ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 20, Number 1, Winter 1990

CONTROLLING CONJUGACY CLASSES IN EMBEDDINGS OF LOCALLY FINITE GROUPS

SUSAN E. SCHUUR

Several recent papers in the theory of locally finite groups discuss the concept of π-homogeneity [**1**, **2**, **3**]. A locally finite group G is π-*homogeneous* for the set of primes π if, for every isomorphism $\mu : H \to K$ between finite π-subgroups of G, there is an $x \in G$ with $h\mu = h^x$ for all $h \in H$. The group G is a π-*ULF group* if it is locally finite, π-homogeneous and contains a copy of every finite group. One of the results of [**3**] is that any locally finite group G can be embedded in a π-ULF group of cardinality $\max\{\aleph_0, |G|\}$ in which, for each $p \notin \pi$, there are exactly two conjugacy classes of elements of order p. The purpose of this paper is to extend this result as follows:

THEOREM. *Let G be a locally finite group and π a non-empty set of primes. Let $K = \{k_p | p \in \pi_1'\}$ be a set of positive integers, where $\pi_1' \subseteq \pi'$. Then there is a π-ULF group $\overline{G}$ satisfying*

(i) *$G \subseteq \overline{G}$ and $|\overline{G}| = \max\{\aleph_0, |G|\}$;*

(ii) *if $p \in \pi'$, $n \geq 1$, and $\nu(p^n, \overline{G})$ is the set of $\overline{G}$-conjugacy classes of elements of order p^n, then*

$$|\nu(p^n, \overline{G})| = \begin{cases} k_p + 1, & \text{if } p \in \pi_1' \text{ and } n > k_p, \\ n + 1, & \text{otherwise .} \end{cases}$$

This theorem was suggested by one of the constructions in [**3**] (cf. Theorem 3).

We need the following notation. In any group G, $\sim_G$ denotes G-conjugacy of elements and subgroups. If $H \subseteq G$, $(G : H)$ denotes the set of right cosets of H in G and $\Sigma = \Sigma(G : H)$ the full symmetric group on $(G : H)$. The representation $\varphi = \varphi(G : H)$ of G into $\Sigma(G : H)$ is defined by

$$Hx(g\varphi) = Hxg \quad \text{for } x, g \in G;$$

Received by the editors on December 29, 1986.

the kernel of φ is $\mathrm{Core}_G(H)$. The constricted symmetric group $C = C(G : H)$ is defined as follows:

$$C = C(G:H) = \{\tau \in \Sigma(G:H) \mid \text{there is a finite subgroup } T_\tau \text{ of } G \text{ such that } (HsT_\tau)\tau \subseteq HsT_\tau \text{ for all } s \in G\}.$$

By Proposition 1 of [**3**], if G is locally finite, so is $C(G:H)$; $G\varphi \subseteq C(G:H)$; and if K and $K\beta$ are isomorphic finite subgroups of G which intersect every conjugate of H trivially, then there is a σ in $C(G:H)$ such that $k\beta\varphi = (k\varphi)^\sigma$ for all $k \in K$. In the following proposition we extend this result to certain subgroups which intersect a conjugate of H non-trivially (for H as in the statement of the proposition). We remark that the proof combines features of Proposition 1 and Lemma 2 of [**3**].

PROPOSITION. *Let G be a locally finite group, $H \subseteq G$ an abelian subgroup, and let φ be the representation of G in $\Sigma = \Sigma(G:H)$. For p a prime suppose that the Sylow p-subgroup of H is isomorphic either to C_{p^k} or to C_{p^∞}. Denote by $\langle w_n \rangle$ the unique subgroup of H of order p^n, where $1 \le n \le k$ in the C_{p^k} case and $n \ge 1$ in the C_{p^∞} case, and, for $0 \le t \le n$, let*

$$C(t, w_n) = C(t, w_n, G) = \left\{ y \in G \,\middle|\, |y| = p^n \text{ and } t \text{ is minimal with respect to } \left\langle y^{p^t} \right\rangle \sim_G \left\langle w_n^{p^t} \right\rangle \right\}.$$

Then the following two properties hold for elements y and z of G of order p^n:

(i) *For $1 \le n \le k$ in the C_{p^k} case and all $n \ge 1$ in the C_{p^∞} case, if there is a t, $0 \le t \le n$, such that $y, z \in C(t, w_n)$, then there is a σ in $C = C(G:H)$ such that $z\varphi = (y\varphi)^\sigma$. Further, $C(t, w_n, G)\varphi \subseteq C(t, w_n\varphi, \Sigma)$; thus if y, z belong to different sets $C(t, w_n)$, then $y\varphi \nsim_\Sigma z\varphi$;*

(ii) *In the C_{p^k} case, if $n > k$, then $z\varphi \sim_C y\varphi$ if and only if $z^{p^{n-k}}\varphi \sim_C y^{p^{n-k}}\varphi$.*

PROOF. (i). Suppose $y, z \in C(t, w_n)$. Then, by definition of $C(t, w_n)$, there exist G-conjugates y_0, z_0 of y, z, respectively, such that $\left\langle y_0^{p^t} \right\rangle =$

$\left\langle z_0^{p^t}\right\rangle = \left\langle w_n^{p^t}\right\rangle$. Let $V = \langle\langle y_0\rangle, \langle z_0\rangle\rangle$ and write G as the double coset decomposition

$$G = \dot\cup\{HsV \mid s \in S\}.$$

For each $s \in S$, the number of right cosets of H in HsV is the index $[V : H^s \cap V]$, which is finite since V is finite. Now, for fixed $s \in S$, write

$$HsV = \dot\cup\{Hsx\langle y_0\rangle \mid x \in X_s \subset V\};$$

this can be obtained from a decomposition of V into left cosets of $\langle y_0\rangle$, and can also be viewed as a decomposition of HsV into a finite number of disjoint $y_0\varphi$-orbits. (There is a similar decomposition of HsV into disjoint $z_0\varphi$-orbits.) For any $v \in V$, the order of the orbit $Hsv\langle y_0\varphi\rangle$ (i.e., the number of right cosets of H in $Hsv\langle y_0\rangle$) equals

$$(1) \qquad [\langle y_0\rangle : H^{sv} \cap \langle y_0\rangle] = [\langle y_0\rangle : \langle w_n\rangle^{sv} \cap \langle y_0\rangle];$$

(1) follows easily from the definition of H. But, as in the proof of Lemma 2(i) of [3],

$$(2) \qquad \langle w_n\rangle^{sv} \cap \langle y_0\rangle = \left\langle w_n^{p^t}\right\rangle^{sv} \cap \left\langle y_0^{p^t}\right\rangle = \left\langle w_n^{p^t}\right\rangle^{sv} \cap \left\langle w_n^{p^t}\right\rangle.$$

Hence the order of the orbit $Hsv\langle y_0\varphi\rangle$ equals

$$\frac{p^n}{\left|\left\langle w_n^{p^t}\right\rangle^{sv} \cap \left\langle w_n^{p^t}\right\rangle\right|},$$

which is at least p^t and at most p^n. The crucial observation, however, is that this is also the order of the orbit $Hsv\langle z_0\varphi\rangle$ (repeat the argument with y_0 replaced by z_0). Thus if Hsv is in a $y_0\varphi$-orbit of order p^i, $t \le i \le n$, then Hsv falls into a $z_0\varphi$-orbit of order p^i, and conversely. It follows that the number of distinct $y_0\varphi$-orbits of order p^i in HsV equals the number of distinct $z_0\varphi$-orbits of order p^i in HsV.

Hence there is a subset $X' = X'_s$ of V and a bijection $x \to x'$ of $X = X_s$ to X' such that

$$HsV = \dot\cup\{Hsx\langle y_0\rangle \mid x \in X\} = \dot\cup\{Hsx'\langle z_0\rangle \mid x' \in X'\},$$

where if $x \to x'$, then the orbits $Hsx\langle y_0\varphi\rangle$ and $Hsx'\langle z_0\varphi\rangle$ have the same order. For each $s \in S$ and $x \in X$, define

$$\tau_s : Hsx\langle y_0\rangle \to Hsx'\langle z_0\rangle$$

by $(Hsxy_0^j)\tau_s = Hsx'z_0^j$, $1 \le j \le p^n$. This is a bijection, so the function τ on $(G:H)$, defined by

$$(Hu)\tau = Hu\tau_s \quad \text{if } Hu \in HsV,$$

is an element of $\Sigma(G:H)$. In fact, $\tau \in C(G:H)$ since $(HsV)\tau \subseteq HsV$ for all $s \in G$. Furthermore, $(y_0\varphi)^\tau = z_0\varphi$. To see this, let $Hu \in HsV$, $s \in S$; so $Hu = Hsx'z_0^j$ for some $x' \in X'$ and some j. Then

$$\begin{aligned}(Hu)(y_0\varphi)^\tau &= (Hsx'z_0^j)\tau^{-1}y_0\varphi\tau = (Hsxy_0^j)(y_0\varphi\tau)\\ &= (Hsxy_0^{j+1})\tau = Hsx'z_0^{j+1} = (Hsx'z_0^j)(z_0\varphi) = (Hu)(z_0\varphi),\end{aligned}$$

as desired.

Finally, $y \sim_G y_0$, $z \sim_G z_0$ implies $y\varphi \sim_{G\varphi} y_0\varphi$ and $z\varphi \sim_{G\varphi} z_0\varphi$. Since $G\varphi \subseteq C = C(G:H)$,

$$y\varphi \sim_C y_0\varphi \sim_C z_0\varphi \sim_C z\varphi,$$

and this completes the proof of the first assertion of (i).

To finish the proof of (i) we must show that $C(t, w_n, G)\varphi \subseteq C(t, w_n\varphi, \Sigma)$. Let $y \in C(t, w_n, G)$, $0 \le t \le n$. Since $\langle y^{p^t}\rangle \sim_G \langle w_n^{p^t}\rangle$, we certainly have $\langle y\varphi^{p^t}\rangle \sim_\Sigma \langle w_n\varphi^{p^t}\rangle$. Thus it suffices to show that, for $t \ne 0, \langle y\varphi^{p^{t-1}}\rangle \nsim_\Sigma \langle w_n\varphi^{p^{t-1}}\rangle$. But this follows from [**3**, (5.1.6)], which holds for H whose Sylow p-subgroup is either C_{p^k} or C_{p^∞} by (1) and (2).

(ii). Clearly $z\varphi \sim_C y\varphi$ implies $z^{p^{n-k}}\varphi \sim_C y^{p^{n-k}}\varphi$. For the converse, suppose $z^{p^{n-k}}\varphi \sim_C y^{p^{n-k}}\varphi$. Since $y^{p^{n-k}}$ and $z^{p^{n-k}}$ are of order p^k, there is a t, $0 \le t \le k$, such that $y^{p^{n-k}}$, $z^{p^{n-k}} \in C(t, w_k)$, by (i). Hence there are $g, h \in G$ such that if $y_0 = (y^{p^{n-k}})^g$ and $z_0 = (z^{p^{n-k}})^h$, then $\langle y_0^{p^t}\rangle = \langle z_0^{p^t}\rangle = \langle w_k^{p^t}\rangle$. Now set $y_1 = y^g$ and $z_1 = z^h$; note that $y_1^{p^{n-k}} = (y^g)^{p^{n-k}} = y_0$ and $z_1^{p^{n-k}} = z_0$. Further let $V = \langle\langle y_1\rangle, \langle z_1\rangle\rangle$. With this V, just as in the proof of (i), $G = \dot\cup\{HsV | s \in S\}$ and, for fixed $s \in S$, $HsV = \{Hsx\langle y_1\rangle \,|\, x \in X \subset V\}$. The order of the orbit

$Hsx\langle y_1\varphi\rangle$ equals

$$\begin{aligned}[\langle y_1\rangle : &H^{sx}\cap\langle y_1\rangle]\\ &= [\langle y_1\rangle : \langle w_k\rangle^{sx}\cap\langle y_1\rangle] = \left[\langle y_1\rangle : \langle w_k\rangle^{sx}\cap\left\langle y_1^{p^{n-k}}\right\rangle\right]\\ &= [\langle y_1\rangle : \langle w_k\rangle^{sx}\cap\langle y_0\rangle] = \left[\langle y_1\rangle : \left\langle w_k^{p^t}\right\rangle^{sx}\cap\langle y_0^{p^t}\rangle\right]\\ &= \left[\langle y_1\rangle : \left\langle w_k^{p^t}\right\rangle^{sx}\cap\left\langle w_k^{p^t}\right\rangle\right] = \frac{p^n}{\left|\left\langle w_k^{p^t}\right\rangle^{sx}\cap\left\langle w_k^{p^t}\right\rangle\right|},\end{aligned}$$

which is also the order of the orbit $Hsx\langle z_1\varphi\rangle$. Hence we can proceed as in (i) to find a $\tau\in C(G:H)$ such that $(y_1\varphi)^\tau = z_1\varphi$ and conclude that

$$y\varphi\sim_C y_1\varphi\sim_C z_1\varphi\sim_C z\varphi,$$

as desired. □

PROOF OF THE THEOREM. Let

$$H = \mathrm{Dr}\{C_{p^{k_p}}|p\in\pi_1'\}\times\mathrm{Dr}\{C_{p^\infty}|p\in\pi'-\pi_1'\},$$

$\pi_1'\subseteq\pi'$, and

$$G_0 = (G\times\mathrm{Sym}(\aleph_0))\,\mathrm{wr}\,H,$$

where the wreath product is restricted and $\mathrm{Sym}(\aleph_0)$ is the countable group of finitary permutations. Then G_0 is f-universal, since $\mathrm{Sym}(\aleph_0)$ contains an isomorphic copy of every finite symmetric group, and $|G_0| = \max\{\aleph_0, |G|\}$. Further $\mathrm{Core}_{G_0}(H) = 1$, so the representation $\varphi_0 = \varphi(G_0:H)$ of G_0 into $\Sigma(G_0:H)$ is an embedding.

Let G_1 be the subgroup of $C(G_0:H)$ generated by:

(a) $G_0\varphi_0$;

(b) for each isomorphism $\beta: K\to K\beta$ of finite π-subgroups of G_0, an element $\sigma\in C(G_0:H)$ such that $k\beta\varphi_0 = (k\varphi_0)^\sigma$ for all $k\in K$;

(c) for $p\in\pi'$ and $0\le t\le n$, where $1\le n\le k_p$ if $p\in\pi_1'$ and $n\ge 1$ otherwise, and each pair $y,z\in C(t,w_n,G_0)$, an element of $C(G_0:H)$ conjugating $y\varphi_0$ to $z\varphi_0$;

(d) for $p\in\pi_1'$, $n>k_p$, and each pair y, $z\in G_0$ such that $|y|=|z|=p^n$ and $y\varphi_0$ is conjugate to $z\varphi_0$ in $C(G_0:H)$, a conjugating element.

(The notation here is as in the proposition.) Proposition 1 of [2] and the above proposition ensure that the conjugating elements in (b) and (c), respectively, can be found in $C(G_0 : H)$; the number of conjugating elements in (b), (c) and (d) is at most $\max\{\aleph_0, |G_0|\}$, so $|G_1| \geq \max\{\aleph_0, |G_0|\}$. It also follows from part (i) of the proposition that for $p \in \pi'$ and n as in (c), if $|y| = |z| = p^n, y \in C(t, w_n, G_0), z \in C(j, w_n, G_0)$ and $j \neq t$, then $y\varphi_0 \nsim_{G_1} z\varphi_0$.

Similarly, for $i = 1, 2, \ldots$ define G_{i+1} and $\varphi_i : G_i \to G_{i+1}$ inductively by $\varphi_i = \varphi(G_i : H_i)$, where $H_i = H\varphi_0 \cdots \varphi_{i-1}$, *and* G_{i+1} is the subgroup of $C(G_i : H_i)$ generated by:

(a) $G_i\varphi_i$;

(b) for each isomorphism $\beta : K \to K\beta$ of finite π-subgroups of G_i, an element $\sigma \in C(G_i : H_i)$ such that $k\beta\varphi_i = (k\varphi_i)^\sigma$ for all $k \in K$;

(c) for $p \in \pi'$ and $0 \leq t \leq n$, where $1 \leq n \leq k_p$ if $p \in \pi_1'$ and $n \geq 1$ otherwise, and each pair $y, z \in C(t, w_n\varphi \cdots \varphi_{i-1}, G_i)$, an element of $C(G_i : H_i)$ conjugating $y\varphi_i$ to $z\varphi_i$;

(d) for $p \in \pi_1'$, $n > k_p$, and each pair y, $z \in G_i$ such that $|y| = |z| = p^n$ and $y\varphi_i$ is conjugate to $z\varphi_i$ in $C(G_i : H_i)$, a conjugating element.

Note that $\mathrm{Core}_{G_i}(H_i) = 1$, $H_i \simeq H$ for all i and $|G_{i+1}| \leq \max\{\aleph_0, |G_i|\}$. Further, for $p \in \pi'$ and n as in (c), if $|y| = |z| = p^n$, $y \in C(t, w_n\varphi_0 \cdots \varphi_{i-1}, G_i)$, $z \in C(j, w_n\varphi_0 \cdots \varphi_{i-1}, G_i)$ and $j \neq t$, then $y\varphi_i \nsim_{G_{i+1}} z\varphi_i$.

Let $\overline{G}$ be the direct limit of the groups G_i. Clearly $\overline{G}$ is a π-*ULF* group of cardinality $|\overline{G}| = \max\{\aleph_0, |G|\}$; we may assume that $\overline{G}$ is the union of the G_i, so $G \subset \overline{G}$. Let y, z be elements of $\overline{G}$ of order p^n, $p \in \pi'$. There is then an i with $y, z \in G_i$. There are two cases. First, for $1 \leq n \leq k_p$ if $p \in \pi_1'$ and all $n \geq 1$ if $p \in \pi' - \pi_1'$, it follows from the above remarks that $y \sim_{\overline{G}} z$ if and only if, for some t, $0 \leq t \leq n$, we have $y, z \in C(t, w_n\varphi_0 \cdots \varphi_{i-1}, G_i)$. Each of the $n+1$ sets $C(t, w_n\varphi_0 \cdots \varphi_{i-1}, G_i)$ is non-empty, because G_0, and hence G_i, contains finite symmetric groups of arbitrarily large degree. Hence in this case $|\nu(p^n, \overline{G})| = n+1$. A similar argument, combined with part (ii) of the proposition, shows that if $p \in \pi_1'$ and $n > k_p$, then $y \sim_{\overline{G}} z$ if and only if $y^{p^{n-k_p}}, z^{p^{n-k_p}} \in C(t, w_{k_p}\varphi_0 \cdots \varphi_{i-1}, G_i)$ for some t, $0 \leq t \leq k_p$. Since there are $k_p + 1$ such sets, we have $|\nu(p^n, \overline{G})| = k_p + 1$. This completes the proof of the theorem.

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Department of Mathematics, Michigan State University, E. Lansing, MI 48824