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CURVATURE AND PROPER HOLOMORPHIC MAPPINGS BETWEEN BOUNDED DOMAINS IN $\mathbf{C}^n$

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ABSTRACT. In this paper we discuss some connections between proper holomorphic mappings between domains in $\mathbf{C}^n$ and the boundary behaviors of certain canonical invariant metrics (Cheng-Yau-Einstein Kähler metric, Bergman metric, intrinsic measures, etc.). Some compactness theorems have been proved (Theorem 4, Theorem 5). This generalizes an earlier result proved by the second author.

Introduction. A continuous mapping $f : X_1 \to X_2$ between two topological spaces is called proper if $f^{-1}(K) \subset X_1$ is compact whenever $K \subset X_2$ is compact. Proper holomorphic mappings between analytic spaces stand out for their beauty and simplicity. For instance, if $g : D_1 \to D_2$ is a proper holomorphic mapping between two bounded domains in C^n, a theorem of Remmert says that (D_1, g, D_2) is a finite branching cover. The branching locus in D_1 is described by $\{z \in D_1 | \det(dg(z)) = 0\}$. For the past ten years, there was a great amount of activity in characterizing the proper holomorphic mappings between pseudoconvex domains. It has been known for a long time that there are numerous proper holomorphic maps between unit disks in $\mathbf{C}^1$. The simplest example is $g : \Delta = \{z \in \mathbf{C}^1 | |z| < 1\} \to \Delta, g(z) = z^n$, where n is any positive integer. Nevertheless, such a phenomenon is no longer true in higher dimensional cases. H. Alexander was able to verify the following interesting fact.

THEOREM 1. [**1**]. *Let $B_n = \{(z_1, z_2, \ldots, z_n) | \sum_{i=1}^n |z_i|^2 < 1\}$ be the unit ball in $\mathbf{C}^n, n \geq 2$. Suppose $f : B_n \to B_n$ is a proper holomorphic mapping; then f must be a biholomorphism.*

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The essential step in proving Theorem 1 is to show that the set of branching loci $\mathcal{B} = \{z \in B_n | \det(df(z)) = 0\}$ is empty. Thus, $f : B_n \to B_n$ is a finite covering. Since B_n is simply connected, f can only be a biholomorphism.

Complex analysts are always interested in generalizing the results from Euclidean balls to strongly pseudoconvex domains. The following result due to S. Pinčuk is a significant extension of Alexander's theorem.

THEOREM 2. [**15**]. *Let D_1 and D_2 be two strongly pseudoconvex bounded domains with smooth boundaries in $\mathbf{C}^n$, $n \geq 2$. Suppose $f : D_1 \to D_2$ is a proper holomorphic mapping; then f is a covering.*

An example of the situation in Theorem 2 can be described as follows:

$$D_1 = \left\{(z_1, z_2) \in \mathbf{C}^2 \middle| |z_1|^4 + \frac{1}{|z_1|^4} + |z_2|^2 < 3\right\},$$
$$D_2 = \left\{(z_1, z_2) \in \mathbf{C}^2 \middle| |z_1|^2 + \frac{1}{|z_1|^2} + |z_2|^2 < 3\right\};$$

the mapping $(z_1, z_2) \to (z_1^2, z_2)$ is a 2 to 1 covering map from D_1 to D_2. This example was due to Y-T Siu, according to [**1**].

In general, the set of proper holomorphic mappings, denoted by $P(D_1, D_2)$, between two strongly pseudoconvex bounded domains is very small. It can be proved easily that, for generic members of strongly pseudoconvex bounded domains D_1 and D_2 in $\mathbf{C}^n, n \geq 2$, with $\pi_1(D_1)$ a normal subgroup of $\pi_1(D_2), P(D_1, D_2)$ is empty. For example, this would happen in the interesting cases when either $\pi_1(D_1)$ is trivial or $\pi_1(D_2)$ is abelian. The underlying reason is, for generic strongly pseudoconvex bounded domains D in $\mathbf{C}^n, n \geq 2$, the groups of biholomorphisms, namely $\mathrm{Aut}(D)$, consist of identity elements only [**3, 8**]. If there is a proper holomorphic map $f : D_1 \to D_2$, it is a finite holomorphic cover by Pinčuk's theorem. Furthermore, $\pi_1(D_2)/\pi_1(D_1)$ will induce a biholomorphic group action on D_1 if $\pi_1(D_1)$ is sitting in $\pi_1(D_2)$ as a subgroup. This would be a contradiction if the order of $\pi_1(D_2)/\pi_1(D_1)$ is greater than one. There is another obvious topological obstruction to the existence of a proper holomorphic map between two strongly pseudoconvex domains D_1 and D_2, namely $\pi_i(D_1) = \pi_i(D_2)$ for all $i \geq 2$.

In [**20**] the second author proved the following result concerning biholomorphic groups of strongly pseudoconvex domains.

THEOREM 3. [**20**]. *Let D be a strongly pseudoconvex bounded domain with smooth boundary in $\mathbf{C}^n$. Then* $\mathrm{Aut}(D)$ *is noncompact if and only if D is biholomorphic to $B_n, n = \dim_{\mathbf{C}} D$.*

Theorem 3 is interesting only when D is an Eilenberg-Maclane space (i.e., $\pi_i(D) = 0$ for all $i \geq 1$) due to the following observation of the authors.

LEMMA. *Let D be a bounded domain in $\mathbf{C}^n$ with ∂D embedded as a C^1 closed submanifold of dimension $2n-1$. Suppose* $\mathrm{Aut}(D)$ *is noncompact and $\pi_i(D) \neq 0$ for some $i \geq 1$; then ∂D admits a complex analytic variety of positive dimension.*

PROOF. Let S be a sphere representing a nontrivial element in $\pi_i(D)$. As $\mathrm{Aut}(D)$ is noncompact, by a normal family argument, there exists a sequence $\{g_j\} \subset \mathrm{Aut}(D)$ such that it will converge on compacta to a holomorphic mapping, $g : D \to \mathbf{C}^n$, with the image $g(D)$ sitting on ∂D. Suppose ∂D does not accommodate any complex variety of positive dimension. Then g must be a constant map. Let us assume $g(D) = p \in \partial D$. Since ∂D is a C^1 closed submanifold, one can always find an open set V containing p in $\mathbf{C}^n$ such that $V \cap D$ is contractible. It is clear that, for sufficiently large j, $g_j(S) \subset V \cap D$. Each g_j is a homeomorphism, thus $g_j(S)$ represents a nontrivial element in the free homotopy class of $\pi_i(D)$. This contradicts the fact that $V \cap D$ is contractible. □

In view of a lot of recent attention on the topic of proper holomorphic mappings, the authors feel that it might be worthwhile to point out the following startling fact which generalizes Theorem 3 significantly.

THEOREM 4. *Let D_1 and D_2 be two strongly pseudoconvex bounded domains with smooth boundaries in $\mathbf{C}^n, n \geq 2$. Then $P(D_1, D_2)$ is noncompact if and only if both D_1 and D_2 are biholomorphic to B_n.*

It follows from Pinčuk's theorem that Theorem 4 is an immediate consequence of the local version stated below, which is the principal result of this note.

THEOREM 5. *Let D_1 and D_2 be bounded domains in $\mathbf{C}^n$. We let $P_0(D_1, D_2)$ be the set of all unbranching proper holomorphic maps from D_1 to D_2. Suppose the following two conditions are fulfilled.*

(1) *There is a strongly pseudoconvex boundary point $p \in \partial D_2$.*

(2) *There exists a point $x \in D_1$ and a sequence $\{f_j\} \subseteq P_0(D_1, D_2)$ such that $\{f_j(x)\}$ converges to p.*

Then both D_1 and D_2 are biholomorphic to B_n.

We shall present three proofs of Theorem 5. The first proof depends on the recent works of Cheng-Yau [**5**] and Mok-Yau [**14**] on the canonical Einstein Kähler metrics on domains of holomorphy. The second and third proofs are a modification of those given in [**20**]. The main techniques of our proofs are basically differential geometry. They involve some known results on boundary behaviors and curvature estimates of some canonical Kähler metrics (Bergman metrics, Cheng-Yau Einstein Kähler metrics) and intrinsic measures (Eisenman measures, Kobayashi-Royden differential metrics, Caratheodory-Reiffen differential metrics). The underlying idea goes back to an old paper of L. Ahlfors (Trans. Amer. Math Soc. **43** (1938), 359–364), who interpreted the classical Schwarz lemma in a differential geometric setting. His theorem can be recorded as follows.

THEOREM (AHLFORS-SCHWARZ LEMMA). *Let (N, ds_N^2) be a hermitian manifold with holomorphic curvature bounded above by $-b^2$. Suppose (M, ds_M^2) is a complete hermitian manifold of dimension one whose holomorphic curvature is bounded from below by $-a^2$. Let $f : M \to N$ be any holomorphic mapping; then $f^*(ds_N^2) \leq \frac{a^2}{b^2} ds_M^2$.*

Some generalizations of the Ahlfors-Schwarz lemma have been considered by Grauert-Reckziegel (Math Z. **89** (1965), 108-125), R.L. Royden [**17**], S.T. Yau [**23**], and others.

A. Some preliminaries and related results. Let M be a complex manifold of dimension $n, x \in M, k$ an integer between one and n.

DEFINITION. The Eisenman differential k-measure on M is a function $E_M^k : \Lambda^k T(M) \to \mathbf{R}$ such that, for all $(x, v) \in \Lambda^k T_x(M)$,

$$E_M^k(x,v) = \inf\Big\{R^{-2k} | \text{there exists a holomorphic map } f : B_k(R) \to M \text{ such that } f(0) = x \text{ and } df_0\Big(\frac{\partial}{\partial w_1} \wedge \frac{\partial}{\partial w_2} \wedge \cdots \wedge \frac{\partial}{\partial w_k}(0)\Big) = v\Big\},$$

where $B_k(R) = \{w = (w_1, w_2, \ldots, w_k) \in \mathbf{C}^k | \sum_{i=1}^k |w_i|^2 < R\}$.

When $k = 1$, it is called a Kobayashi-Royden differential metric [**16**], denoted by $K_M = \sqrt{E_M^1}$. When $k = n$, it is a volume form, denoted by $E_M^n = |E_M^n| dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$, where $|E_M^n|$ is a function on M. The Kobayashi pseudo-distance function d_M^k can be defined as the integrated form of K_M. If d_M^k is nontrivial everywhere (i.e., for all $x \neq y \in M, d_M^k(x,y) \neq 0$), then M is called a hyperbolic manifold. A complete hyperbolic manifold means d_M^k is Cauchy complete.

On the other hand, the Caratheodory differential k-measure C_M^k is defined as follows.

DEFINITION. $C_M^k : \Lambda^k T_x(M) \to \mathbf{R}$, $(x, v) \in \Lambda^k T_x(M)$, $C_M^k(x, v) = \sup\{1/R^{2k} |$ there exists a holomorphic mapping $f : M \to B_k(R)$ such that $f(x) = 0, df_x(v) = \frac{\partial}{\partial w_1} \wedge \cdots \wedge \frac{\partial}{\partial w_k}(0)\}$.

When $k = 1$, it is called a Caratheodory-Reiffen differential metric, denoted by $C_M = \sqrt{C_M^1}$. When $k = n$, it is a volume form $C_M^n = |C_M^n| dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$, where $|C_M^n|$ is a function on M.

One can also define E_M^k and C_M^k relative to a polydisc instead of a ball. They are different measures, but enjoy similar properties. In this section, we shall use I_M^k to represent either E_M^k or C_M^k.

The following theorem follows almost immediately from the definitions.

THEOREM (a). (1) *$E_M^k \geq C_M^k$ on any complex manifold M.*

(2) *Let $f : M_1 \to M_2$ be a holomorphic mapping between complex manifolds M_1 and M_2. Then one has $I_{M_1}^k \geq f^*(I_{M_2}^k)$, a measure-decreasing property under f.*

(3) *Let X be a domain of a complex manifold M. Then $I_X^k \geq I_M^k$, a monotone property, holds.*

(4) *Any biholomorphism f of a complex manifold X is measure-preserving relative to I_X^k, that is, $I_X^k = f^*(I_X^k)$.*

(5) *Let $\tilde{M}$ be the universal cover of a complex manifold M. Let $\pi : \tilde{M} \to M$ be the covering projection. Then $E_{\tilde{M}}^k = \pi^*(E_M^k)$.*

THEOREM (b). [**16**] (ROYDEN'S LOCALIZATION LEMMA). *Let D be a bounded domain on a hyperbolic complex manifold M and $\tilde{D} = V \cap D$, $p \in \partial D, p \in V$, where V is an open set of M. Then the following inequality holds*:

$$|E_{\tilde{D}}^k(z,v)| \leq |\cot h d_{D-\tilde{D}}(z)|^{2k} \cdot |E_D^k(z,v)|,$$

where $z \in \tilde{D}, (z,v) \in \Lambda^k T_z(D)$.

In the statement of Theorem (b), $d_{D-\tilde{D}}(z) = \inf_w\{d_D(z,w)| \, w \in D - \tilde{D}\}$, where $d_D(z,w)$ is defined as follows.

DEFINITION. $d_D(z,w) = \inf\{d_{B_k(a,b)} \,|\, f \in \mathrm{Hol}(B_k,D), f(a) = z, f(b) = w, d_{B_k}$ is the Kobayashi distance function on $B_k\}$, $B_k = \{(z_1,z_2,\ldots,z_k) \in \mathbf{C}^k \,|\, \sum_{i=1}^k |z_i|^2 < 1\}$.

It is clear from the definition that $d_{D-\tilde{D}}(z)$ is not less than $\inf_w\{d_D^k(z,w)|w \in D-\tilde{D}\}$, where d_D^k is the Kobayashi distance function on D. We remark that $E_{\tilde{D}}^k(z,v)/E_D^k(z,v) \to 1$ when $z \to p$ under the situation $d_{D-\tilde{D}}^k(z) \to \infty$. When ∂D is strongly pseudoconvex, this is true [**6**]. Theorem (b) was proved by H.L. Royden for the case $k = 1$. For the general case $1 \leq k \leq n$, the proof is similar.

Using a curvature argument, the second author was able to give the following intrinsic characterization of a Euclidean ball.

THEOREM (c.1). [**20**]. *Let M be a simply-connected hyperbolic manifold with $C_M = K_M$. Moreover, either C_M or K_M is assumed to be a Kähler metric. Then M is biholomorphic to the Euclidean ball.*

Recently, Charles Stanton generalized the above theorem using a different method.

THEOREM (c.2). [**19**]. *Let M be a complete hyperbolic manifold. Suppose there is one point $x \in M$ at which $C_M = K_M$ and one of these two metrics is hermitian of class C^∞. Then M is biholomorphic to the Euclidean ball.*

THEOREM (d). [**7, 21**]. *Let D be a bounded domain in $\mathbf{C}^n$ with a strongly pseudoconvex boundary point $p \in \partial D$. Let $\tilde{D} = V \cap D$, where $p \in V$ is a sufficiently small ball in $\mathbf{C}^n$. The following are true*:

(1) $|E^n_{\tilde{D}}(z)|/|C^n_{\tilde{D}}(z)|$ *approaches one as* $z \to p$;

(2) $K_{\tilde{D}}(z,v)/C_{\tilde{D}}(z,v)$ *approaches one as* $z \to p$.

In [**20**], the next theorem was proved for the special case where D is completely hyperbolic. Actually, a similar proof can yield a slightly more general statement as follows.

THEOREM (e). [**20**]. *Let D be a bounded domain in $\mathbf{C}^n$. Suppose there is one point $x \in D$ such that $|E^n_D(x)| = |C^n_D(x)|$. Then D is biholomorphic to the Euclidean ball.*

S.Y. Cheng and S.T. Yau constructed an invariant complete Einstein Kähler metric on a bounded strongly pseudoconvex domain in [**5**]. We summarize part of their results in the theorem stated below which will be used in the future.

THEOREM (f). [**5**]. *Let D be a strongly pseudoconvex bounded domain in $\mathbf{C}^n$. Then the following statements are true.*

(1) *The boundary limits of $\sqrt{ds_D^2(z,v)}/k_D(z,v)$ and $\sqrt{ds_D^2(z,v)}/C_D(z,v)$ are equal to one, where (z,v) is a nonzero vector and $\sqrt{ds_D^2}$ the Finsler metric associated with the canonical Einstein metric ds_D^2 on D.*

(2) *The holomorphic curvature of ds_D^2 is asymptotically equal to -2 close to the boundary.*

Similar results are true for Bergman metrics with the reservation that a multiple of $\sqrt{n+1}$ and $1/(n+1)$ will appear in the limits of (1) and (2) respectively.

Let D be any bounded domain of holomorphy; it is well known that one can approximate D from the interior by an increasing sequence of strongly pseudoconvex bounded domains $\{D_i\}$. Let $ds_{D_i}^2$ be the Cheng-Yau Einstein metric on D_i. In [**5**], it was proved that the increasing limit $\lim_{i\to\infty} ds_{D_i}^2$ is also an Einstein Kähler metric ds_D^2 on D. N. Mok and S.T. Yau later proved that ds_D^2 is actually complete [**14**]. This gives rise to a differential geometric characterization of the Stein open sets in $\mathbf{C}^n$.

THEOREM (g). [**14**]. *Let D be a bounded domain in $\mathbf{C}^n$. Then D is a domain of holomorphy if and only if it admits a complete Kähler Einstein metric.*

These metrics on domains of holomorphy D are again invariant under biholomorphisms, and they are the unique complete Einstein Kähler metrics on D up to multiples of scalars.

One can also prove the following local version of the Cheng-Yau metric and the Bergman metric around a strongly pseudoconvex boundary point. The result is implicit in [**2, 5, 6, 7, 9**]. Wong is grateful to Professor Shing-Tung Yau for a conversation about this fact during his stay at the Institute for Advanced Study.

THEOREM (h). *Let $p \in \partial D$ be a strongly pseudoconvex point of a domain D in $\mathbf{C}^n$. Let V be a sufficiently small open neighborhood of p which is biholomorphic to B_n, and let $\tilde{D} = V \cap D$. Then the following statements are true*:

(1) *The boundary limits of $\sqrt{ds^2_{\tilde{D}}(z,v)}/C_{\tilde{D}}(z,v)$ and $\sqrt{ds^2_{\tilde{D}}(z,v)}/k_{\tilde{D}}(z,v)$ are equal to one as $z \to p$, where $ds^2_{\tilde{D}}$ is the Cheng-Yau metric on $\tilde{D}$.*

(2) *The boundary limits of $\sqrt{B_{\tilde{D}}(z,v)}/C_{\tilde{D}}(z,v)$ and $\sqrt{B_{\tilde{D}}(z,v)}/k_{\tilde{D}}(z,v)$ are equal to $\sqrt{1+n}$ as $z \to p$, where $B_{\tilde{D}}$ is the Bergman metric on $\tilde{D}$.*

(3) *The holomorphic curvature of $ds^2_{\tilde{D}}$ is asymptotically equal to -2 close to the boundary point p.*

(4) *The holomorphic curvature of $B_{\tilde{D}}$ is asymptotically equal to $-2/(n+1)$ close to the boundary point p.*

THEOREM (i). [**4**] (CARTAN'S FIXED POINT THEOREM). *Let (X, ds^2) be a simply-connected complete Riemannian manifold with nonpositive sectional curvature. Suppose G is a compact Lie group acting on X as isometries; then G has a fixed point.*

In particular, any finite group H acting on X isometrically must fix at least one point.

THEOREM (j). *Let D_1 and D_2 be bounded domains in $\mathbf{C}^n$. Suppose that*

1. *there is a strongly pseudoconvex point $p \in \partial D_2$;*

2. *one can find $x \in D_1$ and a sequence of holomorphic mappings $\{f_j\} \subset \mathrm{Hol}(D_1, D_2)$ such that $\{f_j(x)\} \to p$.*

Then there exists a subsequence of $\{f_j\}$, denoted by the same notation, $\{f_j\}$, satisfying: For any compact set $K \subset D_1$ and any open set

$\tilde{D} = V \cap D_2$, where $p \in V$ is an open set in $\mathbf{C}^n$, there is a j_0 such that $f_j(K) \subset \tilde{D}$ for all $j \geq j_0$.

PROOF. Since $\{f_j(x)\} \to p$, by a normal family argument one can find a subsequence of $\{f_j\}$ converging on compacta to a holomorphic mapping $f : D_1 \to \mathbf{C}^n$ so that $f(x) = p$ and $f(D_1) \subseteq \partial D_2$. By assumption ∂D_2 is strongly pseudoconvex at p and contains no complex analytic variety of positive dimension through p. This implies f is a constant mapping which brings the whole D onto a single point. Our claim in Theorem (j) should now be clear. □

The following theorem is implicit in [**12**].

THEOREM (k). (K.H. LOOK). *Let D be a bounded domain in $\mathbf{C}^n$; then $\sqrt{B_D} \geq \sqrt{(n+1)}C_D$, where B_D =Bergman metric on D.*

THEOREM (l). (K.H. LOOK) [**13**]. *Let D be a bounded domain in $\mathbf{C}^n$ carrying a complete Bergman metric with constant negative holomorphic curvature. Then D is biholomorphic to the ball.*

THEOREM (m). (LEMPERT) [**10, 11, 18**]. *Let D be a convex bounded domain in $\mathbf{C}^n$. Then $K_D = C_D$.*

THEOREM (n). (LEMPERT) [**10, 11**]. *Let D be a convex bounded domain in $\mathbf{C}^n$. Then $d_D^k = d_D^C$, where d_D^C is the classical Caratheodory distance function on D.*

Theorems (b) and (n) will not be used in our proofs.

(B). A proof of Theorem 5 depending on the Einstein Kähler metric.

Claim 1. *D_1 is simply-connected.*

Proof. Let us denote by $\{f_j\}$ the sequence of holomorphic mappings in the proof of Theorem (j) starting from the original sequence of unbranching proper holomorphic mappings stated in Theorem 5. Thus, $\{f_j\}$ are all finite holomorphic coverings. Since $p \in \partial D_2$ is strongly pseudoconvex, one can choose an open set V containing p such that $\tilde{D} = V \cap D_2$ is contractible. Suppose $\pi_1(D_1)$ is nontrivial. Let δ be a closed loop at the base point x representing a nontrivial element in $\pi_1(D_1)$. By Theorem (j), for sufficiently large j, $f_j(\delta)$ is a closed loop $\beta \subset \tilde{D}$ with base point $f_j(x)$. $\tilde{D}$ is simply-connected, β is therefore homotopic to a point in $\tilde{D}$. Nevertheless, $f_j : D_1 \to D_2$ is a covering, $(f_j)_* : \pi_1(D_1) \to \pi_1(D_2)$ is an injection. This is a contradiction to the fact that $f_j(\delta)$ must be a nontrivial element in $(f_j)_*(\pi_1(D_1)) \subset \pi_1(D_2)$. □

Claim 2. *Both D_1 and D_2 admit a complete Einstein Kähler metric.*

Proof. Again $p \in \partial D_2$ is strongly pseudoconvex, hence we can choose an open set V containing p such that $\tilde{D} = V \cap D_2$ has an increasing sequence of Stein open sets $\{\tilde{D}_i\}_{i=1}^{\infty}$ satisfying the following properties:

(i) $\tilde{D}_{i+1} \subset\subset \tilde{D}_i$ for all i;

(ii) $\tilde{D}_i$ is connected and simply-connected for all i;

(iii) $\cup_{i=1}^{\infty} \tilde{D}_i = \tilde{D}$.

By a rearrangement of the sequences $\{f_j\}$ and $\{\tilde{D}_i\}$, including taking subsequences and altering the indices, we can assume $f_i(x) \in \tilde{D}_i$ for all i (Theorem (j)). Let E_i be a subset of D_1 constructed as follows: E_i is the lifting of $\tilde{D}_i$ in D_1 with fixed base points $x \in D_1$ and $x_i = f_i(x) \in \tilde{D}_i$. To avoid confusion, we must present the details of the construction of E_i. Let τ be a path in D_2, starting from x_i and ending at an arbitrary point $y \in \tilde{D}_i$. Since D_1 is simply-connected, there is a unique lifting of τ starting from x and terminating at some point $z \in D_1$. This point is uniquely determined depending only on $y \in \tilde{D}_i$. E_i is the totality of such z and clearly is biholomorphic to $\tilde{D}_i$ for each i.

We claim that $\{E_i\}$ will exhaust D_1. The argument goes as follows. Let K be any relatively compact connected open set in D_1. We can assume K contains our fixed point x. By Theorem (j), $f_i(K)$ will be contained in $\tilde{D}_i$ for some i. Let τ be any path in K joining x to an arbitrary point $z \in K$. The image of τ under f_i, namely $\beta = f_i(\tau)$, is a path contained in $\tilde{D}_i \subset \tilde{D}$, linking $x_i = f_i(x)$ to some point $y \in \tilde{D}_i$. According to our previous discussion, one can conclude $z \in E_i$. This proves that $K \subset E_i$ and thus $\{E_i\}$ will exhaust the whole D_1.

Finally, by taking a subsequence of $\{\tilde{D}_i\}$, one can assume that $E_{i+1} \subset\subset E_i$ for every i. Hence, we have succeeded in exhibiting a sequence of connected open subsets in D_1, namely $\{E_i\}$, with the following properties:

(i) $E_{i+1} \subset\subset E_i$ for all i;

(ii) $\cup_{i=1}^{\infty} E_i = D_1$;

(iii) each E_i is biholomorphic to the Stein open set $\tilde{D}_i$.

A classical theorem of Behnke-Stein can now assure us that D_1 is a bounded domain of holomorphy.

By Theorem (g) there exists a complete Kähler Einstein metric $ds^2_{D_1}$ on D_1. Let us fix any $f_j : D_1 \to D_2$, which is a holomorphic covering with D_1 simply-connected. Since $ds^2_{D_1}$ is invariant under biholomorphisms, f_j will induce a complete Einstein Kähler metric $ds^2_{D_2}$ on D_2. □

An alternate way to construct a complete Einstein Kähler metric D_1 is described next.

One can choose $\tilde{D}_i$ to be strongly pseudoconvex for each i. We let $ds^2_{\tilde{D}_i}$ be the Cheng-Yau metric on each $\tilde{D}_i$. Since E_i is biholomorphic to $\tilde{D}_i$, there is a complete Einstein Kähler metric $ds^2_{E_i}$ defined on E_i as well. By taking an increasing limit of $\{ds^2_{E_i}\}$ one obtains an Einstein Kähler metric $ds^2_{D_1}$ by [**5**]. This metric is complete by Mok-Yau [**14**].

Claim 3. *The holomorphic curvature of* $ds^2_{D_1}$ *is equal to* -2 *everywhere.*

Proof . One can choose $\tilde{D} = V \cap D_2$, where V is a sufficiently small open set biholomorphic to B_n. By Theorem (h) the holomorphic curvature of $ds^2_{\tilde{D}}$ is asymptotically equal to -2 close to ∂D_2. Let $q \in D_1$ be an arbitrary point. By Theorem (j), $\{f_i(q)\}$ through a subsequence will converge to $p \in \partial D_2$. Let $\{\varepsilon_i\}$ be a monotonic decreasing sequence of positive numbers converging to zero. For each i, we can choose a simply-connected strong pseudoconvex domain $\tilde{D}_i$ with $f_i(q) \in \tilde{D}_i \subset\subset \tilde{D}_{i+1}$ large enough such that $|K_i(f_i(q), t) - K(f_i(q), t)| < \varepsilon_i$, where $K_i(f_i(q), t) =$ holomorphic curvature of $ds^2_{\tilde{D}_i}$ at $f_i(q)$ in the direction of t, $K(f_i(q), t) =$ holomorphic curvature of $ds^2_{\tilde{D}}$ at $f_i(q)$ in the direction of t, $t =$ any unit complex vector at $f_i(q)$ with respect to the Euclidean norm. In this way we construct an increasing sequence of simply-connected strongly pseudoconvex domains $\{\tilde{D}_i\}_{i=1}^{\infty}$ such that

(i) $f_i(q) \in \tilde{D}_i$;

(ii) $\tilde{D}_i \subset\subset \tilde{D}_{i+1}$;

(iii) $\cup_{i=1}^{\infty} \tilde{D}_i = \tilde{D}$;

(iv) The holomorphic curvature at p_i with respect to $ds^2_{\tilde{D}_i}$ tends to -2 as $i \to \infty$, here $p_i = f_i(q)$.

As before we obtain a corresponding increasing sequence of open sets $\{E_i\}$ exhausting D_1. For every i, $f_i : E_i \to \tilde{D}_i$ is an isometry relative to $ds^2_{E_i}$ and $ds^2_{\tilde{D}_i}$. By the Cheng-Mok-Yau procedure, $\{ds^2_{E_i}\}$ will converge to $ds^2_{D_1}$ normally. It would imply that the holomorphic curvature at q is equal to -2. □

Claim 4. *Both D_1 and D_2 are biholomorphic to B_n.*

Proof . Since D_1 is a simply-connected complete Kähler manifold with constant negative holomorphic curvature, it is biholomorphic to B_n by a well-known fact in complex differential geometry. Suppose $f_j : D_1 \to D_2$ is a finite cover with index m. Hence, there is a finite biholomorphic group of order m acting freely on B_n as isometries with respect to the Bergman metric, which has negative sectional curvature. This would contradict Cartan's fixed point theorem (Theorem (i)) if $m > 1$. Therefore D_2 must be biholomorphic to B_n also.

REMARK 1. It is possible to use the Bergman metric instead of the Einstein Kähler metric. If one decides to do so, it is necessary to prove that the Bergman metric B_{D_1} on D_1 is complete. One can prove this fact depending on the completeness of $ds^2_{D_1}$. For example, the completeness of B_{D_1} is a consequence of §D(1) (or the remark in §D), §D(2), §D(3), and a theorem of K.H. Look (Theorem (k)). It should not be hard to find a proof of this result independent of the Einstein Kähler metric.

For the rest of the proof, we can apply the same argument as in Claim 3 of §B and Theorem (h)(4) to conclude that B_{D_1} has constant negative holomorphic curvature. Finally, by Claim 1 of §B that D_1 is simply-connected, it follows immediately that D_1 must be biholomorphic to B_n. Alternately, one can invoke a theorem due to K.H. Look (Theorem (l)) to draw the same conclusion without appealing to the simple connectivity of D_1. □

REMARK 2. The proof of Theorem 4 is much easier; the basic result we need is Theorem (f).

C. A proof depending on intrinsic volume forms. Let us assume $|E^n_{D_1}(x)| = |C^n_{D_1}(x)|$ for the given point x in D_1. By Theorem (e), this implies that D_1 must be biholomorphic to B_n. If the order of the covering $f_j : B_n = D_1 \to D_2$ is greater than one, this would contradict Cartan's fixed point theorem (Theorem (i)). Thus D_2 is also biholomorphic to B_n. Therefore the whole proof depends on the following assertion.

Claim. $|E^n_{D_1}(x)| = |C^n_{D_1}(x)|$.

Proof. For each $j, f_j : D_1 \to D_2$ is a covering. From Theorem (a)(5) we have

$$E^n_{D_1}(x,v) = E^n_{D_2}(x_j, df_j(v)),$$

where $x_j = f_j(x)$ and (x,v) is a nonzero n-vector at x. Let $(D_1)_k$ be an increasing sequence of domains such that $\cup_{k=1}^{\infty}(D_1)_k = D_1, x \in (D_1)_k$ for each k, and $(D_1)_k \subset\subset (D_1)_{k+1}$. For each j, let $(D_2)^j_k = f_j((D_1)_k)$. For a fixed k, we obtain, by Theorem (a)(2)(3), the inequalities

$$C^n_{(D_1)_k}(x,v) \geq C^n_{(D_1)^j_k}(x_j, df_j(v)) \geq C^n_{\tilde{D}}(x_j, df_j(v)).$$

The last inequality on the above chain is valid for sufficiently large j. The reason is that when j is sufficiently large, $f_j((D_1)_k) = (D_2)^j_k \subset \tilde{D}$ by Theorem (j), where $\tilde{D} = V \cap D_2, p \in V$, is an open set in $\mathbf{C}^n$. It follows that, for fixed k and large j, we have the chain

$$\frac{C^n_{(D_1)_k}(x,v)}{E^n_{D_1}(x,v)} \geq \frac{C^n_{(D_2)^j_k}(x_j, df_j(v))}{E^n_{D_2}(x_j, df_j(v))} \geq \frac{C^n_{\tilde{D}}(x_j, df_j(v))}{E^n_{D_2}(x_j, df_j(v))}$$

of inequalities (Theorem (a)(5) has been used here).

Observe that:

(i) By the volume decreasing property under holomorphic mappings, we have $E^n_{\tilde{D}}(x_j, df_j(v)) \geq E^n_{D_2}(x_j, df_j(v))$ since the inclusion map $\tilde{D} \hookrightarrow D_2$ is holomorphic. Therefore we have

$$\frac{C^n_{(D_1)_k}(x,v)}{E^n_{D_1}(x,v)} \geq \frac{C^n_{\tilde{D}}(x_j, df_j(v))}{E^n_{\tilde{D}}(x_j, df_j(v))}.$$

(ii) Again by the strong pseudoconvexity of $p \in \partial D_2$, one obtains

$$\frac{C^n_{\tilde{D}}(x_j, df_j(v))}{E^n_{\tilde{D}}(x_j, df_j(v))} \to 1 \quad \text{as } x_j \to p$$

by Theorem (d)(1).

(iii) If we let $k \to \infty$, then $C^n_{(D_1)_k}(x,v) \to C^n_{(D_1)}(x,v)$. This approximation property can be proved by an elementary normal family argument.

(iv) It is always true that $C^n_{(D_1)}(x,v)/E^n_{(D_1)}(x,v) \leq 1$ by Theorem (a)(1).

Combining (i)–(iv), letting $j \to \infty$ and then $k \to \infty$, one concludes $1 \geq C^n_{(D_1)}(x,v)/E^n_{(D_1)}(x,v) \geq 1$, proving our claim.

D. A proof depending on the intrinsic metrics.

1. $C_{D_1} = K_{D_1}$.

Proof. Let $q \in D_1$ be an arbitrary point. By Theorem (j), $\{f_j(q)\}$ will converge to $p \in \partial D_1$ as before. Applying Theorem (d)(2) and a parallel argument as in §C, one obtains $K_{D_1} = C_{D_1}$ at q.

2. There exists a complete Einstein Kähler metric $ds^2_{D_1}$ on D_1 (Claim 2 of §B).

3. $K_{D_1} = \sqrt{ds^2_{D_1}}$.

Proof. Let $q \in D_1$ be an arbitrary point and $\{f_i(q)\}$ the sequence as before. With the same notations as in Claim 3 of §B we can exhibit an increasing sequence of simply-connected open sets $\{\tilde{D}_i\}_{i=1}^{\infty}$ in $\tilde{D}$ with the following properties:

(i) $\tilde{D}_i$ is a relatively compact strongly pseudoconvex domain in $\tilde{D}$,

(ii) $\tilde{D}_i \subset\subset \tilde{D}_{i+1}$,

(iii) $p_i = f_i(q) \in \tilde{D}$, and

(iv) each $\tilde{D}_i$ is chosen large enough so that

$$\lim_{i\to\infty} \frac{K_{\tilde{D}_i}(p_i, df_i(v))}{K_{\tilde{D}}(p_i, df_i(v))} = 1, \qquad \lim_{i\to\infty} \frac{\sqrt{ds^2_{\tilde{D}_i}(p_i, df_i(v))}}{\sqrt{ds^2_{\tilde{D}}(p_i, df_i(v))}} = 1,$$

where v is a nonzero vector at $q \in D_1$.

By Theorem (h)(1) $\lim_{i\to\infty} K_{\tilde{D}}(p_i, df_i(v))/\sqrt{ds^2_{\tilde{D}}(p_i, df_i(v))} = 1$; hence we can conclude from (iv) that

$$\begin{aligned} \frac{K_{D_1}(q,v)}{\sqrt{ds^2_{D_1}(q,v)}} &= \lim_{i\to\infty} \frac{K_{E_i}(q,v)}{\sqrt{ds^2_{E_i}(q,v)}} \\ &= \lim_{i\to\infty} \frac{K_{\tilde{D}_i}(p_i, df_i(v))}{\sqrt{ds^2_{\tilde{D}_i}(p_i, df_i(v))}} = 1. \end{aligned}$$

(Remember, here f_i is an isometry from E_i to $\tilde{D}_i$ with respect to $K_{E_i}, ds^2_{E_i}$, and $K_{\tilde{D}_i}, ds^2_{\tilde{D}_i}$, respectively.)

There are two methods to conclude the proof.

1. By Claim 1 of §B, D_1 is simply-connected. One can apply Theorem (c.1) to finalize the proof.

2. We can invoke Theorem (c.2) to conclude the proof without using the simple connectivity of D_1.

The rest of the proof is the same as Claim 4 of §B.

REMARK. One can prove $C_{D_1} = K_{D_1}$ by means of Theorem (m).

Proof . We follow the same notations as in §B. Let us choose V to be a sufficiently small open set containing p so that $\tilde{D} = D_2 \cap V$ is biholomorphic to a bounded convex set in $\mathbf{C}^n$. One can choose an increasing sequence of simply-connected open sets $\{\tilde{D}_i\}_{i=1}^{\infty}$ such that

(i) $p_i = f_i(q) \in \tilde{D}_i$,

(ii) $\tilde{D}_{i+1} \subset\subset \tilde{D}_i$,

(iii) $\cup_{i=1}^{\infty} \tilde{D}_i = \tilde{D}$, and

(iv) each $\tilde{D}_i$ is biholomorphic to a convex bounded open set in $\mathbf{C}^n$.

The corresponding sequence of open sets $\{E_i\}_{i=1}^{\infty}$ of D_1 in (B) satisfies the following properties:

(i) $q \in E_i$,

(ii) $E_{i+1} \subset\subset E_i$,

(iii) $\cup_{i=1}^{\infty} E_i = D_1$, and

(iv) each E_i is biholomorphic to $\tilde{D}_i$ under f_i.

By Theorem (m), $K_{\tilde{D}_i}$ is equal to $C_{\tilde{D}_i}$ at p_i; thus $K_{E_i} = C_{E_i}$ at q, letting $i \to \infty, K_{E_i} \to K_{D_1}, C_{E_i} \to C_{D_1}$ at q, respectively. □

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