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ENTROPY OF CERTAIN NONCOMMUTATIVE SHIFTS

HONG-SHENG YIN

ABSTRACT. The entropy of the noncommutative shift of the hyperfinite II_1-factor associated with a sequence of Jones' projections is computed.

0. Introduction. By the work of V.F.R. Jones [**3**] and M.Pimsner-S. Popa [**4**], for any real number λ in the set $(0, 1/4] \cup \{(\sec^2 \pi/n)/4 : n \geq 3\}$, we can find a doubly infinite sequence of projections $\{e_i : i \in \mathbf{Z}\}$ in the hyperfinite II_1-factor R such that the e_i's generate R as a von Neumann algebra and satisfy the conditions: (a) $e_i e_j e_i = \lambda e_i$ if $|i-j| = 1$; (b) $e_i e_j = e_j e_i$ if $|i-j| \geq 2$; and (c) $\mathrm{tr}(we_n) = \lambda \mathrm{tr}(w)$, if w is a word on 1 and $\{e_i : i < n\}$. Here tr is the unique normal normalized trace on R. Then $e_i \to e_{i+1}$ defines an ergodic automorphism Θ_λ of R [**4**, §5]. The Connes-Stormer entropy [**2**] of $\Theta_\lambda, H(\Theta_\lambda)$, was computed by Pimsner-Popa in [**4**, §5], among other things. The results are (i) $H(\Theta_\lambda) = (-\ln \lambda)/2$ if $\lambda = (\sec^2 \pi/n)/4, n \geq 3$, and (ii) $H(\Theta_\lambda) = \eta(t) + \eta(1-t)$ if $\lambda < 1/4$, where η is the function $\eta(x) = -x \ln x, x \in \mathbf{R}$, and where $t(1-t) = \lambda$. The case $\lambda = 1/4$ is left open [**4**, p. 92]. In this note we prove $H(\Theta_{1/4}) = \ln 2$, thus completing the circle. Pimsner-Popa showed that, when $\lambda < 1/4$, the Θ_λ are just the noncommutative Bernoulli shifts of Connes-Størmer and Krieger [**2**] with weights $\{t, 1-t\}$, and thus obtained the entropy by the computation in [**2**]. Of course, their approach provides more results than merely the entropy. However, it seems worthwhile to give a direct computation of the entropy. We include such a computation hereon.

Our computation is based on the explicit knowledge of the structure of the finite dimensional algebras $A_n = \{e_1, e_2, \ldots, e_n\}''$ provided by Jones [**3**].

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1. Preliminaries. Let $\{e_i : i \in \mathbf{Z}\}$ be a sequence of projections satisfying the conditions (a), (b), (c) of §0 for some λ in the Jones

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index set, and let $R = \{e_i : i \in \mathbf{Z}\}''$. Let $A_{m,n}$ $(m \leq n)$ be the von Neumann subalgebra of R generated by $e_m, e_{m+1}, \ldots, e_n$, and let A_n be $A_{1,n}$. By [**3**], all $A_{m,n}$ are finite dimensional. Using the Kolmogorov-Sinai type theorem of Connes-Størmer [**2**], we can show, as in the first paragraph of the proof of [**4**, 5.7], that $H(\Theta_\lambda) = \overline{\lim}_{n\to\infty} H(A_{2n})/2n$ where $H(A_{2n})$ is the entropy of A_{2n} defined in [**2**]. Here the point is that $\Theta_\lambda^{s(2n+1)}(A_{2n})$, $s = 0, 1, 2, \ldots,$ mutually commute and are trace-independent. Therefore we only need to study the asymptotical behaviour of $H(A_{2n})/2n$. It turns out that $\lim_{n\to\infty} H(A_{2n})/2n$ will always exist.

In the remainder of this note, we always assume $\lambda \leq 1/4$. The structure of the algebras A_n was determined in [**3**, §5]. Recall that $A_n = \oplus_{k=0}^{[(n+1)/2]} Q_k^n$, where Q_k^n is the matrix algebra of order $\binom{n+1}{k} - \binom{n+1}{k-1}$. Here $\binom{a}{b}$ is the binomial coefficient with the convention $\binom{a}{-1} = 0$. As in [**3**, §5], we let $\left\{ {n \atop k} \right\} = \binom{n}{k} - \binom{n}{k-1}$. The trace of a minimal projection in Q_k^n is $\lambda^k P_{n+2-2k}(\lambda)$, $k = 0, 1, 2, \ldots, [(n+1)/2]$, where $P_j(x)$ is defined by $P_0(x) = 0, P_1(x) = 1$ and $P_{j+1}(x) = P_j(x) - xP_{j-1}(x)$. Let p_k^n denote a minimal projection in Q_k^n. We call the numbers $\{\mathrm{tr}(p_k^n)\}, k = 0, 1, 2, \ldots, [(n+1)/2]$ (in this order), the weights of the trace on A_n.

2. $\mathbf{H(\Theta_{1/4}) = \ln 2}$.

LEMMA 1. *Suppose* $\lambda = 1/4$. *Then the weights of the trace on* A_{2n-1} *and* A_{2n} *are respectively* $\{(2n - 2k + 1)/4^n\}, k = 0, 1, 2, \ldots, n$ *and* $\{(n - k + 1)/4^n\}, k = 0, 1, 2, \ldots, n$.

PROOF. For $n = 1$, we can verify this directly. Then use induction. By [**3**, §5].

$$\mathrm{tr}(p_k^{2n-1}) = \mathrm{tr}(p_k^{2n-2}) - \lambda \mathrm{tr}(p_k^{2n-3}),$$

if $0 \leq k \leq n-1$, and $\mathrm{tr}(p_n^{2n-1}) = \lambda \mathrm{tr}(p_{n-1}^{2n-3})$. This proves the statement for A_{2n-1}. For A_{2n}, the relations are

$$\mathrm{tr}(p_k^{2n}) = \mathrm{tr}(p_k^{2n-1}) - \lambda \mathrm{tr}(p_k^{2n-2}),$$

if $0 \leq k \leq n-1$, and $\mathrm{tr}(p_n^{2n}) = \mathrm{tr}(p_n^{2n-1})$. □

PROPOSITION 2. $H(\Theta_{1/4}) = \ln 2$.

PROOF. We show $\lim_{n\to\infty} H(A_{2n})/2n = \ln 2$. By Lemma 1 above and the property (D) of the function $H(\cdot)$ in [**2**], we have

$$\begin{aligned}\frac{H(A_{2n})}{2n} &= \frac{1}{2n}\sum_{k=0}^{n}\left\{\begin{matrix}2n+1\\ k\end{matrix}\right\}\eta\left(\frac{n-k+1}{4^n}\right)\\ &= -\frac{1}{2n}\sum_{k=0}^{n}\left\{\begin{matrix}2n+1\\ k\end{matrix}\right\}\frac{(n-k+1)}{4^n}\ln(n-k+1)\\ &\quad+\ln 2\sum_{k=0}^{n}\left\{\begin{matrix}2n+1\\ k\end{matrix}\right\}\frac{(n-k+1)}{4^n}.\end{aligned}\tag{1}$$

However,

$$\begin{aligned}&\sum_{k=0}^{n}\left\{\begin{matrix}2n+1\\ k\end{matrix}\right\}\frac{(n-k+1)}{4^n}\\ &\quad= \frac{1}{4^n}\sum_{k=0}^{n}\left(\binom{2n+1}{k}-\binom{2n+1}{k-1}\right)(n-k+1)\\ &\quad= \frac{1}{4^n}\left(\binom{2n+1}{0}+\binom{2n+1}{1}+\cdots+\binom{2n+1}{n}\right) = \frac{1}{4^n}\cdot 4^n = 1.\end{aligned}$$

Thus the second term in (1) is $\ln 2$. For the first term, notice that

$$\begin{aligned}0 &\le \frac{1}{2n}\sum_{k=0}^{n}\left\{\begin{matrix}2n+1\\ k\end{matrix}\right\}\frac{n-k+1}{4^n}\ln(n-k+1)\\ &\le \frac{\ln(n+1)}{2n}\sum_{k=0}^{n}\left\{\begin{matrix}2n+1\\ k\end{matrix}\right\}\frac{(n-k+1)}{4^n} = \frac{\ln(n+1)}{2n}\to 0. \quad\square\end{aligned}$$

3. $\mathbf{H(\Theta_\lambda)}, \boldsymbol{\lambda} < \mathbf{1/4}$. In the following, $\lambda < 1/4$. As in [**3**, §4], let $t = (1+\sqrt{1-4\lambda})/2$. Note that $t(1-t) = \lambda$. We show $\lim_{n\to\infty} H(A_{2n})/(2n+1) = \eta(t)+\eta(1-t)$. By [**3**, §5] (see §1) and the property (D) of [**2**],

$$H(A_{2n}) = \sum_{k=0}^{n}\left\{\begin{matrix}2n+1\\ k\end{matrix}\right\}\eta(\lambda^k P_{2n+2-2k}(\lambda)).$$

By [**3**, 4.2.4], we obtain

$$\begin{aligned}\lambda^k P_{2n+2-2k}(\lambda) &= \lambda^k (t-(1-t))^{-1}(t^{2n+2-2k}-(1-t)^{2n+2-2k}) \\ &= (1-r)^{-1}(1-t)^k t^{2n+1-k}(1-r^{2n+2-2k}),\end{aligned} \tag{2}$$

where $r=(1-t)/t$. To continue the computation we need the following results, which might be well-known to probabilists. We include a brief proof for the completeness.

LEMMA 3. (i) $\lim_{n\to\infty}\sum_{k=0}^{n}\binom{2n+1}{k}(1-t)^k t^{2n+1-k}=1$;

(ii) $\lim_{n\to\infty}(2n+1)^{-1}\sum_{k=0}^{n}\binom{2n+1}{k}(1-t)^k t^{2n+1-k}(2n+1-k)=t$;

(iii) $\lim_{n\to\infty}(2n+1)^{-1}\sum_{k=0}^{n}\binom{2n+1}{k}(1-t)^k t^{2n+1-k}k=1-t$.

PROOF. (i). The proof is contained in the proof of [**3**, 5.3.6]. In fact, the sum is the probability of $\geq n$ successes in $2n+1$ Bernoulli trials, with the probability of success in one trial being $t>1/2$. Since $(2n+1)t>n$, the sum must tend to 1.

(ii). Differentiating (with respect to Z) the identity

$$((1-t)+tZ)^{2n+1}=\sum_{k=0}^{2n+1}\binom{2n+1}{k}(1-t)^k(tZ)^{2n+1-k}$$

and then setting $Z=1$, we obtain

$$t=\left(\sum_{k=0}^{n}+\sum_{k=n+1}^{2n+1}\right)\frac{1}{2n+1}\binom{2n+1}{k}(1-t)^k t^{2n+1-k}(2n+1-k).$$

For the second sum we have

$$\begin{aligned}0 &\leq \sum_{k=n+1}^{2n+1}\frac{1}{2n+1}\binom{2n+1}{k}(1-t)^k t^{2n+1-k}(2n+1-k) \\ &\leq \sum_{k=n+1}^{2n+1}\binom{2n+1}{k}(1-t)^k t^{2n+1-k} \\ &= 1-\sum_{k=0}^{n}\binom{2n+1}{k}(1-t)^k t^{2n+1-k}\to 0.\end{aligned}$$

Therefore, the first sum tends to t.

(iii). The sum of the left sides of (ii) and (iii) tends to 1 by (i). □

PROPOSITION 4. $H(\Theta_\lambda) = \eta(t) + \eta(1-t), \lambda < 1/4$.

PROOF. By (2) we have

$$\begin{aligned} &H(A_{2n})/(2n+1) \\ &\quad = -(1-r)^{-1} \sum_{k=0}^{n} \left\{ \begin{matrix} 2n+1 \\ k \end{matrix} \right\} (1-t)^k t^{2n+1-k} (1 - r^{2n+2-2k}) \\ &\qquad \cdot (2n+1)^{-1} (-\ln(1-r) + k \ln(1-t) \\ &\qquad + (2n+1-k) \ln t + \ln(1 - r^{2n+2-2k})). \end{aligned} \tag{3}$$

Notice that

$$\begin{aligned} &\sum_{k=0}^{n} \left\{ \begin{matrix} 2n+1 \\ k \end{matrix} \right\} (1-t)^k t^{2n+1-k} r^{2n+2-2k} \\ &\quad = r \sum_{k=0}^{n} \binom{2n+1}{k} (1-t)^{2n+1-k} t^k \\ &\qquad - \sum_{k=0}^{n-1} \binom{2n+1}{k} (1-t)^{2n+1-k} t^k \\ &\quad \to 0 \end{aligned}$$

due to Lemma 3(i). Since the second factor of (3) is bounded, we conclude that the $r^{2n+2-2k}$ in the first factor of (3) has no contribution to the limit. Furthermore, since

$$(2n+1)^{-1}(-\ln(1-r) + \ln(1 - r^{2n+2-2k})) \to 0 \quad \text{as } n \to \infty$$

and the first factor converges, this portion has no contribution to the limit, either. Therefore

$$\begin{aligned} &\lim_{n\to\infty} H(A_{2n})/(2n+1) \\ &= \lim_{n\to\infty} -(1-r)^{-1} \sum_{k=0}^{n} \left\{ \begin{matrix} 2n+1 \\ k \end{matrix} \right\} (1-t)^k t^{2n+1-k} \\ &\qquad \cdot (2n+1)^{-1} (k \ln(1-t) + (2n+1-k) \ln t). \end{aligned}$$

Using Lemma 3(ii), (iii) and the fact that $t(1-t)=\lambda<1/4$ implies $\binom{2n+1}{n}\lambda^n \to 0$, we obtain

$$\begin{aligned}(2n+1)^{-1}\sum_{k=0}^{n}\left\{\begin{matrix}2n+1\\k\end{matrix}\right\}(1-t)^k t^{2n+1-k}k \\ &= (2n+1)^{-1}\sum_{k=0}^{n}\binom{2n+1}{k}(1-t)^k t^{2n+1-k}k \\ &\quad - r(2n+1)^{-1}\sum_{k=0}^{n-1}\binom{2n+1}{k}(1-t)^k t^{2n+1-k}(k+1) \\ &\to (1-t)-r(1-t), \quad n\to\infty,\end{aligned}$$

and similarly, $(2n+1)^{-1}\sum_{k=0}^{n}\left\{\begin{smallmatrix}2n+1\\k\end{smallmatrix}\right\}(1-t)^k t^{2n+1-k}(2n+1-k)\to t-rt$. Hence $\lim_{n\to\infty} H(A_{2n})/(2n+1) = -(1-t)\ln(1-t)-t\ln t$. □

REMARK. In the above, the shifts Θ_λ are two-sided, that is, they are on $\{e_i : i\in \mathbf{Z}\}''$. If we restrict Θ_λ to $\{e_i : i \geq 0\}''$, which is also the hyperfinite II_1-factor by [**3**], we get one-sided shifts. The Connes-Størmer's definition [**2**] of entropy of automorphisms obviously works for endomorphisms of the hyperfinite II_1-factor, and we still have the noncommutative Kolmogorov-Sinai theorem and the property $H(\Theta^k)=kH(\Theta), k\geq 0$, of [**2**]. From these facts, it is easy to see that our computation also gives the entropy of one-sided shifts. With similar ideas, the entropy of the shifts considered in [**1**] has also been computed. The results will appear in a forthcoming paper.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WATERLOO, WATERLOO, ONTARIO, CANADA N2L 3G1