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ASYMPTOTIC CONDITIONS FOR THE SOLVABILITY OF A FOURTH ORDER BOUNDARY VALUE PROBLEM WITH PERIODIC BOUNDARY CONDITIONS

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ABSTRACT. This paper concerns the existence of solutions of the fourth order periodic boundary value problem

$$\begin{aligned} -\frac{d^4u}{dx^4} + f(u(x))u'(x) + g(x, u(x)) = e(x), \quad x \in [0, 2\pi], \\ u(0) - u(2\pi) = u'(0) - u'(2\pi) \\ = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0, \end{aligned}$$

under some nonuniform resonance and nonresonance conditions on the asymptotic behavior of $u^{-1}g(x, u)$ for $|u| \to \infty$.

1. Introduction. Fourth order boundary value problems arise in the study of the equilibrium of an elastic beam under an external load (e.g., see [**1, 2, 5, 6, 16**]), where the existence, uniqueness and iterative methods to construct the solutions have been studied extensively. The author studied in [**7**] the following fourth order boundary value problems with periodic boundary conditions:

$$\begin{aligned} \frac{d^4u}{dx^4} + f(u)u' + g(x, u) = e(x), \quad x \in [0, 2\pi], \\ u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) \\ = u'''(0) - u'''(2\pi) = 0 \end{aligned}$$

(1.1)

and

$$\begin{aligned} -\frac{d^4u}{dx^4} + \alpha u' + g(x, u) = e(x), \quad x \in [0, 2\pi], \\ u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) \\ = u'''(0) - u'''(2\pi) = 0, \end{aligned}$$

(1.2)

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where $f : \mathbf{R} \to \mathbf{R}$ is continuous, $g : [0, 2\pi] \times \mathbf{R} \to \mathbf{R}$ satisfies Caratheodory's conditions, $e \in L^1[0, 2\pi]$ and $\alpha \in \mathbf{R}$. The purpose of this paper is to study the analogue of (1.2) when α is replaced by $f(u)$, viz. the boundary value problem

$$\begin{aligned} -\frac{d^4u}{dx^4} + f(u)u' + g(x,u) &= e(x), \quad x \in [0, 2\pi], \\ u(0) - u(2\pi) = u'(0) - u'(2\pi) &= u''(0) - u''(2\pi) \\ &= u'''(0) - u'''(2\pi) = 0 \end{aligned} \tag{1.3}$$

under more general conditions on the asymptotic behavior of $u^{-1}g(x,u)$ relative to the two first eigenvalues 0 and 1 of the linear problem

$$\begin{aligned} -\frac{d^4u}{dx^4} + \lambda u &= 0, \\ u(0) - u(2\pi) = u'(0) - u'(2\pi) &= u''(0) - u''(2\pi) \\ &= u'''(0) - u'''(2\pi) = 0. \end{aligned} \tag{1.4}$$

Instead of assuming, as in [**7**], that $\limsup u^{-1}g(x,u) \le \beta < 1, \beta \in \mathbf{R}$, uniformly for a.e. $x \in [0, 2\pi], |u| \to \infty$, we assume in this paper that there exists a function $\Gamma : [0, 2\pi] \to \mathbf{R}$ with $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty$ where $\Gamma_0(x) \le 1$ for a.e. $x \in [0, 2\pi]$, with strict inequality on a subset of $[0, 2\pi]$ of positive measure, $\Gamma_1 \in L^1[0, 2\pi]$, $\Gamma_\infty \in L^\infty[0, 2\pi]$ with $|\Gamma_1|_{L^1}$ and $|\Gamma_\infty|_{L^\infty}$ sufficiently small such that

$$\limsup_{|u|\to\infty} u^{-1}g(x,u) \le \Gamma(x) \tag{1.5}$$

uniformly for a.e. $x \in [0, 2\pi]$. Accordingly, the expression $\limsup_{|u|\to\infty} u^{-1}g(x,u)$ can cross any number of eigenvalues n^4 of the linear problem (1.4) as far as those crossing take place in subsets of $[0, 2\pi]$ of sufficiently small measure.

The methods and results of this paper are motivated by the paper of Gupta-Mawhin [**8**] (see also [**12, 13**]) for the second order boundary value problem with periodic boundary conditions:

$$\begin{aligned} \frac{d^2u}{dx^2} + f(u)u' + g(x,u) &= e(x), \quad x \in [0, 2\pi]. \\ u(0) - u(2\pi) = u'(0) - u'(2\pi) &= 0. \end{aligned} \tag{1.6}$$

We present in §2 some lemmas giving a priori inequalities that are needed to apply degree-theoretic arguments to obtain existence of solutions for the problem (1.3). In §3, nonresonance conditions for the existence of solutions of (1.3) are studied and in §4 we study (1.3) when it is at resonance. We study in §5 the boundary value problem (1.2) when g satisfies asymptotic conditions (1.5) and obtain a theorem which partially extends the theorem of §4. This requires a rather different lemma, similar to the second order case [**8**], which makes use of an inequality of E. Schmidt [**15**] for periodic absolutely-continuous functions. The result of §5 is an improvement over the result of §4 when $\Gamma_0 = \Gamma_\infty = 0$ and $f \equiv \alpha$; but still is not as sharp as Theorem 2.4 of [**7**] when applied to the case of a constant Γ. But then Theorem 3 of §5 allows $u^{-1}g(x,u)$ to cross infinitely many eigenvalues of (1.4).

We note that, in addition to using the classical spaces $C[0,2\pi]$, $C^k[0,2\pi]$, $L^k[0,2\pi]$ and $L^\infty[0,2\pi]$ of continuous, k-times continuously differentiable, measurable real-valued functions, with k-th powers of the absolute values Lebesgue integrable or measurable functions that are essentially bounded on $[0,2\pi]$, we shall use the Sobolev-spaces $H^k[0,2\pi]$, $k = 2, 3$, or 4, defined by

$$H^k[0,2\pi] = \{u : [0,2\pi] \to \mathbf{R} \mid u^{(j)} \quad \text{absolutely continuous on } [0,2\pi], \\ j = 0, 1, \ldots, k-1, \quad u^{(k)} \in L^2[0,2\pi]\},$$

with the inner product defined by

$$(u,v)_{H^k} = \sum_{j=1}^{k} \frac{1}{2\pi} \int_0^{2\pi} u^{(j)}(x) v^{(j)}(x)\, dx \\ + \left(\frac{1}{2\pi}\int_0^{2\pi} u(x)\, dx\right)\left(\frac{1}{2\pi}\int_0^{2\pi} v(x)\, dx\right)$$

and the corresponding norm by $|\cdot|_{H^k}$. We also define, for the sake of convenience, the norm in $L^k[0,2\pi]$ by

$$|u|_{L^k} = \left(\frac{1}{2\pi}\int_0^{2\pi} |u(x)|^k\, dx\right)^{\frac{1}{k}}.$$

We also use the Sobolev-space $W^{4,1}[0,2\pi]$ defined by $W^{4,1}[0,2\pi] =$

$\{u\ : [0,2\pi] \to \mathbf{R}/u, u', u'', u'''$ absolutely continuous on $[0,2\pi]\}$ with norm

$$|u|_{W^{4,1}} = \sum_{j=0}^{4} \int_0^{2\pi} |u^{(j)}(t)|\, dt.$$

2. A priori inequalities. For $u \in L^1[0,2\pi]$, let us write

$$\overline{u} = \frac{1}{2\pi}\int_0^{2\pi} u(x)\, dx, \quad \tilde{u}(x) = u(x) - \overline{u}, \tag{2.1}$$

so that $\int_0^{2\pi} \tilde{u}(x)\, dx = 0$. Let $\tilde{H}^2[0,2\pi] = \{u \in H^2[0,2\pi] | \overline{u} = 0\}$.

LEMMA 1. *Let $\Gamma \in L^1[0,2\pi]$ be such that, for a.e. $x \in [0,2\pi]$,*

$$\Gamma(x) \le 1, \tag{2.2}$$

with the strict inequality holding on a subset of $[0,2\pi]$ of positive measure. Then there exists a $\delta = \delta(\Gamma) > 0$ such that, for all $\tilde{u} \in \tilde{H}^2[0,2\pi]$ with $\tilde{u}(0) - \tilde{u}(2\pi) = \tilde{u}'(0) - \tilde{u}'(2\pi) = 0$,

$$B_\Gamma(\tilde{u}) = \frac{1}{2\pi}\int_0^{2\pi} [(\tilde{u}''(x))^2 - \Gamma(x)\tilde{u}^2(x)]\, dx \ge \delta|\tilde{u}|^2_{H^2}. \tag{2.3}$$

PROOF. Using (2.2) and Wirtinger's inequality [**3**], we see that, for all $\tilde{u} \in \tilde{H}^2[0,2\pi]$ with $\tilde{u}(0) - \tilde{u}(2\pi) = \tilde{u}'(0) - \tilde{u}'(2\pi) = 0$,

$$B_\Gamma(\tilde{u}) \ge \frac{1}{2\pi}\int_0^{2\pi} [(\tilde{u}''(x))^2 - \tilde{u}^2(x)]\, dx \ge 0, \tag{2.4}$$

and, moreover,

$$B_\Gamma(\tilde{u}) = 0 \tag{2.5}$$

if and only if

$$\tilde{u}(x) = A\sin(x+\theta), \tag{2.6}$$

for some $A, \theta \in \mathbf{R}$. But then, by (2.5), (2.6), we get

$$\begin{aligned} 0 = B_\Gamma(\tilde{u}) &= \frac{1}{2\pi}\int_0^{2\pi}[1-\Gamma(x)]\tilde{u}^2(x)\,dx \\ &= \frac{A^2}{2\pi}\int_0^{2\pi}[1-\Gamma(x)]\sin^2(x+\theta)\,dx, \end{aligned}$$

so that by our assumption (2.2) on Γ we have $A = 0$ and hence $\tilde{u} = 0$.

Let us next assume that the conclusion of the lemma is false. Then there exists a sequence $\{\tilde{u}_n\}, \tilde{u}_n \in \tilde{H}^2[0, 2\pi]$ for every $n = 1, 2, 3, \ldots$ such that

$$\begin{aligned} &B_\Gamma(\tilde{u}_n) \to 0 \quad \text{as} \quad n \to \infty, \\ &|\tilde{u}_n|_{H^2} = 1, \quad \text{for every} \quad n = 1, 2, \ldots. \end{aligned} \tag{2.7}$$

It now follows from (2.7) and the compact imbedding $H^2[0, 2\pi] \subset\hookrightarrow C^1[0, 2\pi]$ that there exists a $\tilde{u} \in \tilde{H}^2[0, 2\pi]$ such that

$$\begin{aligned} &\tilde{u}_n \to \tilde{u} \quad \text{weakly in} \quad H^2[0, 2\pi], \\ &\tilde{u}_n \to \tilde{u} \quad \text{in} \quad C^1[0, 2\pi]. \end{aligned} \tag{2.8}$$

Now (2.8) implies that $\tilde{u}(0) - \tilde{u}(2\pi) = \tilde{u}'(0) - \tilde{u}'(2\pi) = 0$ and $|\tilde{u}|_{H^2} \leq \liminf_{n\to\infty} |\tilde{u}_n|_{H^2}$. Hence we get that

$$0 \leq B_\Gamma(\tilde{u}) \leq \liminf_{n\to\infty} B_\Gamma(\tilde{u}_n) = 0. \tag{2.9}$$

It now follows, from (2.9) and the first part of this proof, that $\tilde{u} = 0$. Also (2.7)–(2.9) imply that

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi}[\tilde{u}_n''(x)]^2\,dx &= B_\Gamma(\tilde{u}_n) + \frac{1}{2\pi}\int_0^{2\pi}\Gamma(x)\tilde{u}_n^2(x)\,dx \\ &\longrightarrow \frac{1}{2\pi}\int_0^{2\pi}\Gamma(x)\tilde{u}^2(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi}[\tilde{u}''(x)]^2\,dx, \end{aligned}$$

so that $\tilde{u}_n \to \tilde{u}$ in $H^2[0, 2\pi]$ and $|\tilde{u}|_{H^2} = 1$. We have thus arrived at a contradiction.

Hence the lemma is true. □

LEMMA 2. *Let* $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty$ *where* $\Gamma_\infty \in L^\infty[0, 2\pi], \Gamma_1 \in L^1[0, 2\pi]$ *and* $\Gamma_0 \in L^1[0, 2\pi]$ *is such that* $\Gamma_0(x) \leq 1$ *for a.e.* $x \in [0, 2\pi]$ *with strict inequality holding on a subset of* $[0, 2\pi]$ *of positive measure. Let* $\delta(\Gamma_0) > 0$ *be as given by Lemma* 1. *Then, for every* $\tilde{u} \in \tilde{H}^2[0, 2\pi]$ *with* $\tilde{u}(0) - \tilde{u}(2\pi) = \tilde{u}'(0) - \tilde{u}'(2\pi) = 0$,

$$B_\Gamma(\tilde{u}) \geq [\delta(\Gamma_0) - \frac{\pi^2}{3}|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty}]|\tilde{u}|^2_{H^2}. \tag{2.10}$$

PROOF. We have

$$\begin{aligned} B_\Gamma(\tilde{u}) =& \frac{1}{2\pi}\int_0^{2\pi} [(\tilde{u}''(x))^2 - \Gamma_0(x)\tilde{u}^2(x)]\,dx \\ & - \frac{1}{2\pi}\int_0^{2\pi} \Gamma_1(x)\tilde{u}^2(x)\,dx - \frac{1}{2\pi}\int_0^{2\pi} \Gamma_\infty(x)\tilde{u}^2(x)\,dx. \end{aligned}$$

Using, now, the fact that $H^2[0, 2\pi] \subset C^1[0, 2\pi]$ and the well-known inequalities (see, e.g., [**14**])

$$|\tilde{u}|_{L^2} \leq |\tilde{u}'|_{L^2} \leq |\tilde{u}|_{H^2}, \quad |\tilde{u}|_{L^\infty} \leq \frac{\pi}{\sqrt{3}}|\tilde{u}'|_{L^2} \leq \frac{\pi}{\sqrt{3}}|\tilde{u}|_{H^2}$$

for $\tilde{u} \in \tilde{H}^2[0, 2\pi]$ with $\tilde{u}(0) - \tilde{u}(2\pi) = \tilde{u}'(0) - \tilde{u}'(2\pi) = 0$, as well as Lemma 1, we get that

$$\begin{aligned} B_\Gamma(\tilde{u}) &\geq \delta(\Gamma_0)|\tilde{u}|^2_{H^2} - |\Gamma_1|_{L^1}|\tilde{u}|^2_{L^\infty} - |\Gamma_\infty|_{L^\infty}|\tilde{u}|^2_{L^2} \\ &\geq [\delta(\Gamma_0) - \frac{\pi^2}{3}|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty}]|\tilde{u}|^2_{H^2}. \ \square \end{aligned}$$

REMARK 1. The best value for $\delta(0)$ is easily seen to be $1/2$, so that $B_{\Gamma_1}(\tilde{u}) \geq (\frac{1}{2} - \frac{\pi^2}{3}|\Gamma_1|_{L^1})|\tilde{u}|^2_{H^2}$ for all $\tilde{u} \in \tilde{H}^2[0, 2\pi]$ with $\tilde{u}(0) - \tilde{u}(2\pi) = \tilde{u}'(0) - \tilde{u}'(2\pi) = 0$.

LEMMA 3. *Let* $\gamma \in L^1[0, 2\pi], \Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty$ *be as in Lemma* 2, $\delta(\Gamma_0)$ *be given by Lemma* 1. *Then, for all measurable functions* $p(x)$ *on* $[0, 2\pi]$ *such that* $\overline{\gamma} \leq \overline{p}, p(x) \leq \Gamma(x)$ *a.e. on* $[0, 2\pi]$, *all continuous* *functions* $f : \mathbf{R} \to \mathbf{R}$ *and all* $u \in W^{4,1}[0, 2\pi]$ *with* $u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0$, *yields*

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} &[\overline{u} - \tilde{u}(x)][-\tilde{u}^{(iv)}(x) + f(u(x))u'(x) + p(x)u(x)]\,dx \\ &\geq \overline{\gamma}\cdot\overline{u}^2 + \big[\delta(\Gamma_0) - \frac{\pi^2}{3}|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty}\big]|\tilde{u}|^2_{H^2}.\end{aligned} \tag{2.11}$$

PROOF. For $u \in W^{4,1}[0, 2\pi]$ with

$$u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0,$$

we have, on integrating by parts and using Lemma 2, that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} &[\overline{u} - \tilde{u}(x)][-u^{(iv)}(x) + f(u(x))u'(x) + p(x)u(x)]\,dx \\ &\geq \overline{p}\cdot\overline{u}^2 + \frac{1}{2\pi}\int_0^{2\pi}[(\tilde{u}''(x))^2 - p(x)\tilde{u}^2(x)]\,dx \\ &\geq \overline{\gamma}\cdot\overline{u}^2 + \Big[\delta(\Gamma_0) - \frac{\pi^2}{3}|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty}\Big]|\tilde{u}|^2_{H^2}. \square\end{aligned}$$

3. Asymptotic conditions for nonresonance. Let $f : \mathbf{R} \to \mathbf{R}$ be continuous and let $g : [0, 2\pi] \times R \to \mathbf{R}$ be a function satisfying Caratheodory's conditions, viz.:

(i) for each $u \in \mathbf{R}$, the function $x \in [0, 2\pi] \to g(x, u) \in \mathbf{R}$ is measurable on $[0, 2\pi]$;

(ii) for a.e. $x \in [0, 2\pi]$, the function $u \in \mathbf{R} \to g(x, u) \in \mathbf{R}$ is continuous on $\mathbf{R}$; and

(iii) for each $r > 0$, there exists a function $\alpha_r(x) \in L^1[0, 2\pi]$ such that $|g(x, u)| \leq \alpha_r(x)$ for a.e. $x \in [0, 2\pi]$ and all $u \in \mathbf{R}$ with $|u| \leq r$.

THEOREM 1. *Let* $\gamma \in L^1[0, 2\pi]$ *with* $\overline{\gamma} > 0$ *be given. Also let* $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty$ *with* $\Gamma_1 \in L^1[0, 2\pi], \Gamma_\infty \in L^\infty[0, 2\pi], \Gamma_0$ *measurable on* $[0, 2\pi], \Gamma_0(x) \leq 1$ *with strict inequality holding on a subset of* $[0, 2\pi]$

of positive measure, and $\frac{\pi^2}{3}|\Gamma_1|_{L^1} + |\Gamma_\infty|_{L^\infty} < \delta(\Gamma_0)$, *where* $\delta(\Gamma_0)$ *is given by Lemma* 1. *Assume that the inequalities*

$$\gamma(x) \le \liminf_{|u|\to\infty} u^{-1}g(x,u) \le \limsup_{|u|\to\infty} u^{-1}g(x,u) \le \Gamma(x), \tag{3.1}$$

hold uniformly for a.e. $x \in [0, 2\pi]$.

Then, for every given $e(x) \in L^1[0, 2\pi]$, *the boundary value problem*

(3.2)

$$-u^{(iv)}(x) + f(u(x))u'(x) + g(x, u(x)) = e(x), \quad x \in [0, 2\pi],$$
$$u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0$$

has at least one solution.

Proof. Let $\eta = 1/2 \min\{\overline{\gamma}, \delta(\Gamma_0) - \pi^2/3|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty}\} > 0$. Then, by (3.1), we can find an $r > 0$ such that, for a.e. $x \in [0, 2\pi]$ and every $u \in \mathbf{R}$ with $|u| \ge r$,

$$\gamma(x) - \eta \le u^{-1}g(x,u) \le \Gamma(x) + \eta. \tag{3.3}$$

Next, define $\tilde{\gamma} : [0, 2\pi] \times \mathbf{R} \to \mathbf{R}$ by

$$\tilde{\gamma}(x,u) = \begin{cases} u^{-1}g(x,u), & \text{if } |u| \ge r, \\ r^{-1}g(x,r), & \text{if } 0 < u < r, \\ -r^{-1}g(x,-r), & \text{if } -r < u < 0, \\ \Gamma(x), & \text{if } u = 0. \end{cases}$$

Note that $\tilde{\gamma}(x,u)u$ satisfies Caratheodory's conditions and, from (3.3),

$$\gamma(x) - \eta \le \tilde{\gamma}(x,u) \le \Gamma(x) + \eta, \tag{3.4}$$

for a.e. $x \in [0, 2\pi]$ and all $u \in \mathbf{R}$. Now, define $h : [0, 2\pi] \times \mathbf{R} \to \mathbf{R}$ by

$$h(x,u) = g(x,u) - \tilde{\gamma}(x,u)u,$$

for $x \in [0, 2\pi], u \in \mathbf{R}$. We then see that

$$|h(x,u)| \le \sup_{|u|\le r} |g(x,u) - \tilde{\gamma}(x,u)u| \le \alpha(x), \tag{3.5}$$

for $x \in [0, 2\pi], u \in \mathbf{R}$, where $\alpha(x) \in L^1[0, 2\pi]$ depends on γ, Γ and α_r.

Now, the equation in (3.2) is equivalent to the equation

$$-u^{(iv)}(x) + f(u(x))u'(x) + \tilde{\gamma}(x, u(x))u(x) + h(x, u(x)) = e(x),$$

to which we shall apply coincidence degree theory [**4, 9**] in a manner similar to the one used in Theorem 1 of [**12**]. Let $X = C^1[0, 2\pi], Z = L^1[0, 2\pi], \operatorname{dom} L = \{u \in W^{4,1}[0, 2\pi\} \mid u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0\}$, and

$$\begin{aligned} L &: \operatorname{dom} L \subset X \to Z, \quad u \to -u^{(iv)}, \\ F &: X \to Z, \quad u \to f(u(\cdot))u'(\cdot), \\ G &: X \to Z, \quad u \to \tilde{\gamma}(\cdot, u(\cdot))u(\cdot), \\ H &: X \to Z, \quad u \to h(\cdot, u(\cdot)) - e(\cdot), \\ A &: X \to Z, \quad u \to \tilde{\gamma}(\cdot, 0)u(\cdot) = \Gamma(\cdot)u(\cdot). \end{aligned}$$

It is easy to check that F, G, H and A are well-defined and L-compact on bounded subsets of X and that L is a linear Fredholm mapping of index zero (see Lemma 2.1 of [**7**]). We consider the homotopy $\Phi : \operatorname{dom} L \times [0, 1] \to Z$ defined by

$$\Phi(u, \lambda) \equiv Lu + \lambda Fu + (1 - \lambda)Au + \lambda Gu + \lambda Hu,$$

for $u \in \operatorname{dom} L, \lambda \in [0, 1]$. Now, in order to apply Theorem IV.5 of [**9**] (see also [**10, 11**]), it suffices to show that the set of possible solutions of the family of equations

(3.6)
$$\begin{aligned} -u^{(iv)}(x) + \lambda f(u(x))u'(x) + [(1 - \lambda)\Gamma(x) + \lambda\tilde{\gamma}(x, u(x))]u(x) \\ + \lambda h(x, u(x)) - \lambda e(x) = 0, \\ u(0) - u(2\pi) = u'(0) - u'(2\pi) - u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) \end{aligned}$$

is a priori bounded in $C^1[0, 2\pi]$ independently of $\lambda \in [0, 1]$. If u is a solution of (3.6), then multiplying (3.6) by $\overline{u} - \tilde{u}$, integrating over $[0, 2\pi]$ and using (3.4), (3.5) together with Lemma 3 with Γ_∞ replaced by $\Gamma_\infty + \eta$ and γ by $\gamma - \eta$, obtains

$$\begin{aligned}
0 &= \frac{1}{2\pi}\int_0^{2\pi} [\overline{u} - \tilde{u}(x)]\{-u^{(iv)}(x) + \lambda f(u(x))u'(x) + [(1-\lambda)\Gamma(x) \\
&\qquad + \lambda\tilde{\gamma}(x, u(x))]u(x) + \lambda h(x, u(x)) - \lambda e(x)\}\, dx \\
&\geq (\overline{\gamma} - \eta)\overline{u}^2 + \big[\delta(\Gamma_0) - \frac{\pi^2}{3}|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty} - \eta\big]|\tilde{u}|_{H^2}^2 \\
&\qquad - (|\alpha|_{L^1} + |e|_{L^1})|\overline{u} - \tilde{u}|_{L^\infty} \\
&\geq \frac{1}{2}\overline{\gamma}\cdot\overline{u}^2 + \frac{1}{2}\big[\delta(\Gamma_0) - \frac{\pi^2}{3}|\Gamma_1|_{L^\infty} - |\Gamma_\infty|\big]|\tilde{u}|_{H^2}^2 - \beta|u|_{H^2} \\
&\geq \eta|u|_{H^2}^2 - \beta|u|_{H^2}.
\end{aligned}$$

Hence $|u|_{H^2} \leq \beta/\eta$ which implies that $|u|_{C^1[0,1]} \leq C$, where C is a constant independent of $\lambda \in [0,1]$, in view of the compact imbedding $H^2[0,2\pi] \subset C^1[0,2\pi]$.

This completes the proof of the Theorem. □

4. Asymptotic conditions at resonance.

Let $f : \mathbf{R} \to \mathbf{R}$ be continuous and $g : [0,2\pi] \times \mathbf{R} \to \mathbf{R}$ be a function satisfying Caratheodory's conditions.

THEOREM 2. *Let $\Gamma \in L^1[0,2\pi]$ be such that*

$$\limsup_{|u|\to\infty} \frac{g(x,u)}{u} \leq \Gamma(x), \tag{4.1}$$

uniformly a.e. in $x \in [0,2\pi]$ and $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty$, where $\Gamma_\infty \in L^\infty[0,2\pi], \Gamma_1 \in L^1[0,2\pi]$ and $\Gamma_0 \in L^1[0,2\pi]$ are such that $\Gamma_0(x) \leq 1$ for a.e. $x \in [0,2\pi]$, with strict inequality holding on a subset of $[0,2\pi]$ of positive measure and $|\Gamma_\infty|_{L^\infty} + (\pi^2/3)|\Gamma_1|_{L^1} < \delta(\Gamma_0)$, where $\delta(\Gamma_0)$ is given by Lemma 1.

Suppose, further, that there exist real numbers a, A, r and R with $a \leq A$ and $r < 0 < R$ such that

$$g(x,u) \geq A \tag{4.2}$$

for a.e. $x \in [0, 2\pi]$ *and all* $u \geq R$, *and*

$$g(x, u) \leq a \tag{4.3}$$

for a.e. $x \in [0, 2\pi]$ *and all* $u \leq r$.

Then the periodic boundary value problem

$$\begin{aligned} -\frac{d^4 u}{dx^4} + f(u(x))u'(x) + g(x, u(x)) = e(x), \quad x \in [0, 2\pi], \\ u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) \\ = u'''(0) - u'''(2\pi) = 0 \end{aligned} \tag{4.4}$$

has at least one solution for each given $e \in L^1[0, 2\pi]$ *with*

$$a \leq \bar{e} \leq A. \tag{4.5}$$

Proof. Define $g_1 : [0, 2\pi] \times \mathbf{R} \to \mathbf{R}$ by $g_1(x, u) = g(x, u) - (1/2)(a + A)$
and $e_1 \in L^1[0, 2\pi]$ by $e_1(x) = e(x) - (1/2)(a + A)$, so that, for a.e. $x \in [0, 2\pi]$, and using (4.2), (4.3), (4.5),

$$g_1(x, u) \geq \frac{1}{2}(A - a) \geq 0, \quad \text{if} \quad u \geq R, \tag{4.6}$$

$$g_1(x, u) \leq \frac{1}{2}(a - A) \leq 0, \quad \text{if} \quad u \leq r, \tag{4.7}$$

$$\frac{1}{2}(a - A) \leq \bar{e}_1 \leq \frac{1}{2}(A - a). \tag{4.8}$$

Now, the equation in (4.4) is clearly equivalent to

$$-\frac{d^4 u}{dx^4} + f(u(x))u'(x) + g_1(x, u(x)) = e_1(x). \tag{4.9}$$

Moreover, we have

$$\limsup_{|u| \to \infty} u^{-1} g_1(x, u) \leq \Gamma(x),$$

uniformly a.e. in $x \in [0, 2\pi]$ and, for $|u| \geq \max(R, -r)$ and a.e. $x \in [0, 2\pi], u^{-1}g_1(x, u) \geq 0$, so that $\Gamma(x) \geq 0$ for a.e. $x \in [0, 2\pi]$.

Let, now, $\eta = \frac{1}{2}[\delta(\Gamma_0) - (\pi^2/3)|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty}] > 0$. Then there exists an $r_1 > 0$ such that, for a.e. $x \in [0, 2\pi]$ and all $u \in \mathbf{R}, |u| \geq r_1$, we have

$$0 \leq u^{-1}g_1(x, u) \leq \Gamma(x) + \eta. \tag{4.10}$$

Proceeding as in the proof of Theorem 1 (of §3) we can write the equation (4.9) in the equivalent form

$$-\frac{d^4u}{dx^4} + f(u(x))u'(x) + \tilde{\gamma}(x, u(x))u(x) + h(x, u(x)) = e_1(x), \tag{4.11}$$

where $0 \leq \tilde{\gamma}(x, u) \leq \Gamma(x) + \eta, |h(x, u)| \leq \alpha(x)$, for a.e. $x \in [0, 2\pi]$, all $u \in \mathbf{R}$ and some $\alpha \in L^1[0, 2\pi]$. Once again, degree arguments will ensure the existence of a solution for (4.4) if the set of all possible solutions of the family of equations

(4.12)
$$\begin{aligned} -\frac{d^4u}{dx^4} + \lambda f(u(x))u'(x) + [(1-\lambda)(\Gamma(x) + \eta) + \lambda\tilde{\gamma}(x, u(x))]u(x) \\ + \lambda h(x, u(x)) = \lambda e_1(x), \quad \lambda \in [0, 1], \end{aligned}$$

$u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0$, is, a priori, bounded in $C^1[0, 2\pi]$ independently of $\lambda \in [0, 1]$. If, now, $u(x)$ is a possible solution of (4.12) for some $\lambda \in [0, 1]$, then integrating the equation in (4.12) over $[0, 2\pi]$ after multiplying it by $\overline{u} - \tilde{u}$, we get, on using Lemma 3 with $\gamma = 0$ and Γ_∞ replaced by $\Gamma_\infty + \eta$,

$$\begin{aligned} 0 = \frac{1}{2\pi}\int_0^{2\pi} [\overline{u} - \tilde{u}(x)]\Big\{ -\frac{d^4u}{dx^4} + \lambda f(u(x))u'(x) + [(1-\lambda)(\Gamma(x) + \eta) \\ + \lambda\tilde{\gamma}(x, u(x))]u(x) + \lambda h(x, u(x)) - \lambda e_1(x)\Big\}\,dx \\ \geq \Big[\delta(\Gamma_0) - \frac{\pi^2}{3}|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty} - \eta\Big]|\tilde{u}|_{H^2}^2 - (|\alpha|_{L^1} + |e_1|_{L^1})|\overline{u} - \tilde{u}|_{L^\infty} \\ \geq \eta|\tilde{u}|_{H^2}^2 - \beta(|\overline{u}| + |\tilde{u}|_{H^2}), \end{aligned}$$

for some constant β, independent of $\lambda \in [0, 1]$. Hence,

$$|\tilde{u}|_{H^2}^2 \leq (\beta/\eta)(|\overline{u}| + |\tilde{u}|_{H^2}). \tag{4.13}$$

Next, we get, on integrating the equation in (4.12) over $[0, 2\pi]$,

(4.14)

$$\frac{1}{2\pi}(1-\lambda)\int_0^{2\pi}(\Gamma(x)+\eta)u(x)\,dx+\frac{1}{2\pi}\lambda\int_0^{2\pi}[g_1(x,u(x))-e_1(x)]\,dx=0.$$

If, now, $u(x) \geq R$ for all $x \in [0, 2\pi]$ then (4.6), (4.8) imply that $(1-\lambda)(\overline{\Gamma}+\eta)R \leq 0$, contradicting $\overline{\Gamma}+\eta \geq \eta > 0$. Similarly, $u(x) \leq r$ for all $x \in [0, 2\pi]$ leads to a contradiction. So there must exist a $\tau \in [0, 2\pi]$ such that

$$r < u(\tau) < R.$$

It is then easy to see from $u(x) = u(\tau) + \int_\tau^x u'(s)\,ds$ that

$$|\overline{u}| \leq \max(R, -r) + 2\pi|\tilde{u}|_{H^2}. \tag{4.15}$$

(4.13) and (4.15) now imply that

$$|\tilde{u}|_{H^2}^2 \leq (3\pi\beta/\eta)|\tilde{u}|_{H^2} + (\beta/\eta)\cdot\max(R, -r),$$

so that there exists a constant ρ, independent of $\lambda \in [0, 1]$ such that

$$|\tilde{u}|_{H^2} \leq \rho. \tag{4.16}$$

Finally (4.15) and (4.16) imply that there is a constant C, independent of $\lambda \in [0, 1]$ such that

$$|u|_{H^2} \leq C,$$

which implies that $|u|_{C^1} \leq C_1$, for some constant C_1, independent of $\lambda \in [0, 1]$.

This completes the proof of Theorem 2. □

REMARK 2. If we take $f(u) \equiv \alpha, \alpha \in \mathbf{R}$ and $\Gamma(x) = \beta < 1$ (i.e., $\Gamma_0 = \beta, \Gamma_1 = \Gamma_\infty = 0$) in Theorem 2 above, we get Theorem 2.4 of [**7**] as a corollary to Theorem 2.

5. An inequality for a linear fourth order operator with periodic boundary conditions. We obtain a partial extension of Theorem 2 of §4 when f is a constant function and $\Gamma_0 = \Gamma_\infty = 0$. We need the following lemma which gives an inequality for a linear fourth order operator with periodic boundary conditions.

LEMMA 4. *Let* $\alpha \in \mathbf{R}, e \in L^1[0,2\pi], \Gamma \in L^1[0,2\pi]$ *with* $\overline{\Gamma} \geq 0$. *Then every possible solution* $u(x)$ *of the problem*

$$\begin{aligned} -\frac{d^4u}{dx^4} + \alpha u'(x) + p(x)u(x) = e(x), \quad x \in [0,2\pi], \\ u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) \\ = u'''(0) - u'''(2\pi) = 0, \end{aligned} \tag{5.1}$$

with $p \in L^1[0,2\pi]$ *such that*

$$\overline{p} \leq \overline{\Gamma}, \quad 0 \leq p(x), \tag{5.2}$$

for a.e. $x \in [0,2\pi]$, *satisfies the inequality*

$$\Big(1 - \frac{\pi^2}{4}\overline{\Gamma}\Big)\Big|\frac{d^4u}{dx^4} - \alpha u'\Big|_{L^1}^2 \leq 2|e|_{L^1}\Big|\frac{d^4u}{dx^4} - \alpha u'\Big|_{L^1} + \overline{\Gamma}|u|_{L^\infty}|e|_{L^1} + 3|e|_{L^1}^2. \tag{5.3}$$

PROOF. Let $p \in L^1[0,2\pi]$ be as above and $u(x)$ be a solution of (5.1). Then, on multiplying the equation in (5.1) by $u(x)/(2\pi)$ and integrating over $[0,2\pi]$, we get

$$-\frac{1}{2\pi}\int_0^{2\pi}(u''(x))^2 + \frac{1}{2\pi}\int_0^{2\pi} p(x)u^2(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} e(x)u(x)\,dx. \tag{5.4}$$

Since, now, $\overline{p} \leq \overline{\Gamma}$, we have, by using Schwarz's inequality,

$$\begin{aligned} \bigg(\frac{1}{2\pi}\int_0^{2\pi}|p(x)u(x)|\,dx\bigg)^2 &\leq \bigg(\frac{1}{2\pi}\int_0^{2\pi}p(x)\,dx\bigg)\bigg(\frac{1}{2\pi}\int_0^{2\pi}p(x)u^2\,dx\bigg) \\ &\leq \overline{\Gamma}\bigg(\frac{1}{2\pi}\int_0^{2\pi}p(x)u^2(x)\,dx\bigg), \end{aligned} \tag{5.5}$$

and hence, using the equation in (5.1),

$$\left(\frac{1}{2\pi}\int_0^{2\pi}\left|e(x)+\frac{d^4u}{dx^4}-\alpha u'\right|dx\right)^2 \le \overline{\Gamma}\left(\frac{1}{2\pi}\int_0^{2\pi}p(x)u^2(x)\,dx\right). \tag{5.6}$$

We next apply an inequality of E. Schmidt [**15**] (see also [**8**]) to $u'''-\alpha\tilde{u}$ to get

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}[u'''-\alpha\tilde{u}]^2\,dx &= \frac{1}{2\pi}\int_0^{2\pi}(u''')^2\,dx+\frac{\alpha^2}{2\pi}\int_0^{2\pi}\tilde{u}^2\,dx\\ &\le \frac{\pi^2}{4}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{d^4u}{dx^4}-\alpha u'\right|dx\right)^2.\end{aligned} \tag{5.7}$$

Now, we get from (5.4), (5.6) and (5.7) that

$$\begin{aligned}&\overline{\Gamma}^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|e(x)+\frac{d^4u}{dx^4}-\alpha u'\right|dx\right)^2+\frac{1}{2\pi}\int_0^{2\pi}(u''')^2\,dx+\frac{\alpha^2}{2\pi}\int_0^{2\pi}\tilde{u}^2dx\\ &\quad\le \frac{1}{2\pi}\int_0^{2\pi}p(x)u^2\,dx+\frac{\pi^2}{4}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{d^4u}{dx^4}-\alpha u'\right|dx\right)^2\\ &\quad= \frac{1}{2\pi}\int_0^{2\pi}(u'')^2\,dx+\frac{1}{2\pi}\int_0^{2\pi}e(x)u(x)\,dx+\frac{\pi^2}{4}\left|\frac{d^4u}{dx^4}-\alpha u'\right|_{L^1}^2.\end{aligned}$$

Hence,

$$\begin{aligned}&-\frac{\pi^2}{4}\left|\frac{d^4u}{dx^4}-\alpha u'\right|_{L^1}^2+\overline{\Gamma}^{-1}\left|e(x)+\frac{d^4u}{dx^4}-\alpha u'\right|_{L^1}^2\\ &\qquad\le \frac{1}{2\pi}\int_0^{2\pi}(u'')^2\,dx-\frac{1}{2\pi}\int_0^{2\pi}(u''')^2\,dx-\frac{\alpha^2}{2\pi}\int_0^{2\pi}\tilde{u}^2\,dx\\ &\qquad\qquad+\frac{1}{2\pi}\int_0^{2\pi}e(x)u(x)\,dx\\ &\qquad\le |e|_{L^1}\cdot|u|_{L^\infty},\end{aligned}$$

in view of Wirtinger's inequality $|u''|_{L^2} \le |u'''|_{L^2}$. Finally, then

$$\begin{aligned}
\Big(1-\frac{\pi^2}{4}\overline{\Gamma}\Big)\Big|\frac{d^4u}{dx^4}-\alpha u'\Big|_{L^1}^2
&= \Big|\frac{d^4u}{dx^4}-\alpha u'+e-e\Big|_{L^1}^2-\frac{\pi^2}{4}\overline{\Gamma}\Big|\frac{d^4u}{dx^4}-\alpha u'\Big|_{L^1}^2\\
&\le \Big|e+\frac{d^4u}{dx^4}-\alpha u'\Big|_{L^1}^2+2|e|_{L^1}\Big|e+\frac{d^4u}{dx^4}-\alpha u'\Big|_{L^1}\\
&\quad+|e|_{L^1}^2-\frac{\pi^2}{4}\overline{\Gamma}\Big|\frac{d^4u}{dx^4}-\alpha u'\Big|_{L^1}^2\\
&\le 2|e|_{L^1}\Big|\frac{d^4u}{dx^4}-\alpha u'\Big|_{L^1}+\overline{\Gamma}|e|_{L^1}\cdot|u|_{L^\infty}+3|e|_{L^1}^2,
\end{aligned}$$

hence the lemma. □

THEOREM 3. *Let $\alpha \in \mathbf{R}$ be given and $g : [0,2\pi]\times\mathbf{R}\to\mathbf{R}$ be a function satisfying Caratheodory's conditions. Assume that there exists a $\Gamma \in L^1[0,2\pi]$ such that*

$$\limsup_{|u|\to\infty} u^{-1}g(x,u) \le \Gamma(x)$$

uniformly a.e. on $[0,2\pi]$ and that $\overline{\Gamma} < 4/\pi^2$. Suppose, further, that there exist real numbers a, A, r, R with $a \le A$ and $r < 0 < R$ such that, for a.e. $x \in [0,2\pi], g(x,u) \ge A$ when $u \ge R$ and $g(x,u) \le a$ when $u \le r$. Then the periodic boundary value problem

$$\begin{aligned}
-\frac{d^4u}{dx^4}+\alpha u'+g(x,u(x)) &= e(x), \quad x\in[0,2\pi],\\
u(0)-u(2\pi)=u'(0)-u'(2\pi) &= u''(0)-u''(2\pi)\\
&= u'''(0)-u'''(2\pi)=0
\end{aligned}$$

(5.8)

has at least one solution for each given $e \in L^1[0,2\pi]$ with $a \le \overline{e} \le A$.

PROOF. We first define g_1 and e_1 as in the proof of Theorem 2 (§4) so that the equation in (5.8) can be written as

$$-\frac{d^4u}{dx^4}+\alpha u'+g_1(x,u(x))=e_1(x), \tag{5.9}$$

with $g_1(x,u) \geq 0$ when $u \geq R$ and $g_1(x,u) \leq 0$ when $u \leq r$ for a.e. $x \in [0,2\pi]$ and $\limsup_{|u|\to\infty} u^{-1}g_1(x,u) \leq \Gamma(x)$ uniformly for a.e. $x \in [0,2\pi]$. Consequently, for a.e. $x \in [0,2\pi]$, $\Gamma(x) \geq 0$. Let $\eta = (1/2)[4/\pi^2 - \overline{\Gamma}] > 0$ so that $\overline{\Gamma} + \eta < 4/\pi^2$, and let $r_1 > 0$ be such that

$$0 \leq u^{-1}g_1(x,u) \leq \Gamma(x) + \eta \tag{5.10}$$

for a.e. $x \in [0,2\pi], |u| \geq r_1$. Proceeding as in the proof of Theorem 1 (§3) we can write (5.9) in the form

$$-\frac{d^4u}{dx^4} + \alpha u' + \tilde{\gamma}(x,u(x))u(x) + h(x,u(x)) = e_1(x), \tag{5.11}$$

where $0 \leq \tilde{\gamma}(x,u) \leq \Gamma(x) + \eta, |h(x,u)| \leq \beta(x)$ for a.e. $x \in [0,2\pi]$ and all $u \in \mathbf{R}$ and some $\beta \in L^1[0,2\pi]$. The same degree arguments will imply the existence of a solution for (5.8) if the set of possible solutions of the family of equations

$$\begin{aligned} -\frac{d^4u}{dx^4} + \alpha u'(x) + [(1-\lambda)(\Gamma(x)+\eta) + \lambda\tilde{\gamma}(x,u(x))]u(x) \\ = -\lambda h(x,u(x)) + \lambda e_1(x), \end{aligned} \tag{5.12}$$

$u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0$, is, a priori, bounded in $C[0,2\pi]$ independently of $\lambda \in [0,1]$. Let $u(x)$ be a solution of (5.12) for some $\lambda \in [0,1]$. Since, now,

$$0 \leq (1-\lambda)(\Gamma(x)+\eta) + \lambda\tilde{\gamma}(x,u(x)) \leq \Gamma(x) + \eta$$

for a.e. $x \in [0,2\pi]$. With $\overline{\Gamma} + \eta < 4/\pi^2$, and since

$$|e_1 - h(\cdot,u(\cdot))|_{L^1} \leq |e_1|_{L^1} + |\beta|_{L^1},$$

it follows from Lemma 4 that

(5.13)

$$\begin{aligned} \Big[1 - \frac{\pi^2}{4}(\overline{\Gamma}+\eta)\Big]\Big|\frac{d^4u}{dx^4} - \alpha u'\Big|^2_{L^1} &\leq 2(|e_1|_{L^1} + |\beta|_{L^1})\Big|\frac{d^4u}{dx^4} - \alpha u'\Big|_{L^1} \\ &\quad + (\overline{\Gamma}+\eta)(|e_1|_{L^1} + |\beta|_{L^1})|u|_{L^\infty} \\ &\quad + 3(|e_1|_{L^1} + |\beta|_{L^1})^2. \end{aligned}$$

Also, we see as in the proof of Theorem 2 (§4) that there exists a $\tau \in [0, 2\pi]$ such that

$$r < u(\tau) < R. \tag{5.14}$$

Next, we use Lemma 2.1 of [**7**] to deduce the existence of constants $\delta = \delta_1(\alpha) > 0, \delta_2 = \delta_2(\alpha) > 0$ such that

$$|\tilde{u}|_{L^\infty} \leq \delta_1 \Big|\frac{d^4\tilde{u}}{dx^4} - \alpha\tilde{u}'\Big|_{L^1} = \delta_1 \Big|\frac{d^4 u}{dx^4} - \alpha u'\Big|_{L^1} \tag{5.15}$$

$$|u'|_{L^\infty} \leq \delta_2 \Big|\frac{d^4\tilde{u}}{dx^4} - \alpha\tilde{u}'\Big| = \delta_2 \Big|\frac{d^4 u}{dx^4} - \alpha u'\Big|_{L^1} \tag{5.16}$$

for every $u \in C^3[0, 2\pi]$ with u''' absolutely continuous and satisfying the periodic boundary conditions in (5.12). Using, next, (5.15) in (5.13), we get

$$\begin{aligned} &\big[1 - \frac{\pi^2}{4}(\overline{\Gamma} + \eta)\big]\Big|\frac{d^4 u}{dx^4} - \alpha u'\Big|_{L^1}^2 \\ &\leq (|e_1|_{L^1} + |\beta|_{L^1})[2 + \delta_1(\overline{\Gamma} + \eta)]\Big|\frac{d^4 u}{dx^4} - \alpha u'\Big|_{L^1} \\ &\qquad + (\overline{\Gamma} + \eta)(|e_1|_{L^1} + |\beta|_{L^1})|\overline{u}| + 3(|e_1|_{L^1} + |\beta|_{L^1})^2. \end{aligned} \tag{5.17}$$

Also, it follows from (5.14), (5.16) that

$$\begin{aligned} |u(x)| = \Big|u(\tau) + \int_\tau^x u'(s)\,ds\Big| &< \max(-r, R) + 2\pi|u'|_{L^\infty} \\ &\leq \max(-r, R) + 2\pi\delta_2\Big|\frac{d^4 u}{dx^4} - \alpha u'\Big|_{L^1} \end{aligned}$$

so that

$$|\overline{u}| \leq \max(-r, R) + 2\pi\delta_2\Big|\frac{d^4 u}{dx^4} - \alpha u'\Big|_{L^1}. \tag{5.18}$$

Finally, it follows from (5.15), (5.17), (5.18) that there exists a constant ρ, independent of $\lambda \in [0, 1]$, such that

$$|u|_{L^\infty} \leq \rho.$$

This completes the proof of Theorem 3. □

REMARK 3. In the case when $\Gamma_0 = \Gamma_\infty = 0$ and $f \equiv \alpha$ in Theorem 2, we see that Theorem 3 improves the condition on Γ from $\overline{\Gamma} < 3/(2\pi^2)$ into $\overline{\Gamma} < 4/\pi^2$, (Note that $\delta(0) = 1/2$ in Lemma 1). In this sense, Theorem 3 is an extension of Theorem 2. However, if Γ is a constant, then Theorem 3 is not as sharp as Theorem 2.

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