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DIVERGENCE OF VECTOR-VALUED RATIONAL INTERPOLANTS TO MEROMORPHIC FUNCTIONS

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Dedicated to Professor W.J. Thron on his seventieth birthday

ABSTRACT. A theorem on convergence of row sequences of Padé approximants for functions which are meromorphic in a disk was established by de Montessus in 1902. Stahl, in 1976, proved a companion theorem on divergence of these sequences outside the disk. In 1984, we extended de Montessus's theorem to the case of convergence in a disk of a row sequence of vector-valued approximants (simultaneous Padé approximants). Here, we establish the associated divergence results for these approximants outside the disk under virtually identical conditions.

1. Introduction. The famous convergence theorem of R. de Montessus de Ballore [**12**] for a row sequence of Padé approximants has become a result of considerable interest and generalization. Recently, attention has been directed to theorems containing generalizations of the interpolation set [**13, 20, 21, 18**], to inverse problems [**4, 5, 9, 2**], to associated divergence results [**14, 17, 18**] and to the adaptability of the theorem to vector-valued rational interpolation [**11, 6, 16**].

We discuss here the scheme of vector-valued rational interpolants whose confluent forms are also known as *simultaneous Padé approximants*. The polynomials constituting the vector-valued rational interpolants are also familiar as solutions of the *German polynomial approximation problem* [**10**]. Simultaneous Padé approximation involves approximation of several functions $\{f_i(z)\}_{i=1}^d$ which are analytic at $z=0$ by rationals of the form $\{P_{N,i}(z)/Q_N(z)\}_{i=1}^d$, where the denominator polynomial $Q_N(z)$ is common to each of the d components. Let nonnegative integers $\rho_1, \rho_2, \dots, \rho_d$ be given, such that $\sum_{i=1}^d \rho_i = M$. It is well known that nontrivial polynomials $\{P_{N,i}(z)\}_{i=1}^d$ and $Q_N(z)$

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can be found with the properties

$$\partial\{P_{N,i}(z)\} \le N - \rho_i, \quad i = 1, 2, \dots, d \tag{1.1}$$

$$\partial\{Q_N(z)\} \le M \tag{1.2}$$

$$P_{N,i}(z) - Q_N(z) f_i(z) = O(z^{N+1}), \quad i = 1, 2, \dots, d \tag{1.3}$$

for each $N \ge M$. We define a vector numerator polynomial by

$$\mathbf{P}_N(z) := (P_{N,1}(z), P_{N,2}(z), \dots, P_{N,d}(z)) \tag{1.4}$$

and take the vector of given functions to be

$$\mathbf{f}(z) := (f_1(z), f_2(z), \dots, f_d(z)). \tag{1.5}$$

Thus, whenever $Q_N(0) \ne 0$, we immediately have a solution of the vector-valued Padé approximation problem, satisfying the characteristic accuracy-through-order property that

$$\frac{\mathbf{P}_N(z)}{Q_N(z)} - \mathbf{f}(z) = O(z^{N+1}). \tag{1.6}$$

This property (1.6) pertains to osculatory rational interpolation at the origin, whereas vector-valued rational interpolation includes interpolation on a more general point set. In the latter case, for each positive integer N, we are given an associated point set

$$\mathcal{S}_N := \{\beta_{N,i},\ i = 0, 1, \dots, N : \beta_{N,i} \in \mathcal{S}\}, \tag{1.7}$$

and we require that $\mathcal{S}_N \subset \mathcal{S}$ for some given compact set $\mathcal{S} \subset \mathbf{C}$. A rational fraction $\mathbf{P}_N(z)/Q_N(z)$ composed from polynomials $\mathbf{P}_N(z)$ and $Q_N(z)$ satisfying (1.1), (1.2), (1.4) and

$$\mathbf{P}_N(\beta_{N,i})/Q_N(\beta_{N,i}) = \mathbf{f}(\beta_{N,i}), \quad i = 0, 1, \dots, N, \tag{1.8}$$

in the Hermite sense, is called a *vector-valued rational interpolant* to $\mathbf{f}(z)$ on the point set $\mathcal{S}_N$. Newton-Padé approximation [**3**] is the name given to rational interpolation schemes specified by (1.7), (1.8) in the special case where $\beta_{N,i} = \beta_i$, independently of N.

An extension of de Montessus's theorem to the simultaneous Padé approximation problem was discovered by Mall [**11**]. Convergence theorems for vector-valued Padé approximants and vector-valued rational interpolants were derived by Graves-Morris and Saff [**6**]. The present paper contains an account of divergence properties for cases considered in [**6**], and so [**6**] is a natural introduction to and logical precursor of the present paper. We refer to it for motivation for the present study and for the background references.

The condition of *polewise independence* was introduced in [**6**] to ensure that the functions $f_1, f_2, \dots, f_d$ are significantly different from each other, and it has subsequently been exploited by van Iseghem [**16**]. This condition is imposed to ensure that degenerate cases and degenerate limiting cases are avoided.

Definition 1.1. Let each of the functions $f_1(z), f_2(z), \dots, f_d(z)$ be meromorphic in the disk $D_R := \{z : |z| < R\}$, and let nonnegative integers $\rho_1, \rho_2, \dots, \rho_d$ be given for which

$$\sum_{i=1}^{d} \rho_i > 0. \tag{1.9}$$

Then the functions $f_i(z)$ are said to be *polewise independent, with respect to the numbers* ρ_i, *in* D_R if there *do not exist* polynomials $\pi_1(z), \pi_2(z), \dots, \pi_d(z)$, at least one of which is nonnull, satisfying

$$\partial\{\pi_i(z)\} \leq \rho_i - 1, \text{ if } \rho_i \geq 1, \tag{1.10a}$$

$$\pi_i(z) \equiv 0, \text{ if } \rho_i = 0 \tag{1.10b}$$

and such that

$$\Phi(z) := \sum_{i=1}^{d} \pi_i(z) f_i(z)$$

is *analytic* throughout D_R.

Using this condition, the following theorem was proved [**6**]:

Theorem 1.1. *Suppose that each of the d functions* $f_1(z), f_2(z), \dots,$ $f_d(z)$ *is analytic in the disk* $D_R := \{z : |z| < R\}$, *except for possible* *poles at the M (not necessarily distinct) points $z_1, z_2, \dots, z_M$ in D_R which are different from the origin. (If z_k is repeated exactly p times, then each $f_i(z)$ is permitted to have a pole of order at most p at z_k.) Let $\rho_1, \rho_2, \dots, \rho_d$ be nonnegative integers such that*

$$M = \sum_{i=1}^{d} \rho_i$$

and such that the functions $f_i(z)$ are polewise independent in D_R with respect to the ρ_i's in the sense of Definition 1.1. Then, for each integer N sufficiently large, there exist polynomials $Q_N(z)$, $\{P_{N,i}(z)\}_{i=1}^{d}$ satisfying (1.1), (1.2) and (1.6) where $\mathbf{P}_N(z)$ and $\mathbf{f}(z)$ are defined by (1.4) and (1.5).

The denominator polynomials (suitably normalized) satisfy

$$\lim_{N\to\infty} Q_N(z) = Q(z) := \prod_{j=1}^{M} (z - z_j), \quad \forall z \in \mathbf{C}. \tag{1.11}$$

Let $D_R^- := D_R - \cup_{j=1}^{M} \{z_j\}$. Then

$$\lim_{N\to\infty} \mathbf{P}_N(z)/Q_N(z) = \mathbf{f}(z), \quad \forall z \in D_R^-, \tag{1.12}$$

the convergence being uniform on compact subsets of D_R^-. More precisely, if K is any compact subset of the plane,

$$\limsup_{N\to\infty} \|Q_N - Q\|_K^{1/N} \le \frac{1}{R} \max_{j=1}^{M} \{|z_j|\} < 1, \tag{1.13}$$

and if E is any compact subset of D_R^-,

$$\limsup_{N\to\infty} \|f_i - P_{N,i}/Q_N\|_E^{1/N} \le \|z\|_E / R < 1 \tag{1.14}$$

for $i = 1, 2, \dots, d$.

In (1.13) and (1.14), the norm is taken to be the sup norm over the indicated set. In the present paper, we are primarily concerned with divergence results which can be established under very similar hypotheses.

Stahl [**14**] proved a divergence theorem for row sequences of Padé approximants, and his result is a natural counterpart to de Montessus's theorem as originally stated. Wallin [**17**] showed that analogous convergence and divergence results hold when Padé approximation is generalized to Newton-Padé approximation.

Kakehashi [**7**] showed that divergence results can be established for the very general polynomial interpolants based on the scheme (1.7), provided that this point set satisfies a strong regularity property. Wallin [**18**] established a generalization of de Montessus's theorem, in which interpolation on the latter point set is used.

We state next our main theorem about divergence of simultaneous Padé approximants. It is expressed as a natural extension of Stahl's theorem [**14**]. We give its proof in Section 2. In Section 3, we state its generalization to an interpolation set which satisfies Kakehashi's strong regularity property.

Theorem 1.2. *Let* $\{\mathbf{P}_N(z)/Q_N(z)\}$ *be a sequence of vector-valued Padé approximants to* $\mathbf{f}(z)$*, satisfying all the hypotheses of Theorem* 1.1*. If, for some* i*,* $f_i(z)$ *is not analytic at some point on* $C_R := \{z : |z| = R\}$*, then the sequence* $\{\mathbf{P}_N(z)/Q_N(z)\}$ *diverges at each point* z *exterior to* C_R*, according to the rule*

$$\limsup_{N\to\infty} |\mathbf{P}_N(z)/Q_N(z)|^{1/N} = |z|/R, \quad |z| > R, \tag{1.15}$$

where $|\cdot|$ *denotes any norm on the vector space* $\mathbf{C}^d$*.*

The proof of Theorem 1.1 is based on the use of certain auxiliary functions $\{F_{k,s}\}$, and these functions are also central to the proof of Theorem 1.2. They are specified in

Lemma 1.1. *With the assumptions of Theorem* 1.1*, write the list* $z_1, z_2, \ldots, z_M$ *in the form* $\{\zeta_k\}_{k=1}^{\nu}$*, where the* ζ_k*'s are distinct and each* ζ_k *is of multiplicity* m_k*, so that*

$$Q(z) = \prod_{j=1}^{M}(z - z_j) = \prod_{k=1}^{\nu}(z-\zeta_k)^{m_k}, \quad \sum_{k=1}^{\nu} m_k = M. \tag{1.16}$$

Then, for each $k = 1, 2, \ldots, \nu$, and each $s = 1, 2, \ldots, m_k$, there exists a function $F_{k,s}(z)$ of the form

$$F_{k,s}(z) = \sum_{i=1}^{d} \pi_i(z) f_i(z), \tag{1.17}$$

where the π_i's satisfy (1.10), *which is analytic in D_R, except for a pole of order s at the point ζ_k.*

Naturally, the polynomials $\pi_i(z)$ in (1.17) will, in general, depend on k and s.

Lemma 1.1 is proved in [**6**]; as a consequence of it, we can write

$$F_{k,s}(z) = \frac{g_{k,s}(z)}{(z-\zeta_k)^s}, \quad k = 1, 2, \ldots, \nu; \; s = 1, 2, \ldots, m_k,$$

where $g_{k,s}(z)$ is analytic in $|z| < R$ and nonzero at $z = \zeta_k$. These conditions ensure that $F_{k,s}(z)$ has a pole of precise order s at ζ_k.

It is tempting to speculate that Theorem 1.2 extends to rational interpolation on distinct points and to Padé-type approximation. We state a result which is a straightforward extension of Theorem 1.2 to the case of rational interpolation, where the interpolation points satisfy Kakehashi's rule, in Section 3. We also give an example which shows that divergence results for Padé-type approximation have a substantially different character from those of ordinary Padé approximation.

2. A proof of Theorem 1.2. To set the scene and establish notation, we begin with a preliminary lemma and then state two lemmas due to Kakehashi. Then we give an inductive proof of Theorem 1.2 in two parts; part A is the initialization and part B is the inductive step.

Lemma 2.1. *Let $\{f_i(z), i = 1, 2, \ldots, d\}$ be a set of d functions which are analytic at the origin and meromorphic in the disk $D_R := \{z : |z| < R\}$, where each $f_i(z)$ has poles of total multiplicity not greater than M in D_R. Let $\{f_i\}_{i=1}^{d}$ be polewise independent in D_R with respect to the given numbers $\rho_1, \rho_2, \ldots, \rho_d$, and let*

$$B_F := \{F_{k,s}(z), k = 1, 2, \ldots, \nu, \; s = 1, 2, \ldots, m_k\} \tag{2.1}$$

be the set of elementary functions specified by Lemma 1.1 *and formed from* $\{f_i(z)\}_{i=1}^d$.

If one or more of the functions f_i *is nonanalytic on* $|z| = R$, *then there is at least one function* $F_{k,s}$ *in the set* B_F *defined by* (2.1) *which is not analytic at some point on* $|z| = R$.

Proof. Suppose the conclusion to be false, so that B_F consists only of functions which are analytic on $|z| = R$. Let V_f be the linear space of functions spanned by

$$f_1, zf_1, \dots, z^{\rho_1 - 1}f_1, f_2, zf_2, \dots, z^{\rho_2 - 1}f_2, \dots, f_d, zf_d, \dots, z^{\rho_d - 1}f_d.$$

Because $\{f_i\}$ are assumed to be polewise independent with respect to $\rho_1, \rho_2, \dots, \rho_d$, the space V_f has dimension $M := \sum_{i=1}^d \rho_i$. Since each $F_{k,s}(z)$ lies in V_f, and the M elements $\{F_{k,s}\}$ of B_F are clearly linearly independent, these functions form a basis for V_f. Therefore, numbers $c_{k,s}^{(i)}$ exist such that

$$f_i(z) = \sum_{k=1}^{\nu} \sum_{s=1}^{m_k} c_{k,s}^{(i)} F_{k,s}(z) \tag{2.2}$$

for each i. Consequently, all $f_i(z)$ are analytic on $|z| = R$, contrary to hypothesis. □

We use integers $\hat{k}, \hat{s}$ to identify an element $F_{\hat{k},\hat{s}}(z)$ in B_F with the property that $\hat{s}$ is the least value of s for which $F_{k,s}(z)$ is not analytic on $|z| = R$.

In the proof of Theorem 1.2, we use two preliminary results:

Lemma 2.2 [7]. *Given a sequence of complex numbers* $\{g_n,\ n = 0, 1, 2, \dots\}$ *with the property that*

$$\limsup_{n\to\infty} |g_n|^{1/n} = R^{-1} \tag{2.3}$$

for some $R > 0$, *then the numbers* $\{g_n\}$ *may be expressed as*

$$g_n = \lambda_n c_n / R^n, \quad n = 0, 1, 2, \dots, \tag{2.4}$$

where $\{\lambda_n\}$ is a sequence of positive numbers, and $\{c_n\}$ is a bounded sequence of complex numbers, with the properties

$$\limsup_{n\to\infty} |c_n| = 1 \tag{2.5}$$

$$\lim_{n\to\infty} \lambda_{n+1}/\lambda_n = 1 \tag{2.6}$$

and either $1 = \lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots$ *or* $1 = \lambda_0 \ge \lambda_1 \ge \lambda_2 \ge \cdots > 0$.

Remark. As Kakehashi observes, a simple consequence of (2.6) is that

$$\lim_{n\to\infty} \lambda_n^{1/n} = 1. \tag{2.7}$$

Lemma 2.3 [**8**]. *Suppose that $g(z)$ is analytic in the annulus $A := \{z : r < |z| < R\}$, but is not analytic on $|z| = R$, so that $g(z)$ has the Laurent expansion*

$$g(z) = \sum_{j=-\infty}^{\infty} g_j z^j, \quad z \in A. \tag{2.8}$$

Then $\{g_j\}$ has the representation (2.4) *for $j \ge 0$ with the properties* (2.5), (2.6). *Let $\{h_n(z),\ n = 1, 2, \dots\}$ be a sequence of functions which are analytic on $C_R := \{z : |z| = R\}$ and such that*

$$\lim_{n\to\infty} h_n(z) = h(z) \tag{2.9}$$

uniformly on a closed region which contains C_R in its interior, with $h(z) \ne 0$ for any $z \in C_R$.

If $h_n(z)g(z)$ has the Laurent expansion

$$h_n(z)g(z) = \sum_{j=-\infty}^{\infty} \gamma_j^{(n)} z^j, \quad n = 1, 2, \dots, \tag{2.10}$$

then

$$0 < \limsup_{n\to\infty} \frac{R^n |\gamma_n^{(n)}|}{\lambda_n} < \infty,$$

where $\{\lambda_n\}$ satisfies (2.6).

Proof of Theorem 1.2. We observe from (1.14) that, for each $i = 1, 2, \dots, d$,

$$\limsup_{N\to\infty} \|P_{N,i}/Q_N\|_{C_R}^{1/N} \le 1,$$

and, hence, from the Bernstein-Walsh lemma [**19**, p. 77], we obtain

$$\limsup_{N\to\infty} |\mathbf{P}_N(z)/Q_N(z)|^{1/N} \le |z|/R, \quad |z| > R.$$

Hence, to establish (1.15) we need only show that

$$\limsup_{N\to\infty} |\mathbf{P}_N(z)/Q_N(z)|^{1/N} \ge |z|/R, \quad |z| > R. \tag{2.11}$$

This will be done in two parts.

A. *Initialization.* Assume that there exists an element $F_{k,1}(z)$ in B_F having a *simple* pole in $|z| < R$ and which is not analytic on $|z| = R$. Let $\tilde{p}_{N,1}(z)$ denote the N^{th} Maclaurin section of $Q_N(z)F_{k,1}(z)$, possessing the accuracy-through-order property

$$Q_N(z)F_{k,1}(z) - \tilde{p}_{N,1}(z) = O(z^{N+1}) \tag{2.12}$$

and the degree bound (cf. (1.3) and (1.17))

$$\partial\{\tilde{p}_{N,1}(z)\} \le N - 1. \tag{2.13}$$

Following [**6**], we define

$$g_{k,1}(z) := (z - \zeta_k)F_{k,1}(z), \tag{2.14}$$

and then $g_{k,1}(z)$ is analytic in $|z| < R$. Substituting (2.12)–(2.14) into Hermite's formula, we obtain

$$(z - \zeta_k)\tilde{p}_{N,1}(z) = \frac{1}{2\pi i}\int_{C_{R'}} \left(1 - \frac{z^{N+1}}{t^{N+1}}\right) \frac{Q_N(t)g_{k,1}(t)}{t - z}\, dt \tag{2.15}$$

for any complex z and any $R' \in (0, R)$. Now fix z with $|z| > R$. Set

$$h_N(t) := \left(\frac{t^{N+1} - z^{N+1}}{z^{N+1}}\right) \frac{Q_N(t)}{t - z}. \tag{2.16}$$

Then

$$\lim_{N\to\infty} h_N(t) = h(t) := \frac{-Q(t)}{t-z}$$

in some neighborhood of $|t| = R$ and $h(t) \neq 0$ for all $t \in C_R$. From (2.15) and (2.16), we find

$$\frac{(z-\zeta_k)\tilde{p}_{N,1}(z)}{z^{N+1}} = \frac{1}{2\pi i}\int_{C_{R'}} \frac{h_N(t)g_{k,1}(t)}{t^{N+1}}\,dt. \tag{2.17}$$

Hence, the left-hand side of (2.17) is a Maclaurin coefficient in the expansion of $h_N(t)g_{k,1}(t)$.

Because $F_{k,1}(z)$ is not analytic on $|z| = R$, neither is $g_{k,1}(z)$. So we can use Lemma 2.3 of Kakehashi and obtain

$$0 < \limsup_{N\to\infty} \left|\frac{R^N(z-\zeta_k)\tilde{p}_{N,1}(z)}{\lambda_N z^{N+1}}\right| < \infty, \tag{2.18}$$

where we have used (2.8) in the form

$$g_{k,1}(z) = \sum_{j=0}^{\infty} \lambda_j c_j (z/R)^j \tag{2.19}$$

and preserved properties (2.5)–(2.7).

From (2.18), we have

$$\limsup_{N\to\infty} |\tilde{p}_{N,1}(z)|^{1/N} = |z|/R. \tag{2.20}$$

From (1.3), (1.17) and (2.12), we obtain

$$\tilde{p}_{N,1}(z) = \sum_{i=1}^{d} \pi_i(z)P_{N,i}(z). \tag{2.21}$$

Hence, from (2.20) and (2.21), the inequality (2.11) follows for the case of $\hat{s} = 1$.

B. *The inductive step.* Suppose that $F_{k,s}(z)$ is analytic on $|z| = R$ for all k and all $s < \hat{s}$, but $F_{\hat{k},\hat{s}}(z)$ is not analytic at some point on

$|z| = R$, for some $\hat{s} > 1$. From the former of these two assumptions, it follows from an inspection of the proof of Theorem 1 of [**6**] that

$$\limsup_{N\to\infty} |Q_N^{(j)}(\zeta_{\hat{k}})|^{1/N} < |\zeta_{\hat{k}}|/R, \quad j = 0, 1, \ldots, s-1, \tag{2.22}$$

where $Q_N^{(j)}(z)$ denotes the j^{th} derivative of $Q_N(z)$ and the strict inequality in (2.22) follows as a consequence of the fact that the integrals $I_N(z)$ in (2.31) of [**6**] can be taken over $|t| = R'$ for some $R' > R$.

Analogous to (2.12), let $\tilde{p}_{N,s}(z)$ denote the N^{th} Maclaurin section of $Q_N(z)F_{k,s}(z)$,

$$Q_N(z)F_{k,s}(z) - \tilde{p}_{N,s}(z) = O(z^{N+1}). \tag{2.23}$$

From (1.3), (1.17) and (2.23),

$$\tilde{p}_{N,s}(z) = \sum_{i=1}^{d} \pi_i(z) P_{N,i}(z), \tag{2.24}$$

where it should be remembered that $\{\pi_i(z)\}$ depends on s and k. Hence,

$$\partial\{\tilde{p}_{N,s}(z)\} \le N - 1. \tag{2.25}$$

Analogous to (2.14), we define

$$g_{k,s}(z) := (z - \zeta_k)^s F_{k,s}(z), \quad k = 1, 2, \ldots, \nu, \ s = 1, 2, \ldots, m_k, \tag{2.26}$$

so that $\{g_{k,s}(z)\}$ are analytic in $|z| < R$ but $g_{\hat{k},\hat{s}}(z)$ is not analytic on $|z| = R$ and $g_{\hat{k},\hat{s}}(\zeta_{\hat{k}}) \ne 0$. From (2.24)–(2.26), we obtain

$$Q_N(z)\frac{g_{\hat{k},\hat{s}}(z)}{(z - \zeta_{\hat{k}})^{\hat{s}-1}} - (z - \zeta_{\hat{k}})\tilde{p}_{N,\hat{s}}(z) = O(z^{N+1}), \tag{2.27}$$

and, therefore,

$$(z - \zeta_{\hat{k}})\tilde{p}_{N,\hat{s}}(z) = \frac{1}{2\pi i}\int_{C_r} \frac{t^{N+1} - z^{N+1}}{t^{N+1}(t - z)} \cdot \frac{Q_N(t) g_{\hat{k},\hat{s}}(t)}{(t - \zeta_{\hat{k}})^{\hat{s}-1}}\, dt, \tag{2.28}$$

where $0 < r < |\zeta_{\hat{k}}|$. We extend the contour in (2.28) to $C_{R'}$, for some $R' \in (|\zeta_{\hat{k}}|, R)$, by evaluating the contribution of the pole at $t = \zeta_{\hat{k}}$ in the right-hand side of (2.28). We write (2.28) as

$$(2.29) \qquad (z - \zeta_{\hat{k}})\tilde{p}_{N,\hat{s}}(z) = \tilde{I}_N(z) - \tilde{J}_N(z),$$

where

$$(2.30) \qquad \tilde{I}_N(z) := \frac{1}{2\pi i} \int_{C_{R'}} \frac{t^{N+1} - z^{N+1}}{t^{N+1}(t-z)} \cdot \frac{Q_N(t) g_{\hat{k},\hat{s}}(t)}{(t - \zeta_{\hat{k}})^{\hat{s}-1}} \, dt$$

with $R' \in (|\zeta_{\hat{k}}|, R)$ and

$$(2.31) \qquad \tilde{J}_N(z) := \frac{1}{2\pi i} \int_{|t - \zeta_{\hat{k}}| = \varepsilon} \frac{t^{N+1} - z^{N+1}}{t^{N+1}(t-z)} \cdot \frac{Q_N(t) g_{\hat{k},\hat{s}}(t)}{(t - \zeta_{\hat{k}})^{\hat{s}-1}} \, dt.$$

Using the Taylor expansion of $Q_N(t)$ about $t = \zeta_{\hat{k}}$ and the inequalities (2.22), we readily obtain

$$(2.32) \qquad \limsup_{N \to \infty} |\tilde{J}_N(z)|^{1/N} < |z|/R, \quad |z| > R.$$

Also, by using the Kakehashi-Wallin method described and used in Part A, we obtain

$$(2.33) \qquad \limsup_{N \to \infty} |\tilde{I}_N(z)|^{1/N} = |z|/R, \quad |z| > R.$$

From (2.29), (2.32) and (2.33),

$$(2.34) \qquad \limsup_{N \to \infty} |\tilde{p}_{N,\hat{s}}(z)|^{1/N} = |z|/R, \quad |z| > R.$$

From (2.24), it now follows that

$$\limsup_{N \to \infty} |\mathbf{P}_N(z)/Q_N(z)|^{1/N} \geq |z|/R, \quad |z| > R$$

for all cases in which $\hat{s} > 1$. Combining this result with Part A, we have established equation (1.15). □

3. Rational interpolation. Kakehashi [**7**] established divergence results for interpolatory polynomials based on the interpolation set

$\mathcal{S}_N$ defined in (1.7) and satisfying the strong condition (3.2) below. To express this condition, we suppose $\mathcal{S}$ is a bounded continuum with connected complement containing more than one point. Assume $\mathcal{S}_N \subset \mathcal{S}$, and define

$$\omega_N(z) := \prod_{i=0}^{N-1} (z - \beta_{N,i}), \quad N = 1, 2, \dots . \tag{3.1}$$

Let

$$w = \psi_{\mathcal{S}}(z) = \frac{z}{c} + a_0 + \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \cdots$$

be the one-to-one analytic mapping of the complement of $\mathcal{S}$ onto the exterior of the unit disk $\{w : |w| > 1\}$, with $\psi_{\mathcal{S}}(\infty) = \infty$, $\psi'_{\mathcal{S}}(\infty) = 1/c > 0$. Then Kakehashi's condition is expressed as

$$\lim_{N\to\infty} \frac{\omega_N(z)}{c^N[\psi_{\mathcal{S}}(z)]^N} = \lambda(z), \quad z \in \mathbf{C}\backslash\mathcal{S}, \tag{3.2}$$

where $\lambda(z)$ is analytic and nonzero in $\mathbf{C}\backslash\mathcal{S}$, and the convergence of (3.2) is uniform on any compact subset of $\mathbf{C}\backslash\mathcal{S}$.

Let Γ_σ, $\sigma > 1$, denote generically the level curve

$$\Gamma_\sigma := \{z : |\psi_{\mathcal{S}}(z)| = \sigma\}, \tag{3.3}$$

and D_σ denote the interior of Γ_σ. We summarize our convergence and divergence results for a row sequence of vector-valued rational interpolants in the following theorem.

Theorem 3.1. *Assume that the interpolatory point sets $\mathcal{S}_N \subset \mathcal{S}$, as defined in* (1.7)*, are given, and that Kakehashi's condition* (3.2) *holds for them. Let each of the functions $f_1(z), f_2(z), \dots, f_d(z)$ be analytic on $\mathcal{S}$ and also in the larger region D_R, except for possible poles at the points $z_1, z_2, \dots, z_M$ in D_R. Define*

$$D_R^- := D_R - \bigcup_{i=1}^{M} \{z_i\},$$

and let E be any compact subset of D_R^-. Given d nonnegative integers $\rho_1, \rho_2, \dots, \rho_d$ satisfying $M = \sum_{i=1}^d \rho_i$, assume that the $f_i(z)$ are pole-wise independent with respect to the ρ_i's in D_R. Then, for each N large enough, there exist polynomials $Q_N(z)$, $\{P_{N,i}(z)\}_{i=1}^d$, with

$$\partial\{Q_N(z)\} = M, \quad \partial\{P_{N,i}(z)\} \le N - \rho_i, \quad i = 1, 2, \dots, d,$$

such that $P_{N,i}(z)/Q_N(z)$ interpolates as

$$P_{N,i}(z)/Q_N(z) = f_i(z) \quad \forall z \in \mathcal{S}_N,\ i = 1, 2, \dots, d,$$

in the Hermite sense. The denominator polynomials obey

$$\lim_{N\to\infty} Q_N(z) = Q(z) := \prod_{i=1}^{M} (z - z_i), \quad \forall z \in \mathbf{C}. \tag{3.4}$$

Furthermore,

$$\lim_{N\to\infty} P_{N,i}(z)/Q_N(z) = f_i(z), \quad \forall z \in D_R^-,\ i = 1, 2, \dots, d, \tag{3.5}$$

the convergence being uniform on E, which is an arbitrary, compact subset of D_R^-.

Define $\sigma_E(\ge 1)$ to be the infimum of numbers σ (> 1) for which it is true that $E \subset D_\sigma$. Then the rates of convergence of the interpolants are given by

$$\limsup_{N\to\infty} \|f_i - P_{N,i}/Q_N\|_E^{1/N} \le \sigma_E/R, \quad i = 1, 2, \dots, d, \tag{3.6}$$

and, provided some $f_i(z)$ is not analytic at some point on Γ_R, their rate of divergence is given by

$$\limsup_{N\to\infty} |\mathbf{P}_N(z)/Q_N(z)|^{1/N} = |\psi_{\mathcal{S}}(z)|/R, \quad z \in \mathbf{C}\backslash\{D_R \cup \Gamma_R\}, \tag{3.7}$$

where $|\cdot|$ denotes any norm on the vector space $\mathbf{C}^d$.

Proof. The proof of the convergence properties (3.4)–(3.6), and further detail of them, follows a fortiori from Theorem 3 of [**6**]. The proof of the divergence result (3.7) is but a minor variant of that given in Section 2 of this paper. □

Recently, an extended cross-rule was established by van Iseghem [**15**] for vector-valued Padé-type approximants. There has also been some unpublished speculation about which convergence and divergence properties of vector-valued Padé approximants extend to vector-valued *Padé-type* approximants. The following example delimits the extent of progress possible under the standard hypotheses for Padé-type approximation [**1**].

Example. Consider Padé-type approximants $p_{n,1}(z)/q_{n,1}(z)$ of type $[n/1]$ to $f(z)$, where

$$f(z) := \frac{2}{z-1} - \frac{1}{z-2}. \tag{3.8}$$

Let $q_{n,1}(z)$ be defined as the denominator of the ordinary Padé approximant of type $[n/1]$ for $g(z) := 1/\{(z-1)(z-2)\}$, and let $p_{n,1}(z)$ be defined by the two conditions

$$\partial\{p_{n,1}(z)\} \leq n, \tag{3.9}$$

$$p_{n,1}(z) - f(z)q_{n,1}(z) = O(z^{n+1}). \tag{3.10}$$

Then, with a suitable normalization, we have

$$q_{n,1}(z) \to 1 - z, \tag{3.11}$$

$$\lim_{n\to\infty} p_{n,1}(z)/q_{n,1}(z) = f(z), \quad |z| < 2, \tag{3.12}$$

but

$$\limsup_{n\to\infty} |p_{n,1}(z)/q_{n,1}(z)|^{1/n} \neq |z|/2 \text{ for some } |z| > 2. \tag{3.13}$$

Proof. Let $\tilde{p}_n(z)/q_{n,1}(z)$ be the Padé approximant of type $[n-1/1]$ for $g(z)$, and let

$$p_{n,1}(z) := (z-3)\tilde{p}_n(z). \tag{3.14}$$

Properties (3.9) and (3.10) follow directly from the definitions of $\tilde{p}_n(z)$, $p_{n,1}(z)$ and $q_{n,1}(z)$. And de Montessus's theorem applied to the sequence of $[n-1/1]$ Padé approximants of $g(z)$ implies (3.11) and (3.12).

However, (3.14) ensures that $p_{n,1}(3) = 0$, and so $z = 3$ is one point for which (3.13) holds good. □

The extension of this example to a similar one for vector-valued Padé type approximants is trivial. Consequently, there can be no direct analogue of Theorem 2.2 or Theorem 3.1 for Padé type approximation without the imposition of supplementary conditions.

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