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ORDER CONTINUOUS BOREL LIFTINGS

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Introduction. The lifting theorem of A. and C. Ionescu-Tulcea [**3**] can be stated as follows: Every bounded linear operator $T : L^\infty_\mu \to L^\infty_\mu$ has a lifting $\hat{T}$, taking values in M, the space of bounded μ-measurable functions. In other words, $P_\mu \circ \hat{T} = T$, where P_μ is the natural projection of M onto equivalence classes in L^∞_μ.

It is not known whether M may be replaced by the space of Borel functions in the Ionescu-Tulcea theorem. In this paper we study order continuous operators on L^∞_μ and characterize those which have an order continuous lifting $\hat{T}$ which takes values in the Borel functions.

Let X be a compact Hausdorff space and let μ be a positive bounded Baire measure on X. $C(X)$, or C, is the space of continuous functions on X, with first and second normal duals C' and C''. μ may be identified with a positive element of C'. C, C', and C'' are Riesz spaces, or vector lattices, and C may be embedded in C'' in a natural way.

This paper will deal with C, C', and C'' along with various subspaces which are order isomorphic with L^1_μ, L^∞_μ, and the space of Borel functions. For a thorough study of C'' and definitions not included here, see [**4**]. For more information on Riesz spaces, see Schaefer [**6**] or Luxemburg-Zaanen [**5**].

C' may be written as the order direct sum of C'_a, the "atomic" measures (those generated by X as a subset of C') and C'_d, the "diffuse" measures (those order disjoint from C'_a). This yields a corresponding decomposition $C'' = C''_a \oplus C''_d$, where $C''_a = (C'^{\perp}_a)^d$. $C^u(C^l)$ consists of those elements of C'' which are infima (suprema) of subsets of C. $s(X) = C^u - C^u = C^l - C^l$ is the linear subspace generated by C^l or C^u. The σ-order closure of $s(X)$ will be denoted by Bo. Bo is order isomorphic with its projection Bo_a onto C''_a (and thus is determined by its values on $X \subset C'$).

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The same is true for C^u, C^l, and $s(X)$. C^u_a and C^l_a may be identified with the upper and lower semicontinuous functions on X, respectively, and $Bo \cong Bo_a$ with the space of Borel functions. Note that each element of Bo is the limit of a decreasing (increasing) net in $C^l(C^u)$ [**4**]. We will denote C^l and C^u by "lsc" and "usc", respectively.

The band C'_μ of C' generated by μ may be identified under order isomorphism with L^1_μ. The dual band $C''_\mu = (C'^{\perp}_\mu)^d \subset C''$ may similarly be identified with L^∞_μ. Let P_μ denote the projection onto C''_μ. Bo projects onto C''_μ, i.e., $Bo_\mu = P_\mu(Bo) = C''_\mu$. (A note of caution about this identification: If μ is a diffuse measure, as Lebesgue measure on [0,1], then $(C''_\mu)_a = \{0\}$ while $Bo_a \approx Bo$). In the following, we will replace the symbol C'_μ with L^1_μ and C''_μ with L^∞_μ.

C'', as a Dedekind complete AM space with unit, is isometric and order isomorphic to $C(Y)$ for some hyperstonean space Y. L^∞_μ as a band of C'', is isometric and order isomorphic to $C(Y_\mu)$ for a closed and open set $Y_\mu \subset Y$.

If $\mathbf{1}$ represents the constant one function on X and its image, the unit element of C'', then $e \in C''_+$ is a *component of* $\mathbf{1}$, or simply a *component*, if $e \wedge (1 - e) = 0$. The set of all components will be denoted by $\mathcal{E}$. $\mathbf{1}_\mu$, the projection of $\mathbf{1}$ on L^∞_μ, which also corresponds to the constant one function on Y_μ, is a component in C''.

For Riesz spaces E and F, $L^r(E, F)$ consists of all order bounded linear operators which are the difference of two positive operators (operators which map E_+ to F_+). $T \in L^r(E, F)$ is order continuous if $Tz_\alpha \to 0$ for $z_\alpha \to 0$ (order convergence). We designate the set of order continuous operators by $L^c(E, F)$. If F is Dedekind complete, e.g., C', C'', or L^∞_μ, then $L^r(E, F)$ is a Dedekind complete Riesz space, and $L^c(E, F)$ is a band of $L^r(E, F)$.

Let $C_\mu = P_\mu(C)$. We will depart somewhat from the above convention and use the symbol $L^r(C_\mu, C)$ to denote all differences of positive operators in $L^c(L^\infty_\mu, C'')$ which map C_μ to C.

2. The band generated by $\mathbf{L^r(C_\mu, C)}$.

For $T \in L^c(L^\infty_\mu, C'')$, define

$$||T||_\mu = \langle \mu, |T|\mathbf{1}_\mu \rangle.$$

The following is a corollary to 2.1 of [**2**]. (See also [**1**, Theorem 15.10].)

THEOREM 2.1. *Given* $T \in L^c(L^\infty_\mu, C'')$ *satisfying* $0 \le T \le S \in L^r(C_\mu, C)$ *and* $\epsilon > 0$, *there is an operator* $T_1 \in L^r(C_\mu, C)$ with $0 \le T_1 \le S$ and $||T_1 - T||_\mu < \epsilon$.

PROOF. Apply (the proof of) [2,2.1] to the operator $T \circ P_\mu \in L^c(C'', C'')$ to find $\tilde{T}_1 \in L^r(C, C)$ with $0 \le \tilde{T}_1 \le S \circ P_\mu$. Let T_1 be the restriction of $\tilde{T}_1$ to L^∞_μ. Since $P_\mu f = 0$ for every $f \in C''$ disjoint from L^∞_μ, it follows that $\tilde{T}_1(f_\mu) = \tilde{T}_1 f$ for all $f \in C''$. Thus T_1 maps C_μ to C.

COROLLARY 2.2. *Given* $0 \le T \le S \in L^r(C_\mu, C)$ *and* $\epsilon > 0$, *there is a* $T_2 \in L^c(L^\infty_\mu, C'')$ *which maps* $C_{\mu+}$ *to usc satisfying* $0 \le T_2 \le S, ||T_2 - T||_\mu < \epsilon$ *and* $||T_2|| \le 2||T||$.

PROOF. Find T_1 as in 2.1. Let

$$E = \{x \in X; ||T_1^t x|| \le 2||T||\}.$$

E is closed in X, and thus determines an element $e \in \mathcal{E} \cap$ usc [**4**]. Define $T_2 = eT_1$. T_2 maps $C_{\mu+}$ to usc, and, since $||T_2|| = \sup_X \langle |T_2^t x|, 1_\mu\rangle = \sup_X ||T_2^t x||$, we have $||T_2|| \le 2||T||$. Also,

$$\begin{aligned}\langle \mu, |T_2 - T|\mathbf{1}_\mu\rangle &= \langle \mu, |T_1 - T|\mathbf{1}_\mu\rangle \\ &= \langle \mu, |eT_1 - (eT + (1-e)T)\mathbf{1}_\mu\rangle \\ &= \langle \mu, e|T_1 - T|\mathbf{1}_\mu\rangle + \langle \mu, (\mathbf{1} - e)T\mathbf{1}_\mu\rangle.\end{aligned}$$

Since

$$(1-e)T\mathbf{1}_\mu \le (\mathbf{1} - e)||T|| \le (\mathbf{1} - e)T_1 1_\mu - (\mathbf{1} - e)T\mathbf{1}_\mu,$$

by definition of e, we conclude

$$\begin{aligned}\langle \mu, |T_2 - T|1_\mu\rangle &= \langle \mu, e|T_1 - T|1_\mu\rangle + \langle \mu, (1-e)T1_\mu\rangle \\ &\le \langle \mu, e|T_1 - T|1_\mu\rangle + \langle \mu, (1-e)(T_1 1_\mu - T1_\mu)\rangle \\ &\le \langle \mu, e|T_1 - T|1_\mu\rangle + \langle \mu, (1-e)|T_1 - T|1_\mu\rangle \\ &= \langle \mu, |T_1 - T|1_\mu\rangle < \epsilon.\end{aligned}$$

PROPOSITION 2.3. *Given* $T \in L^c(L^\infty_\mu, C'')$ *with* $0 \le T \le S \in L^r(C_\mu, C)$*, then* $T_\mu = \lim T^n_\mu$ *for a sequence* $\{T^n\}$ *of operators in* $L^c(L^\infty_\mu, C'')$ *which map* $C_{\mu+}$ *to usc. We may choose* $\{T^n\}$ *satisfying* $||T^n|| \le 2||T||$.

PROOF. Apply 2.2 to find a sequence $\{T^n\}$, each of which maps $C_{\mu+}$ to usc, $0 \le T^n \le S$, and $||T^n|| \le 2||T||$, such that

$$\langle \mu, |T^n - T| 1_\mu \rangle \le (1/2)^n.$$

The remainder of the proof is identical to the proof of 2.4 in [**2**].

THEOREM 2.4. *If* $T \in L^c(L^\infty_\mu, C'')$ *is in the band generated by* $L^r(C_\mu, C)$*, there is an operator* $\hat{T} \in L^c(L^\infty_\mu, C'')$ *which satisfies* $\hat{T}_\mu = T_\mu, ||\hat{T}|| \le 2||T||$*, and which takes values in Bo.*

PROOF. First assume that $0 \le T \le S \in L^r(C_\mu, C)$. Choose a sequence $\{T^n\}$ as in 2.3, and let

$$\hat{T} = \lim\inf_n T^n = \vee_j \wedge_{k \ge j} T^k \le S.$$

Note that, for each j, $\wedge_{k\ge j} T^k$ maps C_μ to usc. Since $\wedge_{k \ge j} T^k$ is increasing in j, $\hat{T} f$ is the supremum of a sequence of elements in usc for $f \in C_\mu$. Thus $\hat{T} f \in Bo$. Since $\hat{T}$ is order continuous and every $g \in L^\infty_\mu$ is the limit of a sequence in C_μ [**4**, 41.2], we conclude that $\hat{T} g \in Bo$, for Bo is σ-closed. Also,

$$|| \wedge_{k \ge j} T^k || \le 2||T||$$

implies that

$$||\hat{T} 1_\mu|| = ||\hat{T}|| \le 2||T||.$$

The above demonstrates the theorem for T in the ideal generated by $L^r(C_\mu, C)$. Suppose that T is in the band generated by $L^r(C_\mu, C)$, and $T \ge 0$. There is an increasing net $\{T_\alpha\}$ such that $T_\alpha \uparrow T$, where each T_α is in the ideal generated by $L^r(C_\mu, C)$. We may find a subsequence of $\{T_\alpha \mathbf{1}_\mu\}$, which we denote by $\{T^n \mathbf{1}_\mu\}$, such that $(T^n \mathbf{1}_\mu)_\mu \uparrow (T\mathbf{1}_\mu)_\mu$.

It follows that $T_\mu^n \uparrow T_\mu$. For simplicity, we assume $T^n \uparrow T$, as we are concerned only with T_μ. Since

$$\langle \mu, (T - T^n)\mathbf{1}_\mu \rangle \downarrow 0,$$

we may assume, by taking a subsequence if necessary, that

$$\langle \mu, (T - T^n)\mathbf{1}_\mu \rangle \leq (1/2)^n.$$

The T^n are in the ideal generated by $L^r(C_\mu, C)$. Thus, there is, for each T^n, an increasing sequence $\{T^{m,n}\}$ of operators which map $C_{\mu+}$ to usc such that $||T^{m,n}|| \leq 2||T^n|| \leq 2||T||$ and $T_\mu^{m,n} \uparrow T_\mu^n$. By again taking subsequences if necessary, we may assume

$$\langle \mu, |T^n - T^{m,n}|\mathbf{1}_\mu \rangle \leq (1/2)^m.$$

We define $S^k = \wedge_{i,j \geq k} T^{i,j}$. S^k maps $C_{\mu+}$ to usc, $||S^k|| \leq 2||T||$, and $S_\mu^k \leq T_\mu$.

The above implies:

$$\begin{aligned}
\langle \mu, |T - S^k|\mathbf{1}_\mu \rangle &= \langle \mu, |T_\mu - S_\mu^k|\mathbf{1}_\mu \rangle \\
&\leq \langle \mu, \vee_{i,j \geq k} |T_\mu - T_\mu^{i,j}|\mathbf{1}_\mu \rangle \\
&= \langle \mu, \vee_{j \geq k} |T_\mu - T_\mu^{k,j}|\mathbf{1}_\mu \rangle \\
&= \langle \mu, \vee_q \vee_{j=k}^q |T_\mu - T_\mu^{k,j}|\mathbf{1}_\mu \rangle \\
&= \vee_q \langle \mu, \vee_{j=k}^q |T_\mu - T_\mu^{k,j}|\mathbf{1}_\mu \rangle \\
&\leq \vee_q \sum_{j=k}^q \langle \mu, |T_\mu - T_\mu^{k,j}|\mathbf{1}\mu \rangle \\
&= \vee_q \sum_{j=k}^q \langle \mu, |T_\mu - T_\mu^j|\mathbf{1}_\mu + |T_\mu^j - T_\mu^{k,j}|\mathbf{1}_\mu \rangle \\
&\leq 2(1/2)^{k-1} = (1/2)^{k-2}.
\end{aligned}$$

Thus

$$\wedge_k \langle \mu, |T - S^k|\mathbf{1}_\mu \rangle = 0$$

and $S_\mu^k \uparrow T_\mu$. Since the S^k are increasing and $||S^k|| \leq 2||T||, \vee_k S^k = \hat{T}$ exists and the above implies $\hat{T}_\mu = T_\mu$. $\hat{T}f$, as in the first part of the proof, is an element of Bo for each $f \in L_\mu^\infty$.

3. Borel liftings for $\mathbf{L^c(L^\infty_\mu, L^\infty_\mu)}$.

THEOREM 3.1. *If $T \in L^c(L^\infty_\mu, C'')$ takes values in Bo, then T is in the band generated by the finite dimensional operators in $L^c(L^\infty_\mu, C'')$.*

PROOF. By IV.9.7 in [**6**], it suffices to show that, for each $\nu \in C'$, there is a kernel K on $Y_\mu \times Y$ such that, for each $f \in L^\infty_\mu$,

$$\langle y, Tf\rangle = \int K(z,y) f(z) d\mu(z)$$

holds a.e. (ν) on Y.

As T is order continuous, T is determined by $T^t : C' \to L^1_\mu$. For each $x \in X, T^t x \in L^1_\mu$ may be Radon-Nikodym represented as $g_x \mu$.

The natural inclusion $I : C \to C''$ is a Riesz homomorphism determined by the mapping $I^t : Y \to X, If(y) = f(I^t y)$.

Determine $K(z,y) = g_x(z)$ where $x = I^t y$. Thus

$$\langle y, Tf\rangle = \int K(y,z) f(z) d\mu(z)$$

holds for each $y \in Y$ which may be identified with an element of X.

Let $\nu \in C'$ be a diffuse measure and $f \in L^\infty_\mu$. $Tf \in Bo$ implies that we may find $\{g_\alpha\} \subset$ lsc and $\{h_\alpha\} \subset$ usc with $g_\alpha \downarrow Tf$ and $h_\alpha \uparrow Tf$. For $\epsilon > 0$ let

$$E_\epsilon = \{y \in Y; |Tf(y) - Tf(i^t y)| \geq \epsilon\}.$$

Since each $g_\alpha \in$ usc can be represented as the infimum of a decreasing net in C, we conclude $g_\alpha(y) \leq g_\alpha(I^t y)$. Similarly, $h_\alpha(y) \geq h_\alpha(I^t y)$. Thus

$$g_\alpha(y) \geq Tf(i^t y) \geq h_\alpha(y)$$

and

$$0 \leq \nu(E_\epsilon) \leq (1/\epsilon)\langle \nu, g_\alpha - h_\alpha\rangle \downarrow 0.$$

We conclude that the representation holds ν almost everywhere.

We combine 3.1 and 2.4 to conclude.

THEOREM 3.2. *$T \in L^c(L^\infty_\mu, L^\infty_\mu)$ has an order continuous lifting $\hat{T} \in L^c(L^\infty_\mu, C'')$ taking values in Bo if and only if T is in the band generated by the finite dimensional operators in $L^c(L^\infty_\mu, L^\infty_\mu)$.*

PROOF. We need only show that the band generated by finite dimensional operators in $L^c(L^\infty_\mu, C'')$ is contained in $L^r(C_\mu, C)$. If T is defined by

$$Tf = \sum_1^n \langle \mu_i, f\rangle g_i,$$

choose $h_i \in C$ satisfying $h_i \geq |g_i|$ so that

$$|T| \leq \sum \langle |\mu_i|, \cdot \rangle h_i \in L^r(C_\mu, C).$$

In particular, this band includes the weakly compact operators [**6**, IV.9.9].

COROLLARY 3.3. *Every $T \in L^c(L^\infty_\mu, L^\infty_\mu)$ which is weakly compact has an order continuous Borel lifting.*

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