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TWO FAMILIES OF ORTHOGONAL POLYNOMIALS RELATED TO JACOBI POLYNOMIALS

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Dedicated to Wolfgang Thron on his 70th birthday

ABSTRACT. A family of orthogonal polynomials that generalize Jacobi polynomials is introduced. The exceptional case $\alpha+\beta=0$ of Jacobi polynomials is investigated.

1. Introduction. The Jacobi polynomials $\{P_n^{\alpha,\beta}(x)\}$ satisfy the three term recurrence relation

$$\begin{aligned}(1.1)\quad & 2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)P_{n+1}^{\alpha,\beta}(x)\\ &\quad = (2n+\alpha+\beta+1)[(2n+\alpha+\beta)(2n+\alpha+\beta+2)x+(\alpha^2-\beta^2)]\\ &\qquad \cdot P_n^{\alpha,\beta}(x) - 2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)P_{n-1}^{\alpha,\beta}(x)\end{aligned}$$

and the initial conditions

$$(1.2)\qquad P_0^{\alpha,\beta}(x)=1,\quad P_1^{\alpha,\beta}(x)=[x(\alpha+\beta+2)+\alpha-\beta]/2.$$

When $\alpha>-1$ and $\beta>-1$, the Jacobi polynomials are orthogonal on $[-1,1]$ with respect to the beta distribution $(1-x)^\alpha(1+x)^\beta\,dx$. When $\alpha+\beta\neq 0$, the Jacobi polynomials are well defined through (1.1), but when $\alpha+\beta=0$ one must be careful in defining the P_1. If we use (1.2), it is then clear that $P_1=x+\alpha$. On the other hand, if we let $\alpha+\beta=0$ in (1.1), then use (1.1) with the initial conditions $P_{-1}=0$ and $P_0=1$ and compute P_1 from the recursion (1.1), we will see that in addition to the option $P_1=x+\alpha$ we may also choose $P_1=x$. The former choice leads to the standard Jacobi polynomials, [**8, 16**], while the latter choice

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leads to what we call the exceptional Jacobi polynomials. In Section 2 we shall study the exceptional Jacobi polynomials in some detail. We shall prove that the exceptional Jacobi polynomials are orthogonal on $[-1,1]$ with respect to the weight function

(1.3)
$$w(x,\alpha) = \frac{2\sin(\pi\alpha)}{\pi\alpha}\frac{(1-x^2)^\alpha}{(1-x)^{2\alpha}+2\cos(\pi\alpha)(1-x^2)^\alpha+(1+x)^{2\alpha}}.$$

The weight function $w(x,\alpha)$ is normalized to have total mass 1, i.e.,

$$\int_{-1}^{1} w(x,\alpha)\,dx = 1.$$

The above definite integral is equivalent to

(1.4)
$$\int_0^\infty \frac{t^\alpha(1+t)^{-2}}{1+2\cos(\pi\alpha)t^\alpha+t^{2\alpha}}dt = \frac{\Gamma(1-\alpha)\Gamma(1+\alpha)}{4}, \quad -1<\alpha<1.$$

It is not difficult to prove (1.4) by integrating $(1+z)^{-2}(1+e^{-i\pi\alpha}z^\alpha)^{-1}$ over a keyhole contour with a small circle around the origin.

The integral in (1.4) resembles a beta integral. In fact, the form (1.4) suggests the existence of a two-parameter generalization of (1.4). For recent work on beta type integrals, we refer readers to R. Askey's interesting series of articles [**1**–**3**].

In [**19**], J. Wimp defined associated Jacobi polynomials as the solution to

(1.5)
$$\begin{aligned}&2(n+c+1)(n+c+\alpha+\beta+1)(2n+2c+\alpha+\beta)P_{n+1}^{\alpha,\beta}(x;c)\\&=(2n+2c+\alpha+\beta+1)[(2n+2c+\alpha+\beta)(2n+2c+\alpha+\beta+2)x+(\alpha^2-\beta^2)]\\&\cdot P_n^{\alpha,\beta}(x;c)-2(n+c+\alpha)(n+c+\beta)(2n+2c+\alpha+\beta+2)P_{n-1}^{\alpha,\beta}(x;c),\end{aligned}$$

which satisfies the initial conditions

(1.6)
$$P_{-1}^{\alpha,\beta}(x;c)=0, \quad P_0^{\alpha,\beta}(x;c)=1.$$

Associated Jacobi polynomials, as Wimp points out, arise when we replace n by $n+c$ in the coefficients in the recurrence relation defining the Jacobi polynomials. Thus, Wimp's associated Jacobi polynomials reduce to the familiar Jacobi polynomials when $c = 0$ and $\alpha+\beta \neq 0$. For recent work on associated classical polynomials, we refer the interested reader to [**5, 7, 10–13, 19**] and their references.

In Sections 3, 4, and 5 we study a different class of associated Jacobi polynomials. The motivation comes from stochastic processes. A birth and death process with birth rates $\{\lambda_n\}$ and death rates $\{\mu_n\}$ generates a family of orthogonal polynomials $\{Q_n(x)\}$ via

$$Q_{-1}(x) = 0, \quad Q_0(x) = 1,$$
$$\lambda_n Q_{n+1}(x) = (\lambda_n + \mu_n - x)Q_n(x) - \mu_n Q_{n-1}(x), \quad n \geq 0. \tag{1.7}$$

The birth and death rates are assumed to satisfy

$$\lambda_n > 0, \quad \mu_{n+1} > 0 \quad \text{for } n \geq 0, \text{ but } \mu_0 \geq 0. \tag{1.8}$$

A family of birth and death process polynomials is orthogonal on a subset of $[0, \infty)$.

Wimp's associated Jacobi polynomials are birth and death process polynomials with rates

(1.9)
$$\lambda_n = \frac{2(n+c+1+\beta)(n+c+1+\alpha+\beta)}{(2n+2c+1+\alpha+\beta)(2n+2c+2+\alpha+\beta)}, \quad n \geq 0,$$
$$\mu_n = \frac{2(n+c)(n+c+\alpha)}{(2n+2c+\alpha+\beta)(2n+2c+1+\alpha+\beta)}, \quad n \geq 0,$$

$$Q_n(x) = (-1)^n[(1+c)_n/(1+c+\beta)_n]P_n^{\alpha,\beta}(x-1;c). \tag{1.10}$$

When $\mu_0 \neq 0$, it was observed in [**10**] and [**11**] that a companion process arises by redefining μ_0 to be zero, and this naturally leads to a second family of orthogonal polynomials. Thus, we assume that

$$\lambda_n \text{ and } \mu_n \text{ are as in (1.9), but we choose } \mu_0 = 0. \tag{1.11}$$

We shall denote the family of polynomials generated by the rates (1.11) by $\{\mathcal{Q}_n(x)\}$. Let

$$\mathcal{Q}_n(x) = (-1)^n[(1+c)_n/(1+c+\beta)_n]\mathcal{P}_n^{\alpha,\beta}(\mathcal{P}_n^{\alpha,\beta}(x-1;c). \tag{1.12}$$

The $\mathcal{P}_n$'s constitute the second family of associated Jacobi polynomials. Following Wimp [**19**], we use the more convenient polynomials

$$R_n^{\alpha,\beta}(x;c) = P_n^{\alpha,\beta}(2x-1;c) \text{ and } \mathcal{R}_n^{\alpha,\beta}(x;c) = \mathcal{P}_n^{\alpha,\beta}(2x-1;c). \tag{1.13}$$

Throughout the rest of this work we shall use only the R_n's and $\mathcal{R}_n$'s and will refer to them as associated Jacobi polynomials.

In Section 3 we derive explicit representations for the $\mathcal{R}_n$'s. They are stated as Theorem 3.3 and Theorem 3.10. We also give the value of an $\mathcal{R}_n$ at $x = 0$. In Section 4 we derive a generating function for the $\mathcal{R}_n$'s and determine the asymptotic behavior of the $\mathcal{R}_n$'s as $n \to \infty$. In Section 5 we prove that the J-fraction associated with the $\mathcal{R}_n$'s is a quotient of two hypergeometric functions. We also find the weight function of the $\mathcal{R}_n$'s and record the orthogonality relation.

2. Exceptional Jacobi polynomials. Recall that the exceptional Jacobi polynomials $\{\mathcal{P}_n^\alpha(x)\}$ are generated by the recurrence relation (1.1) for $n > 0$, with $\alpha = -\beta$, and the initial conditions

$$\mathcal{P}_0^\alpha(x) = 1, \quad \mathcal{P}_1^\alpha(x) = x. \tag{2.1}$$

It is easy to see that

$$\mathcal{P}_n^\alpha(x) = \lim_{c\to 0+} P_n^{\alpha,-\alpha}(x;c). \tag{2.2}$$

Wimp [**19**] found explicit representations, a generating function, and the weight function for his associated Jacobi polynomials. He also gave a fourth-order differential equation satisfied by his associated Jacobi polynomials. In the case of exceptional Jacobi polynomials, Wimp's formulas simplify considerably, so we will state them. Wimp [**19**] gave two explicit representations for his polynomials. They are (19) and (28) on pages 987 and 988 of [**19**]. When $\alpha + \beta = 0$, both representations reduce to

$$\begin{aligned}\mathcal{P}_n^\alpha(x) = \frac{(-1)^n}{2(n!)}\{&(1+\alpha)_n\, {}_2F_1(-n, n+1; 1+\alpha; (1+x)/2) \\ &+ (1-\alpha)_n\, {}_2F_1(-n, n+1; 1-\alpha; (1+x)/2)\}.\end{aligned} \tag{2.3}$$

Since

$$P_n^{\alpha,\beta}(x) = \frac{(-1)^n(\beta+1)_n}{n!}\, {}_2F_1(-n, n+\alpha+\beta+1; \beta+1; (1+x)/2),$$

we rewrite (2.3) in the form

$$\mathcal{P}_n^{\alpha}(x) = \frac{1}{2}[P_n^{\alpha,-\alpha}(x) + P_n^{-\alpha,\alpha}(x)]. \tag{2.4}$$

It is straightforward to use (2.4) and derive generating functions for the polynomials under consideration and determine their asymptotic development from the corresponding results for Jacobi polynomials [**16**].

The corresponding J-fraction converges to the Stieltjes transform of the measure of orthogonality, $d\mu$, say. Thus,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{d\mu(t)}{z-t} &= \frac{(1-\zeta)^{-\alpha}(1+\zeta)^{\alpha} - (1-\zeta)^{\alpha}(1+\zeta)^{-\alpha}}{\alpha[(1-\zeta)^{\alpha}(1+\zeta)^{-\alpha} + (1-\zeta)^{-\alpha}(1+\zeta)^{\alpha}]},\\ \zeta &= z - \sqrt{z^2-1}, |\zeta| < 1,\ z \notin [-1,1].\end{aligned} \tag{2.5}$$

We then apply the inversion formula for the Stieltjes transform [**15**] and find the weight function. The orthogonality relation is

$$\int_{-1}^{1} \mathcal{P}_n^{\alpha}(x)\mathcal{P}_m^{\alpha}(x) w(x;\alpha)\, dx = \frac{(1+\alpha)_n(1-\alpha)_n}{(2n+1)(n!)^2}\delta_{m,n}, \tag{2.6}$$

where the normalized weight function $w(x;\alpha)$ is given by (1.3).

One way to find a fourth-order differential equation satisfied by the exceptional Jacobi polynomials is to let $\beta = -\alpha$, $\gamma\ (= \alpha+\beta+1) = 1$, and let $c \to 0$ in the differential equation Wimp derived for his polynomials, [**19**, (48), p. 993]. Another way is to use the differential and recursion properties of the Jacobi polynomials as follows. The operator

$$\begin{aligned}D(\alpha,\beta;n)f(x) := (1-x^2)f''(x) &+ [\beta-\alpha-(\alpha+\beta+2)x]f'(x)\\ &+ n(n+\alpha+\beta+1)f(x)\end{aligned} \tag{2.7}$$

annihilates a Jacobi polynomial $P_n^{\alpha,\beta}(x)$, Szegö [**16**, §4.2]. Using the relationship

$$\frac{d}{dx}P_n^{\alpha,\beta}(x) = \frac{1}{2}(\alpha+\beta+n+1)P_{n-1}^{\alpha+1,\beta+1}(x),$$

Szegö [**16**, §4.21], and (2.4) we see that

$$\begin{aligned}&D(1-\alpha, 1+\alpha; n-1)D(\alpha, -\alpha; n)\mathcal{P}_n^{\alpha}(x)\\ &\qquad\qquad = -4\alpha D(1-\alpha, 1+\alpha; n-1)\frac{d}{dx}P_n^{-\alpha,\alpha}(x) = 0.\end{aligned}$$

Therefore, the exceptional Jacobi polynomials satisfy the differential equation

$$D(1-\alpha, 1+\alpha; n-1)D(\alpha, -\alpha; n)\mathcal{P}_n^{\alpha}(x) = 0. \tag{2.8}$$

The fourth-order differential equation (2.8) is the same as the equation resulting from setting $\beta = -\alpha, \gamma = 1$, and letting $c \to 0$ in (48) on page 993 of [**19**].

3. Explicit representations for associated Jacobi polynomials. Both families of associated Jacobi polynomials satisfy the same recurrence relation but different initial conditions. Furthermore, the polynomials $\{R_n^{\alpha,\beta}(x;c)\}$ and $\{R_{n-1}^{\alpha,\beta}(x;c+1)\}$, $n \geq 0$, also satisfy the same recurrence relation and are linearly independent functions of n. Therefore,

$$\mathcal{R}_n^{\alpha,\beta}(x;c) = AR_n^{\alpha,\beta}(x;c) + BR_{n-1}^{\alpha,\beta}(x;c+1)$$

where A and B do not depend on n but may depend on x. The initial conditions

$$\mathcal{R}_0^{\alpha,\beta}(x;c) = 1, \quad \mathcal{R}_1^{\alpha,\beta}(x;c) = \frac{(\beta+c+1)(2x-\lambda_0)}{\lambda_0(c+1)},$$

with λ_0 as in (1.9), give $A = 1$ and $B = c(c+\alpha)(2c+1+\gamma)/[(c+1)(c+\gamma)(2c+\gamma-1)]$, so that

$$\mathcal{R}_n^{\alpha,\beta}(x;c) = R_n^{\alpha,\beta}(x;c) + \frac{c(c+\alpha)(2c+1+\gamma)}{(c+1)(c+\gamma)(2c-1+\gamma)}R_{n-1}^{\alpha,\beta}(x;c+1), \tag{3.1}$$

where

$$\gamma = \alpha + \beta + 1. \tag{3.2}$$

Theorem 3.3. *The $\mathcal{R}_n$'s have the explicit representation*

$$\begin{aligned}(3.4)\quad & \mathcal{R}_n^{\alpha,\beta}(x;c) = \\ & (-1)^n \frac{(\gamma+2c)_n(\beta+c+1)_n}{(c+\gamma)_n n!} \sum_{k=0}^{n} \frac{(-n)_k(\gamma+n+2c)_k}{(1+c)_k(c+1+\beta)_k} x^k \\ & \cdot {}_4F_3\left(\begin{matrix} k-n, n+\gamma+k+2c, c+1+\beta, c \\ k+c+1+\beta, k+c+1, \gamma+2c \end{matrix} \middle| 1\right).\end{aligned}$$

Proof. We use the representation, due to J. Wimp [**19**, p. 987],

$$\begin{aligned}(3.5)\quad & R_n^{\alpha,\beta}(x;c) = \\ & (-1)^n \frac{(\gamma+2c)_n(\beta+c+1)_n}{(c+\gamma)_n n!} \sum_{k=0}^{n} \frac{(-n)_k(\gamma+n+2c)_k}{(1+c)_k(c+1+\beta)_k} x^k \\ & \cdot {}_4F_3\left(\begin{matrix} k-n, n+\gamma+k+2c, c+\beta, c \\ k+c+1+\beta, k+c+1, \gamma+2c-1 \end{matrix} \middle| 1\right),\end{aligned}$$

and the relationship (3.1) to get

(3.6)

$$\begin{aligned}& \mathcal{R}_n^{\alpha,\beta}(x;c) = (-1)^n \frac{(\gamma+2c)_n(\beta+c+1)_n}{(c+\gamma)_n n!} \sum_{k=0}^{n} \frac{(-n)_k(n+2c+\gamma)_k}{(c+1)_{k+1}(c+1+\beta)_{k+1}} x^k \\ & \cdot \Bigg[(c+1+k)(k+c+1+\beta)\, {}_4F_3\left(\begin{matrix} k-n, n+\gamma+k+2c, c+\beta, c \\ k+c+1+\beta, k+c+1, \gamma+2c-1 \end{matrix} \middle| 1\right) \\ & \qquad + \frac{c(c+\alpha)(k-n)(k+n+2c+\gamma)}{(\gamma+2c)(\gamma+2c-1)} \\ & \qquad \cdot {}_4F_3\left(\begin{matrix} k-n+1, n+\gamma+k+2c+1, c+\beta+1, c+1 \\ k+c+2+\beta, k+c+2, \gamma+2c+1 \end{matrix} \middle| 1\right)\Bigg].\end{aligned}$$

Wilson [**18**, (4.3)] proved the following contiguous relation for a ${}_4F_3$ series,

$$(3.7)\quad {}_4F_3(A,B,C,D;E,F,G;1) - {}_4F_3(A+1,B,C,D;E+1,F,G;1) = \frac{(A-E)BCD}{E(E+1)FG}\,{}_4F_3(A{+}1,B{+}1,C{+}1,D{+}1;E{+}2,F{+}1,G{+}1;1),$$

provided that one of the numerator parameters is a negative integer and the ${}_4F_3$ is balanced (Saalschützian), that is, the sum of the denominator parameters exceeds the sum of the numerator parameters by 1. In (3.7) we set

$$A = c+\beta,\quad B = k-n,\quad C = c,\quad D = n+k+2c+\gamma,$$
$$E = \gamma+2c-1,\quad F = k+c+1,\quad G = k+c+\beta+1.$$

Hence $A-E = \beta-\gamma-c+1 = -c-\alpha$. A calculation now gives (3.4), and the proof is complete. □

In his interesting work [**17**], J.A. Wilson outlined a systematic way of deriving contiguous relations like (3.7). Wilson's dissertation [**18**] contains a valuable complete list of contiguous relations for terminating balanced ${}_4F_3$'s.

Corollary 3.8. *We have the explicit evaluation*

$$(3.9)\qquad \mathcal{R}_n^{\alpha,\beta}(0;c) = (-1)^n \frac{(c+1+\beta)_n}{(c+1)_n}.$$

Proof. Set $x=0$ in (3.4) to obtain

$$\mathcal{R}_n^{\alpha,\beta}(0;c) = (-1)^n \frac{(\gamma{+}2c)_n(\beta{+}c{+}1)_n}{(c{+}\gamma)_n n!}\,{}_4F_3\left(\begin{matrix} -n, n{+}\gamma{+}2c, c{+}1{+}\beta, c \\ c{+}1{+}\beta, c{+}1, \gamma{+}2c \end{matrix}\,\middle|\, 1\right)$$
$$= (-1)^n \frac{(\gamma+2c)_n(\beta+c+1)_n}{(c+\gamma)_n n!}\,{}_3F_2\left(\begin{matrix} -n, n+\gamma+2c, c \\ c+1, \gamma+2c \end{matrix}\,\middle|\, 1\right).$$

By Saalschütz's theorem, Bailey [**6**, p. 9], the ${}_3F_2$ above sums to $\{(-n)_n(c+\gamma)_n\}/\{(\gamma+2c)_n(-c-n)_n\}$ and (3.9) follows after simple manipulations. □

We now derive another representation of the $\mathcal{R}_n$'s. The new representation will be used in the next section to determine the asymptotic behavior of $\mathcal{R}_n$ for large n and fixed x.

Theorem 3.10. *We have*

(3.11)

$$(-1)^n \mathcal{R}_n^{\alpha,\beta}(x;c) = \frac{(c+1+\beta)_n}{(c+1)_n}\, {}_2F_1(-n-c, n+c+\gamma; \beta+1; x) \cdot {}_2F_1(c, 1-c-\gamma; -\beta; x) - \frac{c(c+\alpha)_{n+1}}{\beta(\beta+1)(c+\gamma)_n} x \cdot {}_2F_1(n+c+1, 1-n-c-\gamma; 1-\beta; x) \cdot {}_2F_1(1-c, c+\gamma; 2+\beta; x).$$

Proof. Wimp [**19**, (28) p. 988] proved

(3.12)

$$R_n^{\alpha,\beta}(x;c) = \frac{(-1)^n(\gamma+c+1)/\beta}{(\gamma+2c-1)(\beta+c)_{n+1}(c+1)_n}\, {}_2F_1(c, 2-\gamma-c; 1-\beta; x) \cdot {}_2F_1(-n-c, n+c+\gamma; 1+\beta; x) + \frac{(-1)^{n+1}c/\beta}{(\gamma+2c-1)(\gamma+c)_n(\alpha+c)_{n+1}} \cdot {}_2F_1(1-c, c+\gamma-1; 1+\beta; x) \cdot {}_2F_1(n+c+1, 1-n-c-\gamma; 1-\beta; x).$$

We now substitute for the R_n's in (3.1) by their corresponding expressions from (3.12) and establish the representation

$$(-1)^n \mathcal{R}_n^{\alpha,\beta}(x;c) = \frac{\Gamma(n+c+1+\beta)}{\Gamma(n+c+1)} C\, {}_2F_1(-n-c, n+c+\gamma; 1+\beta; x) + \frac{\Gamma(n+c+1+\alpha)}{\Gamma(n+c+\gamma)} D \cdot {}_2F_1(n+c+1, 1-n-c-\gamma; 1-\beta; x),$$

where C and D are given by

$$\begin{aligned}\beta(\gamma+2c-1)\Gamma(c+1+\beta)C &= \Gamma(c+1)[(\beta+c)(\gamma+c-1)\\ &\quad\cdot {}_2F_1(c,2-c-\gamma;1-\beta;x)\\ &\quad - c(c+\alpha)\,{}_2F_1(1+c,1-c-\gamma;1-\beta;x)],\\ \beta(\gamma+2c-1)\Gamma(c+\alpha)D &= c\Gamma(c+\gamma)[\,{}_2F_1(-c,c+\gamma;1+\beta;x)\\ &\quad - {}_2F_1(1-c,\gamma+c-1;1+\beta;x)].\end{aligned}$$

To simplify C and D, we expand the respective hypergeometric functions in powers of x and collect the coefficients of like powers. A fantastic amount of simplification occurs and we find

$$\begin{aligned}C &= [\Gamma(c+1)/\Gamma(c+1+\beta)]\,{}_2F_1(c,1-\gamma-c;-\beta;x),\\ (\beta)_2 D &= -[c\Gamma(\gamma+c)x/\Gamma(c+\alpha)]\,{}_2F_1(1-c,\gamma+c;\beta+2;x).\end{aligned}$$

This completes the proof of Theorem 3.10. □

Note that Corollary 3.8 also follows from (3.11).

4. Asymptotics and a generating function. The following asymptotic formula (Erdélyi et al. [**9**, (17), p. 77])

$$\begin{aligned}(4.1)\quad &{}_2F_1(a+n,b-n;c,\sin^2\theta)\\ &\approx \frac{\Gamma(c)n^{-c+1/2}(\cos\theta)^{c-a-b-1/2}}{\sqrt{\pi}(\sin\theta)^{c-1/2}}\cos\left[2n\theta+(a-b)\theta+\frac{\pi}{2}\left(\frac{1}{2}-c\right)\right],\end{aligned}$$

as $n\to\infty$, $\theta\in(0,\pi)$, is due to G.N. Watson. The first result in this section is an asymptotic formula for the $\mathcal{R}_n$'s.

Theorem 4.2. *For $\theta\in(0,\pi/2)$ and $n\to\infty$, we have*

$$\begin{aligned}(4.3)\quad \mathcal{R}_n^{\alpha,\beta}(\sin^2\theta;c) &\approx \frac{\Gamma(\beta+1)\Gamma(c+1)}{\sqrt{n\pi}\,\Gamma(c+1+\beta)}(\cos\theta)^{-\alpha-1/2}(\sin\theta)^{-\beta-1/2}\\ &\quad W(\theta)\cos[(2n+2c+\gamma)\theta+(n-1/4)\pi-\eta],\end{aligned}$$

with

$$\begin{aligned}(4.4)\quad W(\theta) = |\,&{}_2F_1(c,-c-\beta-\alpha;-\beta;\sin^2\theta)+\mathcal{K}e^{i\pi\beta}(\sin\theta)^{2\beta+2}\\ &\cdot {}_2F_1(c+1+\beta,1-c-\alpha;2+\beta;\sin^2\theta)|\end{aligned}$$

where

$$\mathcal{K} = \frac{\Gamma(c+\gamma)\Gamma(c+1+\beta)}{\Gamma(c)\Gamma(c+\alpha)\Gamma(2+\beta)}\Gamma(-\beta) \tag{4.5}$$

and η, which depends on θ, is given by (4.8) *and* (4.9).

Proof. Apply (4.1) to the n-dependent ${}_2F_1$'s in (3.11) and use $\Gamma(n+a)/\Gamma(n+b) \approx n^{a-b}$ to obtain

(4.6)

$$\begin{aligned}(-1)^n\sqrt{\pi n}\mathcal{R}_n^{\alpha,\beta}(\sin^2\theta;c)\\ &\approx \frac{\Gamma(\beta+1)\Gamma(c+1)}{\Gamma(\beta+c+1)}(\cos\theta)^{-\alpha-1/2}(\sin\theta)^{-\beta-1/2}\\ &\quad\cdot {}_2F_1(c,1-c-\gamma;-\beta;\sin^2\theta)\\ &\quad\cdot \cos(\xi_n-\frac{\pi}{2}\beta)+\frac{c\Gamma(c+\gamma)\Gamma(-\beta)}{(\beta+1)\Gamma(c+\alpha)}(\cos\theta)^{\alpha-1/2}(\sin\theta)^{\beta+3/2}\\ &\quad\cdot {}_2F_1(1-c,c+\gamma;2+\beta;\sin^2\theta)\cos\left(\xi_n+\frac{\pi}{2}\beta\right),\end{aligned}$$

where

$$\xi_n = (2n+2c+\gamma)\theta - \pi/4. \tag{4.7}$$

We then apply the Kummer transformation

$${}_2F_1(a,b;c;x) = (1-x)^{c-a-b}\,{}_2F_1(c-a,c-b;c;x),$$

Bailey [**6**, (2), p. 2], to the function ${}_2F_1(1-c,\gamma+c;\beta+2;\sin^2\theta)$ in (4.6), then write the result in an amplitude phase form. The result is (4.3) with

$$\begin{aligned}W(\theta)\cos\eta = \{\,{}_2F_1(c,-c-\beta-\alpha;-\beta;\sin^2\theta) + \mathcal{K}(\sin\theta)^{2\beta+2}\\ \cdot\, {}_2F_1(c+1+\beta,1-c-\alpha;2+\beta;\sin^2\theta)\}\cos(\pi\beta/2),\end{aligned} \tag{4.8}$$

$$\begin{aligned}W(\theta)\sin\eta = \{\,{}_2F_1(c,-c-\beta-\alpha;-\beta;\sin^2\theta) - \mathcal{K}(\sin\theta)^{2\beta+2}\\ \cdot\, {}_2F_1(c+1+\beta,1-c-\alpha;2+\beta;\sin^2\theta)\}\sin(\pi\beta/2).\end{aligned} \tag{4.9}$$

The next result is a generating function for the $\mathcal{R}_n$'s. It is an easy consequence of the following lemma.

Lemma 4.10. *Let t, x, a, b, c be complex numbers, $x \notin [1, \infty)$, and assume $|t| < |x^{1/2} + (x-1)^{1/2}|^{-2}$. Then*

$$\sum_{n=0}^{\infty} \frac{(c+a)_n (b)_n t^n}{n!(a+b+1)_n} \, {}_2F_1(-n-a, n+b; c; x) = 2^b (Z_2 - t)^{-a-c} (Z_2 + t)^{c+a-b} \cdot {}_2F_1(-a, b; c; (t - Z_1)/2t) \, {}_2F_1(a+c, a+1; a+b+1; 2t/(t - Z_2)) \tag{4.11}$$

where

$$Z_1 = 1 - \Phi, \quad Z_2 = 1 + \Phi, \quad \Phi = [(1-t)^2 + 4xt]^{1/2}. \tag{4.12}$$

Lemma 4.10 is stated in Wimp [**19**] with references to the various authors who contributed to its development.

Theorem 4.13. *Let*

$$\rho := [(1+t)^2 - 4xt]^{1/2}. \tag{4.14}$$

The $\mathcal{R}_n$'s have the generating function

(4.15)

$$\sum_{n=0}^{\infty} \frac{(\gamma+c)_n (c+1)_n t^n}{n!(\gamma+2c+1)_n} \mathcal{R}_n^{\alpha,\beta}(x; c) = \left\{ \frac{2}{1+t+\rho} \right\}^{c+\gamma} {}_2F_1\left(\begin{matrix} c, 1-c-\gamma \\ -\beta \end{matrix} \middle| x \right) {}_2F_1\left(\begin{matrix} -c, c+\gamma \\ 1+\beta \end{matrix} \middle| \frac{1+t-\rho}{2t} \right) \cdot {}_2F_1\left(\begin{matrix} c+1+\alpha, \gamma \\ \gamma+2c+1 \end{matrix} \middle| \frac{2t}{1+t+\rho} \right) - \frac{c(c+\alpha)}{\beta(\beta+1)} \left\{ \frac{2}{1+t+\rho} \right\}^{c+1} x \cdot {}_2F_1\left(\begin{matrix} 1-c, 2+\gamma \\ 2+\beta \end{matrix} \middle| x \right) {}_2F_1\left(\begin{matrix} 1-c-\gamma, c+1 \\ 1-\beta \end{matrix} \middle| \frac{1+t-\rho}{2t} \right) \cdot {}_2F_1\left(\begin{matrix} \beta+c+1, c+1 \\ \gamma+2c+1 \end{matrix} \middle| \frac{2t}{1+t+\rho} \right).$$

Proof. We multiply both sides of (3.11) by $(-t)^n (\gamma+c)_n (c+1)_n / [n!(\gamma+2c+1)_n]$ and sum over nonnegative integral values of n. Using (4.11), this gives, for the sum (4.15),

$$
\begin{aligned}
&2^{c+\gamma}(t+Z_2)^{-c-\beta-1}(Z_2-t)^{-\alpha}\,{}_2F_1\left(\left.\begin{matrix}c, 1-c-\gamma\\ -\beta\end{matrix}\right| x\right)\\
&\cdot\ {}_2F_1\left(\left.\begin{matrix}-c, \gamma+c\\ \beta+1\end{matrix}\right| \frac{t+Z_1}{2t}\right)\,{}_2F_1\left(\left.\begin{matrix}c+\beta+1, c+1\\ \gamma+2c+1\end{matrix}\right| \frac{2t}{t+Z_2}\right)\\
&-\frac{c(c+\alpha)}{\beta(\beta+1)}x2^{c+1}(t+Z_2)^{-c-1-\alpha}(Z_2-t)^{\alpha}\\
&\cdot\ {}_2F_1\left(\left.\begin{matrix}1-c, c+\gamma\\ 2+\beta\end{matrix}\right| x\right)\,{}_2F_1\left(\left.\begin{matrix}1-c-\gamma, c+1\\ 1-\beta\end{matrix}\right| \frac{t+Z_1}{2t}\right)\\
&\cdot\ {}_2F_1\left(\left.\begin{matrix}c+1+\alpha, c+\gamma\\ \gamma+2c+1\end{matrix}\right| \frac{2t}{t+Z_2}\right),
\end{aligned}
$$

where t is replaced by $-t$ in Z_1 and Z_2 of (4.14). We then apply the Kummer transformation, stated below (4.7), to the third and sixth ${}_2F_1$ in the above expression and transform it to the right-hand side of (4.15). □

When $c = 0$, the generating function (4.15) reduces to

$$
\sum_{n=0}^{\infty}\frac{\gamma t^n}{n+\gamma}P_n^{\alpha,\beta}(2x-1) = \left\{\frac{2}{1+t+\rho}\right\}^{\gamma}\,{}_2F_1\left(\left.\begin{matrix}\alpha+1, \gamma\\ 1+\gamma\end{matrix}\right| \frac{2t}{1+t+\rho}\right),
$$
$$
\gamma := \alpha+\beta+1.
$$

5. The weight function. Let $\{p_n(x)\}$ be a sequence of polynomials generated by

(5.1)
$$
p_0(x)=1,\quad p_1(x)=A_0x+B_0,\quad p_{n+1}(x)=(A_nx+B_n)p_n(x)-C_np_{n-1}(x),
$$

such that the positivity condition

$$
A_{n-1}A_nC_n > 0,\quad n = 1, 2, \dots,
$$

is satisfied. Under these conditions the polynomials $\{p_n(x)\}$ are orthogonal with respect to a finite positive measure, $d\mu$ say, with infinite support. If we normalize μ to have total mass 1, then the orthogonality relation will be

$$\text{(5.2)}\qquad \int_{-\infty}^{\infty} p_m(x)p_n(x)\,d\mu(x) = \zeta_n\delta_{m,n}, \quad \zeta_0 := 1, \zeta_n := \frac{A_0}{A_n}\prod_{j=1}^{n} C_j.$$

The orthonormal polynomials are $\{p_n(x)/\sqrt{\zeta_n}\}$. Nevai [**14**, pp. 141–143] proved that if the series

$$\sum_{n=1}^{\infty}\Big\{|B_n/A_n| + |\sqrt{C_{n+1}/(A_nA_{n+1})} - \frac{\gamma}{2}|\Big\}$$

converges, then $d\mu = \mu' dx + d\mu_j$ where μ' is continuous and positive in $(-\gamma,\gamma)$, the support of μ' is $[-\gamma,\gamma]$ and μ_j is a jump function constant outside $(-\gamma,\gamma)$. Under the same assumptions Nevai also proved that the limiting relation

$$\limsup_{n\to\infty}\{\mu'(x)\sqrt{\gamma^2 - x^2}\,p_n^2(x)/\zeta_n\} = 2/\pi$$

holds almost everywhere on the support of $d\mu$ provided that the total μ mass is 1. We shall prove later that the measure $d\mu$ is absolutely continuous in the case of the polynomials $\{\mathcal{R}_n^{\alpha,\beta}(x)\}$ when $c \ge 0$, $\alpha + c \ge 0$, and $\beta + 1 > 0$. In the present case,

$$\text{(5.3)}\qquad \zeta_n = \frac{(2c+\gamma)(c+1+\alpha)_n(c+1+\beta)_n}{(2c+2n+\gamma)(c+1)_n(c+\gamma)_n} \approx \frac{(2c+\gamma)\Gamma(c+1)\Gamma(c+\gamma)}{2n\Gamma(c+1+\alpha)\Gamma(c+1+\beta)}.$$

Hence, after a change of variable, Nevai's theorem establishes the orthogonality relation

$$\text{(5.4)}\qquad \int_0^1 \mathcal{R}_n^{\alpha,\beta}(x;c)\mathcal{R}_m^{\alpha,\beta}(x;c)\frac{x^\beta(1-x)^\alpha}{W^2(\arcsin\sqrt{x})}\,dx = \frac{\Gamma(c+1)\Gamma^2(\beta+1)\Gamma(c+\alpha+1+n)(c+\beta+1)_n}{(2n+2c+\gamma)\Gamma(c+\gamma+n)\Gamma(c+1+\beta)(c+1)_n}\delta_{m,n},$$

and W is given by (4.4).

We now find a representation for the continued J-fraction whose denominator approximants are the $\mathcal{R}_n$'s. When $|x| > 1$, Wimp [**19**] proved that the continued J-fraction with denominator approximants $\{R_n^{\alpha,\beta}(x;c)\}$ is

$$x^{-1}\,{}_2F_1(c+1,c+\beta+1;2c+\gamma+1;1/x)/\,{}_2F_1(c,c+\beta;2c+\gamma-1;1/x).$$

The numerator approximants of $\{R_n^{\alpha,\beta}(x;c)\}$ are $\{(\gamma+2c)_2[(c+1)(c+\gamma)]^{-1}R_{n-1}^{\alpha,\beta}(x;c+1)\}$. In view of Markov's theorem, we have, for $|x|>1$,

$$\lim_{n\to\infty} R_{n-1}^{\alpha,\beta}(x;c+1)/R_n^{\alpha,\beta}(x;c) = \frac{(c+1)(c+\gamma)\,{}_2F_1(c+1,c+\beta+1;2c+\gamma+1;1/x)}{x(\gamma+2c)_2\,{}_2F_1(c,c+\beta;2c+\gamma-1;1/x)}.$$

The numerator approximants of $\{\mathcal{R}_n^{\alpha,\beta}(x;c)\}$ are also $\{(\gamma+2c)_2[(c+1)(c+\gamma)]^{-1}R_{n-1}^{\alpha,\beta}(x;c+1)\}$. Taking into account (3.1) and the above limit, we get after some manipulations

$$\lim_{n\to\infty} R_{n-1}^{\alpha,\beta}(x;c+1)/\mathcal{R}_n^{\alpha,\beta}(x;c) = \frac{(c+1)(c+\gamma)\,{}_2F_1(c+1,c+\beta+1;2c+\gamma+1;1/x)}{x(\gamma+2c)_2\,{}_2F_1(c,c+\beta+1;2c+\gamma;1/x)}. \tag{5.5}$$

Therefore, when $|x|>1$, the continued J-fraction $F(x)$ whose denominator approximants are the $\mathcal{R}_n$'s has the representation

$$F(x) = \frac{{}_2F_1(c+1,c+1+\beta;2c+1+\gamma;1/x)}{x\,{}_2F_1(c,c+1+\beta;2c+\gamma;1/x)}, \quad |x|>1. \tag{5.6}$$

By Markov's theorem, $F(x)$ is the Stieltjes transform of $d\mu$ when x is outside the support of $d\mu$. When $c\ge 0$, $\alpha+c\ge 0$, $\beta+1>0$, the denominator in (5.6) has no zeros in $(1,\infty)$, hence $d\mu$ has no mass points in $(1,\infty)$. Spectral measures of birth and death processes are always supported on a subset of $[0,\infty)$, so $d\mu$ has no masses in $(-\infty,0)$. Nevai's theorem shows that $(0,1)$ is free of point masses. To show that $x=0$ or 1 do not support a discrete mass, we note that the series

$$\sum_{n=0}^{\infty}[\mathcal{R}_n^{\alpha,\beta}(x;c)]^2/\zeta_n, \quad x=0,1$$

diverges and appeal to Corollary 2.6, pages 45–46 in [**15**].

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