\alpha_2<\cdots<\alpha_k<2,$$

and

$$\frac{1}{2}+\varepsilon>\xi_1>\xi_2>\cdots>\xi_k>\frac{1}{\alpha_1} \tag{1}$$

be given. Then there are numbers $n = n(M, \varepsilon, \eta, \{\alpha_i\}, \{\xi_i\})$ *and* $c_1, c_2, \ldots, c_k$ *such that*

$$|c_1| < \eta, \quad 0 < |c_{i+1}| < \frac{1}{2}|c_i|, \quad i = 1, 2, \ldots, k-1, \tag{2}$$

and, with $f_\alpha(z) = (1 + \alpha z)^{-1}$,

$$(-1)^s T_{n-1}\left(\sum_{i=1}^{k} c_i f_{\alpha_i}, \xi_s\right) > M, \quad s = 1, \ldots, k.$$

PROOF. The functions $\{(1 + \alpha_i z)^{-1}\}_{i=1}^k$ are analytic and linearly independent on every interval $[a, b]$ not containing $-1/\alpha_i, i = 1, \ldots, k$. Since

$$\sum_{i=1}^{k} \frac{d_i}{1 + \alpha_i z} = \frac{1}{\prod_{i=1}^{k}(1 + \alpha_i z)} \sum_{i=1}^{k} d_i \prod_{j \neq i}(1 + \alpha_j z),$$

it follows that nontrivial linear combinations of these functions can have at most $(k - 1)$ zeros, hence $\{(1 + \alpha_i z)^{-1}\}_{i=1}^k$ forms a Chebyshev system. Then the system of equations

$$\sum_{i=1}^{k} \frac{d_i}{1 + \alpha_i \xi_s} = (-1)^s, \quad s = 1, \ldots, k, \tag{3}$$

has a solution and, by the same token, if $\{d_i\}_{i=1}^k$ denotes the solution, then none of the d_i's vanish.

Let

$$c_i = \frac{d_i}{\alpha_i^n} c$$

with

$$c \xi_k^n > M + 1 \tag{4}$$

and

$$c \alpha_1^{-n} < \eta \Big/ \Big(\sum_{i=1}^{k} |d_i|\Big), \tag{5}$$

where n will be chosen in a moment. (1) shows that (4) and (5) can be satisfied with a $c = c(n)$ for every large n. Also, if n is sufficiently large, then (2) holds (cf. (5)).

Now, if n is odd, then

$$T_{n-1}\left(\sum_{i=1}^{k} c_i f_{\alpha_i}; \xi_s\right) = \sum_{i=1}^{k} c_i \sum_{\nu=0}^{n-1} (-1)^\nu (\alpha_i \xi_s)^\nu$$
$$= \sum_{i=1}^{k} c_i \left(\frac{(\alpha_i \xi_s)^n}{1+\alpha_i\xi_s} + \frac{1}{1+\alpha_i\xi_s}\right)$$
$$= c\xi_s^n \sum_{i=1}^{k} \frac{d_i}{1+\alpha_i\xi_s} + c\sum_{i=1}^{k} \frac{d_i\alpha_i^{-n}}{1+\alpha_i\xi_s}.$$

Here the first term equals $(-1)^s c\xi_s^n$ (see (3)), and, for the absolute value of the second one, we get the upper bound

$$c\sum_{i=1}^{k} |d_i|\alpha_i^{-n} \le c\alpha_1^{-n} \sum_{i=1}^{k} |d_i| < 1 \tag{6}$$

if n is large enough. Thus, if we choose n so large that (4)–(6) and (2) are satisfied, then we obtain the statement of the lemma because

$$c\xi_s^n \ge c\xi_k^n > M + 1.$$

The f that we are going to construct will be of the form

$$f(z) = \sum_{k=1}^{\infty} \sum_{i=1}^{k} c_i^{(k)} f_{\alpha_i^{(k)}}(z) = \int_{\mathbf{R}} \frac{d\mu(t)}{1+tz}, \tag{7}$$

where the signed measure $\mu(t)$ is defined by

$$\mu(E) = \sum_{\alpha_i^{(k)} \in E} c_i^{(k)}.$$

Our aim is to define the numbers $c_i^{(k)}$ and $\alpha_i^{(k)}$ in such a way that f satisfies the requirements stated in the beginning of the construction.

First of all, if

$$3/2 \le \alpha_1^{(1)} < \alpha_1^{(2)} < \alpha_2^{(2)} < \alpha_1^{(3)} < \alpha_2^{(3)} < \cdots < 2 \tag{8}$$

and

$$|c_{i+1}^{(k)}| < \frac{1}{2}|c_i^{(k)}|, \quad i = 1, 2, \ldots, k-1, \tag{9}$$

$$|c_1^{(k)}| < \frac{1}{2}|c_{k-1}^{(k-1)}|, \quad k = 2, 3, \ldots, \tag{10}$$

then the series in (7) representing f uniformly converges and f is analytic on $[0, 1]$.

Besides $c_i^{(k)}$ and $\alpha_i^{(k)}$ we shall inductively define two more positive sequences $\{\varepsilon_k\}$ and $\{\eta_k\}$. Let

$$c_1^{(1)} = \varepsilon_1 = \eta_1 = 1, \qquad \alpha_1^{(1)} = 3/2,$$

and suppose that, for some k, all the numbers $\{c_i^{(k-1)}\}_{i=1}^{k-1}$, $\{\alpha_i^{(k-1)}\}_{i=1}^{k-1}$, ε_{k-1} and η_{k-1} are already defined. We set

$$\varepsilon_k = \min\{2 - \alpha_{k-1}^{(k-1)}, (\alpha_{k-1}^{(k-1)})^{-1} - 1/2\}. \tag{11}$$

If

$$2 - \varepsilon_k < \alpha_1^{(k)} < \cdots < \alpha_k^{(k)} < 2 \tag{12}$$

and

$$\frac{1}{2} + \varepsilon_k > \xi_1^{(k)} > \xi_2^{(k)} > \cdots > \xi_k^{(k)} > \frac{1}{\alpha_1^{(k)}} \tag{13}$$

are chosen arbitrarily, then, by the Lemma, to $M = k + 8$, $\varepsilon = \varepsilon_k$ and $\eta = \min\{\eta_{k-1}, \frac{1}{2}|c_{k-1}^{(k-1)}|\}$ there are numbers $c_1^{(k)}, \ldots, c_k^{(k)}$ and $n = n(k)$ such that

$$\begin{gathered} 0 < |c_{i+1}^{(k)}| < \frac{1}{2}|c_i^{(k)}|, \quad i = 1, \ldots, k-1, \\ |c_1^{(k)}| < \min\Big\{\eta_{k-1}, \frac{1}{2}|c_{k-1}^{(k-1)}|\Big\} \end{gathered} \tag{14}$$

and

$$(15)\qquad (-1)^s T_{n-1}\left(\sum_{i=1}^{k} c_i^{(k)} f_{\alpha_i^{(k)}}, \xi_s^{(k)}\right) > k+8, \quad s=1,\dots,k.$$

Now, setting

$$(16)\qquad \eta_k = 2^{-n} = 2^{-n(k)},$$

the definition of the sequences above is complete.

By our construction, (8), (9) and (10) hold true. Let us consider the $(n-1)=(n(k)-1)$-th partial sum of the Taylor expansion of f around 0 at the points $\xi_s^{(k)}$, $s=1,\dots,k$, where k is fixed. We have

$$\begin{aligned} T_{n-1}(f,\xi_s^{(k)}) &= \sum_{l=1}^{\infty} T_{n-1}\left(\sum_{i=1}^{l} c_i^{(l)} f_{\alpha_i^{(l)}}, \xi_s^{(k)}\right) \\ &= \sum_{l=1}^{k-1} + \sum_{l=k} + \sum_{l=k+1}^{\infty} = I_1 + I_2 + I_3. \end{aligned}$$

According to (12), (13) and (11), if $l \le k-1$, then, for every $1 \le i \le l$,

$$0 < \alpha_i^{(l)} \xi_s^{(k)} < \alpha_{k-1}^{(k-1)}\left(\frac{1}{2} + \varepsilon_k\right) \le 1.$$

Therefore,

$$|T_{n-1}(f_{\alpha_i^{(l)}}, \xi_s^{(k)})| = \left|\sum_{\nu=0}^{n-1} (-1)^\nu (\alpha_i^{(l)} \xi_s^{(k)})^\nu\right| \le 1$$

and hence (9) and (10) yield

$$|I_1| \le \sum_{l=1}^{k-1}\sum_{i=1}^{l} |c_i^{(l)}| \le \sum_{l=1}^{k-1} 2|c_1^{(l)}| < 4.$$

Since $n=n(k)$, we get, from (15),

$$(-1)^s I_2 = (-1)^s T_{n-1}\left(\sum_{i=1}^{k} c_i^{(k)} f_{\alpha_i^{(k)}}, \xi_s^{(k)}\right) > k+8, \quad s=1,\dots,k.$$

Finally, let $l > k$. Then, for $1 \le i \le l$,

$$|T_{n-1}(f_{\alpha_i^{(l)}}, \xi_s^{(k)})| \le \sum_{\nu=0}^{n-1} |\alpha_i^{(l)} \xi_s^{(k)}|^\nu \le \sum_{\nu=0}^{n-1} 2^\nu < 2^n.$$

Therefore, (9) shows that

$$\left| T_{n-1}\left(\sum_{i=1}^{l} c_i^{(l)} f_{\alpha_i^{(l)}}, \xi_s^{(k)} \right) \right| \le \sum_{i=1}^{l} |c_i^{(l)}| 2^n \le 2^{n+1} |c_1^{(l)}|$$

and hence we get from (10), (14) and (16) that

$$|I_3| \le 2^{n+1} \sum_{l=k+1}^{\infty} |c_1^{(l)}| \le 2^{n+2} |c_1^{(k+1)}| \le 4.$$

Collecting our estimates we can see that, for $n = n(k)$,

$$(-1)^s T_{n-1}(f, \xi_s^{(k)}) > (k+8) - 4 - 4 = k, \quad s = 1, \dots, k, \quad k = 2, \dots,$$

and, since $\xi_s^{(k)} \in (0,1)$, the proof is complete. □

REFERENCE

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