ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 20, Number 4, Fall 1990

# SOME REMARKS ON THE LUSTERNIK-SCHNIRELMAN METHOD FOR NON-DIFFERENTIABLE FUNCTIONALS INVARIANT WITH RESPECT TO A FINITE GROUP ACTION

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**1. Introduction.** A variational problem with symmetries may have multiple solutions. The problem of finding the best possible estimate on the number of orbits of critical points of an invariant functional was studied by many authors, see for example, [**10, 1, 2, 7–9, 12, 13, 20, 11, 22**].

In our previous paper [**14**] we presented a general and explicit formula for the number of critical points of a functional of class $C^1$, invariant with respect to a finite group action (cf. [**14**, Theorem (1.3)]). The purpose of this paper is to extend this variational minimax method to the class of locally Lipschitzian functionals.

Many nonlinear partial differential equations with discontinuous nonlinearities can be reduced to nondifferentiable variational problems. The Lusternik-Schnirelman method, which is a very efficient tool in finding multiple solutions to differentiable variational problems, was recently applied to some nondifferentiable functionals: see [**5, 16, 21, 23**]. The generalized gradient of F. H. Clarke (cf. [**6**]) is used to extend the concepts of critical point and the Palais-Smale condition.

We would like to emphasize that our lower-estimation on the number of critical points of a locally Lipschitzian functional, invariant with respect to a finite group action (Theorem 3), even in the case of $C^1$-functionals, is an improvement (compare [**14**] with [**10**] and [**3**]).

**2. Locally Lipschitzian G-invariant functionals on Banach manifolds.** Let $M$ be a Banach manifold of class $C^2$ on which a finite group $G$ acts by diffeomorphisms, i.e., $M$ is a *G-manifold*. Suppose that $M$ is endowed with an invariant Finsler structure $\|\cdot\| : TM \to \mathbf{R}$. Such a manifold $M$ will be called a *Finsler G-manifold*. It is known

Research of first author supported by grant from NSERC Canada.

(cf. [**14**]) that any paracompact $G$-manifold $M$ admits a structure of a Finsler $G$-manifold, and, therefore, $M$ is a $G$-metrizable space.

Let $M$ be a Finsler $G$-manifold modelled on a Banach space $\mathbf{E}$ and $f : M \to \mathbf{R}$ a $G$-invariant locally Lipschitzian function. We define, following the definition of F. H. Clarke (cf. [**6**]), the *generalized gradient* of $f$ at $x$, denoted $\partial f(x) \subset T_x M$, as follows: First, we consider the case where $f : U \to \mathbf{R}$, $U \subset \mathbf{E}$ is an open set. The *generalized directional derivative* $f^0(x,v)$ of the function $f$ at $x \in U$ is defined as

$$f^0(x,v) = \overline{\lim_{\substack{h\to 0\\ \lambda\downarrow 0}}} \frac{1}{\lambda}[f(x+h+\lambda v) - f(x+h)], \quad x \in U,\ v \in \mathbf{E}.$$

We have supposed that $f$ is locally Lipschitzian on $U$; this means that, for every $x \in U$, there exists a neighborhood $V_x \subset U$ of $x$ and a constant $k_x > 0$ depending on $V_x$ such that

$$|f(x_1) - f(x_2)| \le k_x \|x_1 - x_2\| \quad \forall x_1, x_2 \in V_x.$$

The function $v \to f^0(x,v)$ is a subadditive, positively homogeneous, and, thus, convex, continuous function. Moreover, $|f^0(x,v)| \le k_x\|v\|$, $|f^0(x,u) - f^0(x,v)| \le k_x \|u-v\|$ and $f^0(x,-v) = (-f)^0(x,v)$ (cf. [**5,6**]). Let $\varphi : U' \to U$ be a $C^1$-diffeomorphism, where $U' \subset \mathbf{E}$. Then

$$(f \circ \varphi)^0(x,v) = f^0(\varphi(x), D\varphi(x)v). \tag{1}$$

Indeed, we have the inequality

$$\begin{aligned}
&|(f\circ\varphi)^0(x,v) - f^0(\varphi(x), D\varphi(x)v)| \\
&= \Bigg| \overline{\lim_{\substack{h\to 0\\ \lambda\downarrow 0}}} \frac{1}{\lambda}[f(\varphi(x+h+\lambda v)) - f(\varphi(x+h))] \\
&\qquad\qquad - \lim_{\substack{h\to 0\\ \lambda\downarrow 0}} \frac{1}{\lambda}[f(\varphi(x+h) + \lambda D\varphi(x)v) - f(\varphi(x+h))]\Bigg| \\
&\le k_x \cdot \lim_{\substack{h\to 0\\ \lambda\downarrow 0}} \frac{1}{\lambda}|\varphi(x+h+\lambda v) - \varphi(x+h) - \lambda D\varphi(x)v)| = 0.
\end{aligned}$$

The *generalized gradient* of $f$ at $x$, denoted $\partial f(x)$, is defined as

$$\partial f(x) = \{w \in \mathbf{E}^* : \langle w, v\rangle \le f^0(x,v)\ \forall v \in \mathbf{E}\}.$$

It is well known that the generalized gradient $\partial f(x)$ is a nonempty convex and $w^*$-compact subset of $\mathbf{E}^*$ such that, for each $w \in \partial f(x)$, $\|w\| \leq k_x$. If $f, g : U \to \mathbf{R}$ are two locally Lipschitzian functionals, then $\partial(f+g)(x) \subset \partial f(x) + \partial g(x)$ and $\partial(\lambda f)(x) = \lambda \partial f(x)$ for all $\lambda \in \mathbf{R}$ (cf. [**6**] and [**5**]). Moreover, we have

$$[D\varphi(x))^{-1}]^*(\partial f(\varphi(x))) = \partial(f \circ \varphi)(x). \tag{2}$$

Indeed, by (1),

$$\begin{aligned}\partial(f\circ\varphi) &= \{w \in \mathbf{E}^* : \langle w, v\rangle \leq (f\circ\varphi)^0(x,v)\ \forall v \in \mathbf{E}\}\\ &= \{w \in \mathbf{E}^* : \langle w, v\rangle \leq f^0(\varphi(x), D\varphi(x)v)\ \forall v \in \mathbf{E}\}\\ &= \{w \in \mathbf{E}^* : \langle w, [D\varphi(x)]^{-1}v^1\rangle \leq f^0(\varphi(x), v^1)\ \forall v^1 \in \mathbf{E}\}\\ &= \{[(D\varphi(x))^{-1}]^* w^1 : \langle w^1, v^1\rangle \leq f^0(\varphi(x), v^1)\ \forall v^1 \in \mathbf{E}\}\\ &= [(D\varphi(x))^{-1}]^*(\partial f(\varphi(x))).\end{aligned}$$

Let us now return to the case where $f : M \to \mathbf{R}$ is a locally Lipschitzian function on a Finsler $G$-manifold modelled on $\mathbf{E}$. Suppose that $x \in M$ and that $(U, \varphi)$ is a chart at $x$. We define

$$\partial f(x) = (T^*\varphi)_{\varphi(x)}(\partial(f\circ\varphi)(\varphi(x))),$$

where $T^*\varphi : T^*U \to T^*M|_U$ denotes the co-tangential map for $\varphi$. By (2), this definition doesn't depend on the choice of local coordinates $(U, \varphi)$. Let us recall that, for each $g \in G$, there is a diffeomorphism $g : M \to M$ defined by $g(x) = g \cdot x$, $x \in M$. The $G$-action on $T^*M$ is defined as follows: if $(x, w) \in T^*M$, i.e., $w \in T_x^*M$, then $g \cdot (x, w) = (gx, w')$, where $w' = (T_{g(x)}g^{-1})^* w$. Suppose that $f : M \to \mathbf{R}$ is a $G$-invariant functional, i.e., $f(gx) = f(x)$ for every $g \in G$, $x \in M$. Then, by (2),

$$g \cdot \partial f(x) = (T^* g^{-1})(\partial f(x)) = (\partial(f \circ g)^{-1})(gx) = \partial f(gx),$$

i.e.,

$$g \cdot \partial f(x) = \partial f(gx). \tag{3}$$

This means that the generalized gradient $\partial f : M \to T^*M$ of the invariant functional $f$ is also invariant.

The function $\lambda(x) = \min\{\|w\| : w \in \partial f(x)\}$ is lower semi-continuous (cf. [**5**]) and if $f$ is invariant, then $\lambda$ is also invariant. A point $x \in M$ is called a *critical point* of $f$ if $\lambda(x) = 0$, and, since there is always a $w_0 \in \partial f(x_0)$ such that $\|w_0\| = \lambda(x_0)$, this is equivalent to $0 \in \partial f(x)$. The set of all critical points of $f$ is denoted by $K$. We put

$$K_c = K \cup f^{-1}(c)$$
$$f_c = \{x \in M : f(x) \leq c\},$$

where $c \in \mathbf{R}$.

DEFINITION . A locally Lipschitzian function $f : M \to \mathbf{R}$ satisfies the *Palais-Smale* ((P.S.) for abbreviation) *condition* if any sequence $\{x_n\} \subset M$ such that

(i) $\{f(x_n)\}$ is bounded;

(ii) $\lambda(x_n) \to 0$ as $n \to \infty$,

possesses a convergent subsequence.

The following equivariant version of the *Deformation Lemma* can be easily obtained by using standard constructions and the "averaging" techniques (cf. [**5, 11, 1**]).

LEMMA 1. *Let $M$ be a complete Finsler $G$-manifold modelled on a reflexive Banach space $\mathbf{E}$ and $f : M \to \mathbf{R}$ a $G$-invariant, locally Lipschitzian, functional satisfying the (P.S.) condition. Suppose that $c \in \mathbf{R}$, $\bar{\epsilon} > 0$ and $U$ is an equivariant neighborhood of $K_c$. Then there exist $\epsilon \in (0, \bar{\epsilon})$ and $\eta \in C([0,1] \times M, M)$ such that*

(1°) *$\eta(0, x) = x$ for every $x \in M$;*

(2°) *$\eta(t, x) = x$ for all $t \in [0,1]$, whenever $f(x) \in [c - \bar{\epsilon}, c + \bar{\epsilon}]$;*

(3°) *$\eta(t, \cdot)$ is a $G$-homeomorphism of $M$ onto $M$ for each $t \in [0,1]$ such that $\eta(t,\cdot)(M^H) = M^H$ for every subgroup $H$ of $G$, where $M^H = \{x \in M : x = hx \ \forall h \in H\}$.*

(4°) *$f(\eta(t,x)) \leq f(x)$ for all $x \in M$ and $t \in [0,1]$.*

(5°) *$\eta(1, f_{c+\epsilon} \backslash U) \subset f_{c-\epsilon}$.*

Suppose now that $G$ is a compact Lie group and $M$ is a $G$-manifold of class $C^2$. Let $x$ be a *symmetric* point of $M$, i.e., $x \in M^G$. There is a natural linear representation of $G$ on $T_xM$ given by $g \mapsto Dg(x)$. The action $G$ is called *linearizable* at $x$ if there is a diffeomorphism $\varphi$ of an open set $U$ of $M$, containing $x$ and $G$-invariant, onto an open equivariant set $\varphi(U)$ in $T_xM$ and a mapping $x$ to the origin such that the map

$$\varphi \circ g \circ \varphi^{-1} : \varphi(U) \to T_xM$$

is the restriction to $\varphi(U)$ of the linear map $Dg(x)$; i.e., $\varphi$ *linearizes* the action of $G$ about $x$ (cf. [**18**]]). Let us remark that, if $(U, \psi)$ is a chart at $x$ such that $U$ is equivariant and $\psi(x) = 0$, where $\psi : U \to \mathbf{E}$ and $\mathbf{E} \cong T_xM$, by identifying $\mathbf{E}$ with $T_xM$, we can define

$$\varphi(y) = \int_G (Dg(y) \cdot \psi)(g^{-1}y)\, dg, \quad g \in U,$$

where $dg$ is a normalized Haar measure on $G$. The map $\varphi$ linearizes the action of $G$ about $x$. This implies that any action of a compact Lie group $G$ by diffeomorphisms on a Banach manifold $M$ is linearizable at symmetric points. Since $\mathbf{E}^G$ is a closed linear subspace of $\mathbf{E}$ and $\varphi$ maps $U \cap M^G$ onto $\varphi(U) \cap \mathbf{E}^G$, $M^G$ is a submanifold.

The following *Principle of Symmetric Criticality* is valid for $G$-invariant locally Lipschitzian functional (see [**18**]):

*Let $x \in M^G$ and $f : M \to \mathbf{R}$ be a locally Lipschitzian $G$-invariant functional. Then $x$ is a critical point of $f$ if and only if $x$ is a critical point of $f^G := f|_{M^G} : M^G \to \mathbf{R}$.*

Indeed, since the action of $G$ is linearizable at $x$, it is sufficient to show the Principle of Symmetric Criticality (P.S.C. for abbreviation) for a special case where $f : U \to \mathbf{R}$, $U \subset \mathbf{E}$, is a locally Lipschitzian $G$-invariant function, $x = 0$ and $\mathbf{E}$ is a linear representation of $G$. Let $A : \mathbf{E} \to \mathbf{E}$ be the *averaging operator* over $G$, defined by

$$Av = \bar{v}; \quad \bar{v} = \int_G g \cdot v\, dg, \ v \in \mathbf{E}.$$

$A$ is a continuous projection on $\mathbf{E}^G$. Since the function $f^0(0,\cdot)$ is a continuous convex function,

$$\begin{aligned} f^0(0, Av) &\le \int_G f^0(0, g\cdot v)\,dg = \int_G (f\circ g)^0(0,v)\,dg \\ &= \int_G f^0(0,v) = f^0(0,v). \end{aligned}$$

Let us remark that $(f^G)^0(0,v) \le f^0(0,v)$ for $v \in \mathbf{E}^G$ and $A^*\mathbf{E}^* = (\mathbf{E}^*)^G \cong (\mathbf{E}^G)^*$. Thus

$$\begin{aligned} \partial f^G(0) &= \{w \in (\mathbf{E}^*)^G : \langle w, v\rangle \le (f^G)^0(0,v) \quad \forall v \in \mathbf{E}^G\} \\ &\subset \{w \in (\mathbf{E}^*)^G : \langle w, v\rangle \le f^0(0,v) \quad \forall v \in \mathbf{E}^G\} \\ &= \{w \in A^*\mathbf{E}^* : \langle w, v\rangle = \langle w, Av\rangle \le f^0(0, Av) \\ &\qquad\qquad\qquad\qquad\qquad \le f^0(0,v) \quad \forall v \in \mathbf{E}\} \\ &\subset A^*\partial f(0). \end{aligned}$$

Therefore, if $0 \in \partial f^G(0)$, then $0 \in A^*\partial f(0)$, and, since $A^*(\partial f(0)) \subset \partial f(0)$, this implies that $0 \in \partial f(0)$ and the (P.S.C.) is satisfied.

**3. Multiplicity result for nondifferentiable functionals.** The following lemma was proved in [**14**].

LEMMA 2. *Let $G$ be a finite group acting freely on a metric space $A$. Assume that $S$ is a $G$-invariant subset of $A$, of dimension $n$, which is also an $n$-dimensional cohomological sphere over $\mathbf{Z}$ or $\mathbf{Z}_p$, where $p$ is a prime number dividing $|G|$. Then*

$$\operatorname{cat}(A/G) \ge n+1,$$

*where* cat *denotes the Lusternik-Schnirelman category.*

Let $G$ be a finite group and $X$ a $G$-space. By $\xi(X)$ we denote a complete list of representatives of all conjugacy classes of isotropy subgroups $H = G_x$ for some $x \in X\backslash X^G$, where $X^G = \{s \in X : gx = x \ \forall g \in G\}$. The set $\xi(X)$ is ordered by the following relation (cf. [**4**]):

$$H_1 \le H_2 \Longleftrightarrow \exists_{g\in G} g^{-1}H_{1g} \supset H_2,$$

where $H_1,\ H_2 \in \xi(X)$. By $\mu(X)$ we denote the set of all minimal elements $H$ in $\xi(X)$ such that the Weyl group $W(H) = N(H)/H$ is nontrivial, and, by $\mu_0(X)$, we denote the set of those minimal elements $H$ in $\xi(X)$ for which $W(H) = \{e\}$.

We consider a complete Finsler $G$-manifold $M$ of class $C^2$. In what follows we put $\mu := \mu(M)$, $\mu_0 := \mu_0(M)$. Suppose that $S \subset M\backslash M^G$ is a $G$-invariant subset $G$-homeomorphic to a sphere $S(V)$ of an orthogonal finite-dimensional representation $V$ of $G$. We put

$$\nu_0(S,M) = \sum_{H\in\mu} \dim V^H + |\mu_0(S) \cap \mu_0|$$

$$\nu(S,M) = \sum_{H\in\mu} \frac{|G|}{|H|} \dim V^H + \sum_{H\in\mu_0(S)\cap\mu_0} \frac{|G|}{|H|}.$$

Now we can state our Main Result.

THEOREM 3. *Let $G$ be a finite group, $M$ a complete Finsler $G$-manifold of class $C^2$, modelled on a reflexive Banach space, and $f : M \to \mathbf{R}$ a $G$-invariant locally Lipschitzian, bounded below, functional satisfying the (P.S.) condition. Suppose that*

(i) *there is a $G$-invariant subset $S \subset M\backslash M^G$ such that $S$ is $G$-homeomorphic to a sphere $S(V)$ of a finite dimensional orthogonal representation $V$ of $G$;*

(ii) *there is a number $r \in \mathbf{R}$ such that*

$$f(s) < r < f(p)$$

*for all $s \in S$ and $p \in M^G$.*

*Then $f$ has at least $\nu_0(S,M)$ distinct critical orbits in $f_r$, i.e., $f$ has at least $\nu(S,M)$ critical points in $f_r$.*

PROOF. We put $A = \{x \in M : f(x) < r\}$ and consider the function $\tilde{f} = a \circ f : A \to \mathbf{R}$, where $a(t) = t - 1/(t-r)$. Let us remark that $\tilde{f}$ is locally Lipschitzian, bounded below, satisfies the (P.S.) condition and has exactly the same critical points as the restriction $f|_A$. Suppose that $H \in \mu$ and $V^H \neq \{0\}$. Then $W(H) = N(H)/H \neq \{e\}$ acts freely on $A^H \neq \phi$. Let $\epsilon^H$ denote the class of closed $W(H)$-sets contained in $A^H$. We define

$$\gamma_j := \{Y \in \epsilon^H : \operatorname{cat}(Y/W(H),\ A^H/W(H)) > j\}$$

for $j = 1, \dots, \dim V^H$.

Since $S^H \subset A^H$ is a cohomological sphere of dimension $\dim V^H - 1$, by Lemma 2, $\operatorname{cat}(A^H/W(H)) \geq \dim V^H$, all classes $\gamma_j$, $j = 1, \dots, \dim V^H =: \eta_H$ are nonempty and $\gamma_1 \supset \gamma_2 \supset \cdots \supset \gamma_{n_H}$. We put

$$c_j = \inf_{Y \in \gamma_j} \sup_{x \in Y} \tilde{f}^H(x), \quad 1 \leq j \leq n_H.$$

Using the Deformation Lemma, Lemma 1, one can show in the standard way that all numbers $c_j$, $1 \leq j \leq n_H$, are critical values of $\tilde{f}^H$ and that $\tilde{f}^H$ has at least $\operatorname{cat}(A^H/W(H)) \geq \dim V^H$ distinct critical orbits in the set $A^{(H)}$. Let $A^{(H)} = G \cdot A^H = \cup_{g \in G} gA^H$. Since $A^H/N(H) = A^{(H)}/G$, this means that $\tilde{f}$ has at least $\dim V^H$ distinct critical orbits in the set $A^{(H)}$. Since $A^{(H_1)}$ and $A^{(H_2)}$ are disjoint for $H_1, H_2 \in \mu$, we have at least $\sum_{H \in \mu} \dim V^H$ distinct critical orbits of $\tilde{f}$.

If we suppose that $H \in \mu_0 \cap \mu_0(S)$, i.e., $S^H \neq \phi$ and $W(H) = \{e\}$, then we can state only the existence of at least one critical point of $\tilde{f}^H$ in $A^H$; thus, $\tilde{f}$ has at least one critical orbit in $A^{(H)}$. This shows that the function $\tilde{f}$, and, consequently, $f$, has at least $\nu_0(S, M)$ distinct critical orbits. □

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