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WILSON POLYNOMIALS AND SOME CONTINUED FRACTIONS OF RAMANUJAN

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In honor of W.J. Thron on his 70th birthday.

ABSTRACT. We obtain the general solution to the recurrence relation for Wilson polynomials for the special cases $a+b+c+d=1,2,\ldots$. We derive a subdominant solution and, thus, from Pincherle's theorem, an explicit expression for the associated continued fraction and weight function. In the cases $a+b+c+d=1,2$ this yields the convergence properties of some continued fractions of Ramanujan. We also indicate how these results may be generalized to the q-Askey-Wilson case.

1. Introduction. Wilson polynomials form the most general class of orthogonal hypergeometric polynomials in the Askey-Wilson chart [**1**]. Here we examine properties associated with these polynomials by obtaining a subdominant solution to their recurrence relation

$$X_{n+1} - (z - a_n)X_n + b_n^2 X_{n-1} = 0 \tag{1}$$

and applying Pincherle's theorem to the corresponding continued fraction

$$CF(z) = z - a_0 + \mathop{\mathbf{K}}_{n=1}^{\infty} \left(\frac{-b_n^2}{z - a_n} \right). \tag{2}$$

Definition 1. $X_n^{(s)}(z)$ is a subdominant solution of (1) at $z \in \mathbf{C}$ iff there exist linearly independent solutions $X_n^{(s)}(z)$, $X_n^{(d)}(z)$ with the property

$$\lim_{n\to\infty} X_n^{(s)}(z)/X_n^{(d)}(z) = 0. \tag{3}$$

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Theorem 2. (Pincherle, 1894, see [**5**]). *Let* $b_n^2 \neq 0$, $n \geq 1$. *Then* $CF(z)$ *converges iff* $X_n^{(s)}(z)$ *exists and, in this case,*

$$\frac{1}{CF(z)} = \frac{X_0^{(s)}(z)}{b_0^2 X_{-1}^{(s)}(z)}. \tag{4}$$

The denominators of the approximants to (4) are monic polynomials $P_n(z)$ which are initial value solutions $(P_0 = 1, P_1 = z - a_0)$ to (1) and orthogonal on the real axis with respect to a probability measure $dw(x)$ when $b_{n+1}^2 > 0$, $a_n \in \mathbf{R}$, $n \geq 0$. That is,

$$\int P_n(x) P_m(x)\, dw(x) = \delta_{nm} b_1^2 b_2^2 \cdots b_n^2. \tag{5}$$

Furthermore, one has the Cauchy representation

$$\frac{1}{CF(z)} = \int \frac{dw(x)}{z-x}, \qquad \operatorname{Im} z \neq 0. \tag{6}$$

From the real axis boundary values of (6), one can recover the weight function $w'(x)$ for x in the absolutely continuous spectrum and, by combining (4) and (6), obtain [**6**]

$$w'(x) = \frac{1}{2\pi i} \frac{W(X_{-1}^{(s)}(x+i0), X_{-1}^{(s)}(x-i0))}{b_0^2 |X_{-1}^{(s)}(x+i0)|^2}, \tag{7}$$

where the numerator in (7) is the Wronskian

$$W(X_n, Y_n) := X_n Y_{n+1} - Y_n X_{n+1}. \tag{8}$$

We recommend the use of (7) as a general means of obtaining $w'(x)$ in cases where $X_n^{(s)}(z)$ can be obtained explicitly and suggest that the term "classical" be reserved for such cases.

In what follows, we construct the subdominant solution to certain special cases of Wilson polynomials by applying a symmetry transformation to the Wilson solution and a transformation due to Whipple.

For two special subcases, with indeterminate ratios defined in terms of limits, we obtain continued fractions of Ramanujan and a weight function that differs from that given by Wilson [**9**].

We also indicate how these results may be extended to the q-Askey-Wilson case [**1**].

2. Wilson polynomials. The recurrence relation for *monic* Wilson polynomials [**8**] is given by (1) with

(9)

$$\begin{aligned} b_n^2(a,b,c,d) &= A_{n-1}(a,b,c,d)B_n(a,b,c,d),\\ a_n(a,b,c,d) &= A_n(a,b,c,d)+B_n(a,b,c,d)-a^2,\\ A_n(a,b,c,d) &= \frac{(n+a+b+c+d-1)(n+a+b)(n+a+c)(n+a+d)}{(2n+a+b+c+d-1)(2n+a+b+c+d)},\\ B_n(a,b,c,d) &= \frac{n(n+b+c-1)(n+b+d-1)(n+c+d-1)}{(2n+a+b+c+d-2)(2n+a+b+c+d-1)}. \end{aligned}$$

The Wilson polynomial solution (renormalized) is

(10)

$$\begin{aligned} X_n^{(1)}(z;a,b,c,d) = \\ &(-1)^n \frac{\Gamma(n+a+b+c+d-1)\Gamma(n+a+b)\Gamma(n+a+c)\Gamma(n+a+d)}{\Gamma(2n+a+b+c+d-1)}\\ &\times\ {}_4F_3\left(\begin{matrix} -n, n+a+b+c+d-1, a-i\sqrt{z}, a+i\sqrt{z}; 1\\ a+b, a+c, a+d\end{matrix}\right). \end{aligned}$$

The coefficients a_n, b_n^2 and the solution $X_n^{(1)}(z)/\Gamma(a+b)\Gamma(a+c)\Gamma(a+d)$ are symmetric functions of the four (in general, complex) parameters a, b, c, d.

3. New solution. An additional symmetry for the coefficients a_n, b_n^2 is given by

$$\begin{aligned} b_{-n}^2(1-a,1-b,1-c,1-d) &= b_n^2(a,b,c,d),\\ a_{-n-1}(1-a,1-b,1-c,1-d) &= a_n(a,b,c,d). \end{aligned} \tag{11}$$

Applying the transformation $n \to -n-1$, $(a,b,c,d) \to (1-a,1-b,1-c,1-d)$ to (1) and (10) and renormalizing yields a second polynomial solution

(12)

$$X_n^{(2)}(z;a,b,c,d) = (-1)^n \frac{\Gamma(n+1)\Gamma(n+b+c)\Gamma(n+b+d)\Gamma(n+c+d)}{\Gamma(2n+a+b+c+d-1)} \times {}_4F_3\left(\begin{matrix} n+1, -n-a-b-c-d+2, 1-a-i\sqrt{z}, 1-a+i\sqrt{z}; 1 \\ 2-a-b, 2-a-c, 2-a-d, \end{matrix}\right), \quad a+b+c+d = 1,2,\dots .$$

4. Subdominant solution. To construct the subdominant solution, one needs an appropriate linear combination of $X_n^{(1)}, X_n^{(2)}$ deduced from their large n behavior. For general values of the parameters, this is best accomplished by expressing solutions in terms of well-poised ${}_7F_6$ hypergeometric functions [**2**]. However, for the special values $a+b+c+d = 1,2,\dots$, the asymptotics may be obtained directly from the Whipple transform [**2**, 7.2(1)]:

$$(13) \quad {}_4F_3\left(\begin{matrix} -n, x, y, z; 1 \\ u, v, w \end{matrix}\right) = \frac{(v-z)_n(w-z)_n}{(v)_n(w)_n} {}_4F_3\left(\begin{matrix} -n, u-x, u-y, z; 1 \\ u, 1-v+z-n, 1-w+z-n \end{matrix}\right), \quad u+v+w = x+y+z-n+1.$$

Applying (13) to (10) yields, for $n \to \infty$,

(14)

$$X_n^{(1)} \sim C_n(a,b,c,d) n^{-2i\sqrt{z}} \frac{\Gamma(-2i\sqrt{z})\Gamma(a+b)\Gamma(a+c)\Gamma(a+d)}{\Gamma(a-i\sqrt{z})\Gamma(b-i\sqrt{z})\Gamma(c-i\sqrt{z})\Gamma(d-i\sqrt{z})},$$

$$C_n(a,b,c,d) = (-1)^n (2\pi)^{3/2} \left(\frac{n}{2}\right)^{a+b+c+d-3/2} \left(\frac{n}{2e}\right)^{2n}, \quad \operatorname{Im}\sqrt{z} > 0.$$

Applying (13) to (12) yields

$$(15) \quad X_n^{(2)} \sim C_n(a,b,c,d) n^{-2i\sqrt{z}} \times \frac{\Gamma(-2i\sqrt{z})\Gamma(2-a-b)\Gamma(2-a-c)\Gamma(2-a-d)}{\Gamma(1-a-i\sqrt{z})\Gamma(1-b-i\sqrt{z})\Gamma(1-c-i\sqrt{z})\Gamma(1-d-i\sqrt{z})}, \quad \operatorname{Im}\sqrt{z} > 0, \quad a+b+c+d = 1,2,\dots .$$

From (14) and (15) we deduce, for $\operatorname{Im}\sqrt{z} > 0$ and $a+b+c+d = 1, 2, \dots,$

(16)
$$X_n^{(s)}(z;a,b,c,d) = \alpha X_n^{(1)}(z;a,b,c,d) - \beta X_n^{(2)}(z;a,b,c,d),$$
$$\alpha = \frac{\Gamma(2-a-b)\Gamma(2-a-c)\Gamma(2-a-d)}{\Gamma(1-a-i\sqrt{z})\Gamma(1-b-i\sqrt{z})\Gamma(1-c-i\sqrt{z})\Gamma(1-d-i\sqrt{z})},$$
$$\beta = \frac{\Gamma(a+b)\Gamma(a+c)\Gamma(a+d)}{\Gamma(a-i\sqrt{z})\Gamma(b-i\sqrt{z})\Gamma(c-i\sqrt{z})\Gamma(d-i\sqrt{z})},$$

with large n behavior

$$X_n^{(s)}(z) \sim D_n(a,b,c,d)\Gamma(2i\sqrt{z})n^{2i\sqrt{z}}f(a,b,c,d,z),$$
$$D_n(a,b,c,d) = \frac{C_n(a,b,c,d)(1-a-b)(1-a-c)(1-a-d)}{\pi\sin\pi(a+b)\sin\pi(a+c)\sin\pi(a+d)},$$
$$f(a,b,c,d,z) = \sin\pi(a+i\sqrt{z})\sin\pi(b+i\sqrt{z})\sin\pi(c+i\sqrt{z})\sin\pi(d+i\sqrt{z}) \\ -\sin\pi(a-i\sqrt{z})\sin\pi(b-i\sqrt{z})\sin\pi(c-i\sqrt{z})\sin\pi(d-i\sqrt{z}).$$

From (4), (6), (10), (12) and (16) we obtain explicit expressions for the continued fraction and hence the weight function associated with the resolvent for the tridiagonal Jacobi matrix having diagonal and off-diagonal elements $(a_0, a_1, \dots)$ and $(b_1, b_2, \dots)$, respectively.

5. Extra special cases. For the two cases $a+b+c+d = 1, 2$ we obtain continued fractions given by Ramanujan and a weight function that differs from that given by Wilson [**9**] by taking limits $n \to 0$ in (9).

Case I. $a+b+c+d = 2$. From (10), (12) and (16) we obtain

$$\frac{X_0^{(s)}(z)}{b_0^2 X_{-1}^{(s)}(z)} = \frac{2(\Pi_1(i\sqrt{z}) - \Pi_2(i\sqrt{z}))/(\Pi_1(i\sqrt{z}) + \Pi_2(i\sqrt{z}))}{(a+b-1)(a+c-1)(a+d-1)} \tag{17}$$

(18)
$$\Pi_1(i\sqrt{z}) = \Gamma(a-i\sqrt{z})\Gamma(b-i\sqrt{z})\Gamma(c-i\sqrt{z})\Gamma(d-i\sqrt{z}),$$
$$\Pi_2(i\sqrt{z}) = \Gamma(1-a-i\sqrt{z})\Gamma(1-b-i\sqrt{z})\Gamma(1-c-i\sqrt{z})\Gamma(1-d-i\sqrt{z}).$$

With a change of notation,

$$\begin{aligned} a &= (1+k+l+m)/2, & b &= (1-k+l-m)/2, \\ c &= (1-k-l+m)/2, & d &= (1+k-l-m)/2, \end{aligned} \tag{19}$$

we obtain Ramanujan's Entry 35 [**3**] from Pincherle's theorem (equation (4)). That is,

$$\frac{2(\Pi_1(i\sqrt{z}) - \Pi_2(i\sqrt{z}))/klm}{(\Pi_1(i\sqrt{z}) + \Pi_2(i\sqrt{z}))} = \frac{1}{z - a_0 + \mathbf{K}_{n=1}^{\infty}\left(\frac{-b_n^2}{z-a_n}\right)}, \quad \operatorname{Im}\sqrt{z} > 0 \tag{20}$$

with

$$\begin{aligned} a_n &= (2n^2 + 2n + 1 - k^2 - l^2 - m^2)/4, \\ b_n^2 &= (n^2 - k^2)(n^2 - l^2)(n^2 - m^2)/4(4n^2 - 1), \end{aligned} \tag{21}$$

and, hence,

Theorem 3. *Ramanujan's Entry* 35 (*given by* (20)) *is valid iff either* $\operatorname{Im}\sqrt{z} > 0$, *or* k^2, l^2 *or* m^2 *is a positive integer squared.*

Case II. $a + b + c + d = 1$. From (10), (12) and (16) we obtain

$$\frac{X_0^{(s)}(z)}{b_0^2 X_{-1}^{(s)}(z)} = \frac{2}{z - a_0 - \Pi_2(i\sqrt{z})/\Pi_1(i\sqrt{z})}. \tag{22}$$

With the change of notation

$$\begin{aligned} a &= (1 + k + l + m)/4, \qquad & b &= (1 - k + l - m)/4, \\ c &= (1 - k - l + m)/4, \qquad & d &= (1 + k - l - m)/4, \end{aligned} \tag{23}$$

we obtain from Pincherle's theorem (equation (4)), for $\operatorname{Im}\sqrt{z} > 0$,

$$\frac{2}{z - a_0 - \Pi_2(i\sqrt{z})/\Pi_1(i\sqrt{z})} = \frac{1}{z - a_0 + \mathbf{K}_{n=1}^{\infty}\left(\frac{-b_n^2}{z-a_n}\right)}, \tag{24}$$

$$\begin{aligned} a_n &= (8n^2 + 1 - k^2 - l^2 - m^2 - 2klm/(4n^2 - 1))/16, \\ b_n^2 &= \frac{((2n-1)^2 - k^2)((2n-1)^2 - l^2)((2n-1)^2 - m^2)}{(16)^2(2n-1)^2}. \end{aligned} \tag{25}$$

For the subcase $m = 0$, we obtain Ramanujan's Entry 39 [**3**], and, hence,

Theorem 4. *Equation* (24) (*and hence Ramanujan's Entry* 39) *is valid iff either* $\operatorname{Im}\sqrt{z} > 0$, *or* k^2, l^2 *or* m^2 *is an odd integer squared.*

Comments. (a) Both (20) and (24) give a weight function that differs from that given by Wilson since, for these special cases, one has the first-degree polynomial $P_1(z) = z + a^2 - A_0 - B_0$, $B_0 \neq 0$ while Wilson [**9**] replaces (A_0, B_0) by $(A_0, 0)$ or $(2A_0, 0)$ in Case I or II, respectively.

(b) The condition $\operatorname{Im}\sqrt{z} > 0$ differs from the condition given by Berndt et al. [**3**] for Entry 39.

(c) Ramanujan's Entries 35 and 39 are two of the entries described by Berndt et al. as "especially enigmatic." They are important entries since Entry 35 implies Entries 18, 30, 31, 32iii, 33, 36, 37 and 38 while Entry 39 implies Entries 25, 26, 28 and 32i [**3**].

(d) From the special form of the coefficients $a_n = A_n + B_n - a^2$, $b_n^2 = A_{n-1}B_n$ (equation (9)), it follows that $1/CF(z)$ is the even part of the S-fraction

$$\cfrac{1}{z + a^2 - B_0 - \cfrac{A_0}{1 - \cfrac{B_1}{z + a^2 - \cfrac{A_1}{1 - \ddots}}}}.$$

(See [**7**, Chapter 12].)

6. Wilson cases. For normal special cases $a + b + c + d = 3, 4, \dots$, one has $B_0 = 0$. These same calculations yield, for $\operatorname{Im}\sqrt{z} > 0$,

(26a)

$$\begin{aligned}\frac{1}{CF(z)} &= -\frac{\Gamma(1-a-b)\Gamma(1-a-c)\Gamma(1-a-d)\Pi_1(i\sqrt{z})\Gamma(a+b+c+d)}{\Gamma(b+c)\Gamma(b+d)\Gamma(c+d)\Pi_2(i\sqrt{z})} \\ &\quad - \frac{(a+b+c+d-1)}{(a+b-1)(a+c-1)(a+d-1)} \\ &\quad \times {}_4F_3\left(\begin{matrix}1, 2-a-b-c-d, 1-a-i\sqrt{z}, 1-a+i\sqrt{z}; 1\\ 2-a-b, 2-a-c, 2-a-d\end{matrix}\right),\end{aligned}$$

and, hence, for $x > 0$,

(26b)

$$w'(x) = \frac{1}{2\pi i} \frac{\Gamma(1-a-b)\Gamma(1-a-c)\Gamma(1-a-d)}{\Gamma(b+c)\Gamma(b+d)\Gamma(c+d)} \left(\frac{\Pi_1(i\sqrt{x})}{\Pi_2(i\sqrt{x})} - \frac{\Pi_1(-i\sqrt{x})}{\Pi_2(-i\sqrt{x})} \right) \times \Gamma(a+b+c+d).$$

These results agree with Wilson [**9**].

The substitution in (26a) and (26b) of $a+b+c+d = 1$ or 2 yields additional Wilson cases with the (A_0, B_0) in Section 5 replaced by $(2A_0, 0)$ or $(A_0, 0)$, respectively (see comments (a) and (b) above).

Note that, for real orthogonality in the Wilson cases, (26a) yields an explicit discrete spectrum in $(-\infty, 0)$ from the pole singularities of Π_1. Thus, mass points exist for $x \in (-\infty, 0)$ if and only if one of the parameters, say a, is negative [**9**].

For the exceptional cases of Section 5, this discrete spectrum is no longer explicit. In Case I, for example, one must examine the location of the zeros of $\Pi_1 + \Pi_2$. However, from the location of the poles of Π_1 and Π_2, it clear that mass points in $(-\infty, 0)$ exist for this exceptional case if two of the parameters, say a and b, satisfy $a < 0$ and $b > 1$. From the positivity of Π_1, Π_2 for $z \leq 0$, there can be no mass points in $(-\infty, 0]$ if all real parameters are between 0 and 1. Also, by considering the case $a = -\varepsilon$, $b = c = d = 2/3 + \varepsilon/3$, $\varepsilon > 0$, one sees that a negative parameter is no longer sufficient for the existence of a mass point in $(-\infty, 0)$.

For the exceptional Case II orthogonality, one has a mass point in $(-\infty, 0)$ if some parameter, say a, satisfies $a > 1$, since this implies that $\Pi_2 = \infty$ for some $z \leq 0$. Hence, (24) has a zero in $(-\infty, 0]$ and therefore a mass point in $(-\infty, 0)$. If all parameters are between 0 and 1, there are no mass points in $(-\infty, 0)$ from the positivity of a_0, Π_1 and Π_2. Here, again, a negative parameter does not imply the existence of a mass point in $(-\infty, 0)$ from a consideration of the case $a = -\varepsilon$, $b = c = d = 1/3 + \varepsilon/3$, $\varepsilon > 0$.

7. q-Askey-Wilson polynomials [**1**]. The classical hypergeometric polynomials have q-series generalizations which are expressed in terms of *basic* hypergeometric functions

$$ {}_{r+1}\phi_{r+j}\left[\begin{matrix} x_0,\dots,x_r \\ y_1,\dots,y_{r+j}\end{matrix};\text{q,t}\right]=\sum_{n=0}^{\infty}\frac{[x_0]_n\cdots[x_r]_n(-1)^{jn}q^{\frac{jn(n-1)}{2}}}{[y_1]_n\cdots[y_{r+j}]_n[q]_n}t^n, $$
$$ [x]_n=(1-x)(1-xq)\cdots(1-xq^{n-1}),\quad [x]_0=1. $$

The q-analogue of the Wilson case is the q-Askey-Wilson case with the recurrence relation (1) having

(27)

$$ \begin{aligned} b_n^2(a,b,c,d;q)&=A_{n-1}(a,b,c,d;q)B_n(a,b,c,d;q),\\ a_n(a,b,c,d;q)&=-A_n(a,b,c,d;q)-B_n(a,b,c,d;q)+\frac{a}{2}+\frac{1}{2a},\\ A_n(a,b,c,d;q)&=\frac{(1-abcdq^{n-1})(1-abq^n)(1-acq^n)(1-adq^n)}{2a(1-abcdq^{2n-1})(1-abcdq^{2n})},\\ B_n(a,b,c,d;q)&=\frac{a(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{2(1-abcdq^{2n-2})(1-abcdq^{2n-1})}. \end{aligned} $$

The Askey-Wilson polynomial solution (renormalized) is

$$ \begin{aligned} X_n^{(1)}(z;a,b,c,d;q)=&\frac{[abcdq^{-1}]_n[ab]_n[ac]_n[ad]_n}{(2a)^n[abcdq^{-1}]_{2n}}\\ &\times\ {}_4\phi_3\left(\begin{matrix} q^{-n},abcdq^{n-1},ae^{i\theta},ae^{-i\theta},q,q \\ ab,ac,ad\end{matrix}\right) \end{aligned} \tag{28} $$

with $z=\cos\theta$.

In addition to the permutation symmetry with respect to the parameters (a,b,c,d), one has

$$ \begin{aligned} b_{-n}^2(q/a,q/b,q/c,q/d;q)&=b_n^2(a,b,c,d;q),\\ a_{-n-1}(q/a,q/b,q/c,q/d;q)&=a_n(a,b,c,d;q). \end{aligned} \tag{29} $$

Applying the transformation $n\to -n-1$, $(a,b,c,d)\to(q/a,q/b,q/c,q/d)$

to (1) and (28) and renormalizing yields a second polynomial solution

(30)
$$X_n^{(2)}(z;a,b,c,d;q) = \frac{[q]_n[bc]_n[cd]_n[bd]_n}{(2q/a)^n[abcd]_{2n-1}} \times {}_4\phi_3\left(\begin{matrix} q^{n+1}, q^{-n+2}/abcd, qe^{i\theta}/a, qe^{-i\theta}/a; q, q \\ q^2/ab, q^2/ac, q^2/ad \end{matrix}\right), \quad abcd = q^1, q^2, \dots .$$

For these special cases $abcd = q^m$, $m = 1, 2, \dots$, $|q| < 1$, one can obtain the large n asymptotics of both $X_n^{(1)}$ and $X_n^{(2)}$ from the Sear's transformation [**4**]

(31)
$$ {}_4\phi_3\left(\begin{matrix} q^{-n}, x, y, z; q, q \\ u, v, w \end{matrix}\right) = \left(\frac{yz}{u}\right)^n \frac{[uv/yz]_n[uw/yz]_n}{[v]_n[w]_n} \times {}_4\phi_3\left(\begin{matrix} q^{-n}, x, u/y, u/z; q, q \\ u, uv/yz, uw/yz \end{matrix}\right), \quad uvw = xyzq^{-n+1}.$$

One can then obtain the subdominant solution $X_n^{(s)}(z)$ to (1) for $z \notin [-1,1]$, $abcd = q^m$, $m = 1, 2, \dots$, and, in particular, the q-analogue of (20), (24) and (26a,b).

For more general values of $abcd \neq 0$, one can express the subdominant solution in terms of well-poised ${}_8\phi_7$ basic hypergeometric functions instead of Saalschützian ${}_4\phi_3$'s using Watson's formulas [**2**, p. 69]. This will yield a general proof of the Askey-Wilson orthogonality which avoids the use of contour integration [**1**].

With solutions to (1) expressed in terms of well-poised ${}_8\phi_7$'s (${}_7F_6$'s) one may also introduce an extra parameter by a translation of n ($n \to n + e$) and do the large n asymptotics to obtain the subdominant solution for the associated q-Askey-Wilson (Wilson) case.

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