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THE ASYMPTOTIC BEHAVIOR OF SEQUENCES AND NEW SERIES TRANSFORMATIONS BASED ON THE CAUCHY PRODUCT

CLAUDE BREZINSKI

ABSTRACT. We first give a review of results concerning the asymptotic behavior of the ratio of the errors and the ratio of the differences for a convergent sequence. Then a review and new results on the asymptotic comparison of ratios of errors and ratios of differences for two converging sequences are given. These results are used for showing how to accelerate the convergence. In particular, new series transformations based on the Cauchy product by an arbitrary given series are discussed and their properties are studied.

Introduction. The construction and the study of convergence acceleration methods for sequences and series needs the knowledge of results on the asymptotic behavior of the ratio of the errors and the ratio of the differences between two consecutive terms of a convergent sequence. It also needs some results on the asymptotic behavior of ratios of the errors and ratios of the differences for two sequences. In the first two sections we shall review such existing results and give some new ones. In the last section we shall use them to show how to accelerate the convergence under certain assumptions. Recently, some old results on the ratio of a term of a series obtained by Cauchy product from a previous one divided by the corresponding term of the initial series were rediscovered and extended by using techniques of nonstandard analysis [**2**]. These results are used to build new series transformations leading to the concept of Cauchy-type approximation. This type of approximation is related to Padé-type approximation in a particular case. Some acceleration properties of this transformation are given and an extension is studied.

Asymptotic behavior of a sequence. We shall give a review of the existing results. (u_n) is a sequence converging to u. The most general result is

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Theorem 1 [**10**]. *Let* $\lambda \in C$, $|\lambda| \neq 1$. *Then*

$$\lim_{n\to\infty} (u_{n+1}-u)/(u_n-u) = \lambda \textit{ if and only if } \lim_{n\to\infty} \Delta u_{n+1}/\Delta u_n = \lambda.$$

As proved by counter-examples, the assumption $|\lambda| \neq 1$ cannot be suppressed. Let us now give some theorems where the condition $|\lambda| \neq 1$ is replaced by another one.

Theorem 2 [**8**]. *Suppose that* (u_n) *is monotone. If* $\exists\,\lambda$, *finite or not,* $\lim_{n\to\infty} \Delta u_{n+1}/\Delta u_n = \lambda$, *then* $\lim_{n\to\infty}(u_{n+1} - u)/(u_n - u) = \lambda$.

This result can be directly deduced from Theorem 8 below.

Theorem 3 [**8**]. *Suppose that* $((-1)^n \Delta u_n)$ *is monotone. If* $\exists\,\lambda$, *finite or not,* $\lim_{n\to\infty} \Delta u_{n+1}/\Delta u_n = \lambda$, *and if* $\lim_{n\to\infty}((1+\Delta u_{n+2}/\Delta u_{n+1})/(1+\Delta u_{n+1}/\Delta u_n)) = 1$, *then* $\lim_{n\to\infty}(u_{n+1}-u)/(u_n-u) = \lambda$.

This result can also be deduced from Theorem 8 below. If $\lambda \neq -1$, $\pm\infty$ the second condition is automatically satisfied. As stated, this theorem is a slight generalization of the original result given in [**8**]; see also [**9**].

Theorem 4 [**11**]. *Let* $\lambda, \mu \in \mathbf{R}$, $0 \leq \lambda < \mu < 1$. *If,* $\forall n$, $\lambda \leq \Delta u_{n+1}/\Delta u_n \leq \mu$, *then* $\forall n$, $\lambda' \leq (u_{n+1} - u)/(u_n - u) \leq \mu'$ *with* $\lambda' = \lambda(\mu - 1)/(\lambda - 1)$ *and* $\mu' = \mu(\lambda - 1)/(\mu - 1)$.

If, $\forall n$, $\lambda \leq (u_{n+1}-u)/(u_n-u) \leq \mu$, *then,* $\forall n$, $\lambda' \leq \Delta u_{n+1}/\Delta u_n \leq \mu'$ *with* $\lambda' = \lambda(\mu - 1)/(\lambda - 1)$ *and* $\mu' = \mu(\lambda - 1)/(\mu - 1)$.

Asymptotic comparison of sequences. We shall give a review of the existing results and extend some of them. (u_n) and (v_n) are sequences converging respectively to u and v. Let us first recall some definitions.

Definition 1. (u_n) converges faster than (v_n) if and only if $\lim_{n\to\infty}(u_n - u)/(v_n - v) = 0$.

Definition 2. (u_n) converges at the same rate as (v_n) if and only if $\exists\, 0 < a \le b$, $\exists\, N$, $\forall n \ge N$, $a \le |u_n - u|/|v_n - v| \le b$.

Theorem 5 [**19**]. *Suppose that* $\exists\, \rho$, $|\rho| < 1$, *and* $\exists\, \lambda$, $|\lambda| < 1$, *such that* $\lim_{n\to\infty} \Delta u_{n+1}/\Delta u_n = \rho$, $\lim_{n\to\infty} \Delta v_{n+1}/\Delta v_n = \lambda$. *Then*

(i) (u_n) *converges faster than* (v_n) *if and only if* (Δu_n) *converges faster than* (Δv_n).

(ii) (u_n) *converges at the same rate as* (v_n) *if and only if* (Δu_n) *converges at the same rate as* (Δv_n).

The faster convergence of (Δu_n) obviously implies $|\rho| < |\lambda|$.

Theorem 6 [**19**]. *Suppose that* $\exists\, \rho$, $0 \le \rho < 1$, $\exists\, \lambda$, $0 \le \lambda < 1/2$, *and* $\exists\, N$ *such that* $\forall n \ge N$,

$$|\Delta u_{n+1}/\Delta u_n| \le \rho, \;\; |\Delta v_{n+1}/\Delta v_n| \le \lambda.$$

If (Δu_n) *converges faster than* (Δv_n), *then* (u_n) *converges faster than* (v_n).

Tucker gave a counter-example showing that, in Theorem 6, $1/2$ cannot be replaced by a larger number.

Theorem 7 [**3**]. *Suppose that* $\exists\, a, b$, $a < 1 < b$, *and* $\exists\, N$, *such that* $\forall n \ge N$,

$$(v_{n+1} - v)/(v_n - v) \notin [a, b].$$

If $\exists\, c$, $\lim_{n\to\infty}(u_n - u)/(v_n - v) = c$, *then* $\lim_{n\to\infty} \Delta u_n/\Delta v_n = c$.

As proved by a counter-example ($u_n = 1/n, v_n = (-1)^n/n$), the reciprocal of this theorem is not true.

Theorem 8 [**8**]. *Suppose that* (v_n) *is monotone. If* $\exists\, c$, *finite or not*, $\lim_{n\to\infty} \Delta u_n/\Delta v_n = c$, *then* $\lim_{n\to\infty}(u_n - u)/(v_n - v) = c$.

The case (v_n) decreasing was proved by Bromwich [**7**], see also [**9**].

Theorem 9 [**3**]. *Suppose that* $\exists\, a, b,\ a < 1 < b$, *and* $\exists\, N$, *such that* $\forall n \geq N$,

$$(v_{n+1} - v)/(v_n - v) \notin [a, b].$$

If $|u_n - u| = O\ (|v_n - v|)$, *then* $|\Delta u_n| = O(|\Delta v_n|)$.

Theorem 10 [**3**]. *Suppose that* $\limsup_{n\to\infty} |\Delta u_n|^{1/n} = 1/R$ *and*

$$\lim_{n\to\infty} |\Delta v_n|^{1/n} = 1/r.$$

If $r < R$, *then* (Δu_n) *converges faster than* (Δv_n).

If the conditions of theorem 5 are satisfied, then $|\rho| = 1/R$ and $|\lambda| = 1/r$ with $|\rho| < |\lambda|$ and (u_n) converges faster than (v_n).

We shall now generalize some of the previous results. Let us begin by an extension of Theorem 5.

Theorem 11. *Suppose that* $\exists\, \rho,\ |\rho| < 1,\ \exists\, \lambda,\ |\lambda| < 1$, *such that* $\lim_{n\to\infty} \Delta u_{n+1}/\Delta u_n = \rho$, $\lim_{n\to\infty} \Delta v_{n+1}/\Delta v_n = \lambda$. *Let* $a \in \mathbf{C}$. *Then*

$$\lim_{n\to\infty} (u_n - u)/(v_n - v) = a \qquad \text{if and only if} \qquad \lim_{n\to\infty} \Delta u_n/\Delta v_n = a.$$

Moreover, if $a \neq 0$, *then* $\lambda = \rho$.

Proof. $\forall k \geq 0$, $\lim_{n\to\infty} \Delta u_{n+k}/\Delta u_n = \rho^k$ and $\lim_{n\to\infty} \Delta v_{n+k}/\Delta v_n = \lambda^k$. We have

$$\frac{u_n - u}{v_n - v} = \frac{\Delta u_n + \Delta u_{n+1} + \cdots}{\Delta v_n + \Delta v_{n+1} + \cdots} = \frac{\Delta u_n}{\Delta v_n} \frac{1 + \frac{\Delta u_{n+1}}{\Delta u_n} + \frac{\Delta u_{n+2}}{\Delta u_n} + \cdots}{1 + \frac{\Delta v_{n+1}}{\Delta v_n} + \frac{\Delta v_{n+2}}{\Delta v_n} + \cdots}.$$

When n tends to infinity, the second ratio in the right-hand side converges to

$$\frac{1 + \rho + \rho^2 + \cdots}{1 + \lambda + \lambda^2 + \cdots} = \frac{1 - \lambda}{1 - \rho}$$

and $\exists\, a$, $\lim_{n\to\infty}(u_n - u)/(v_n - v) = a$, if and only if $\exists\, b$, $\lim_{n\to\infty} \Delta u_n/\Delta v_n = b$, where a and b are related by $a = b(1-\lambda)/(1-\rho)$. Of course, $a \neq 0$ if and only if $b \neq 0$ and $a = 0$ if and only if $b = 0$. If $b \neq 0$, then $(\Delta u_{n+1}/\Delta v_{n+1})/(\Delta u_n/\Delta v_n)$ tends to $b/b = 1$. That is,

$$\begin{aligned}\rho &= \lim_{n\to\infty} \frac{\Delta u_{n+1}}{\Delta v_n} = \lim_{n\to\infty} \frac{\Delta v_{n+1}}{\Delta v_n}\frac{\Delta u_{n+1}}{\Delta u_n}\frac{\Delta v_n}{\Delta v_{n+1}} \\ &= \lim_{n\to\infty} \frac{\Delta v_{n+1}}{\Delta v_n} \lim_{n\to\infty} \frac{\Delta u_{n+1}/\Delta v_{n+1}}{\Delta u_n/\Delta v_n} = \lambda\end{aligned}$$

and, thus, $a = b$. Conversely, if $a = b$, then $\lambda = \rho$. □

Of course, due to Theorem 1, the conditions of Theorem 11 can be replaced by

$$\lim_{n\to\infty}(u_{n+1} - u)/(u_n - u) = \rho \text{ and } \lim_{n\to\infty}(v_{n+1} - v)/(v_n - v) = \lambda.$$

Let us now generalize Theorem 6.

Theorem 12. *Suppose that $\exists\, \rho$, $0 \leq \rho < 1/2$, $\exists\, \lambda$, $0 \leq \lambda < 1/2$, and $\exists\, N$, such that $\forall n \geq N$,*

$$|\Delta u_{n+1}/\Delta u_n| \leq \rho, \qquad |\Delta v_{n+1}/\Delta v_n| \leq \lambda.$$

(Δu_n) converges faster than (Δv_n) if and only if (u_n) converges faster than (v_n).

Proof. We have, $\forall k \geq 0$, $-\lambda^k \leq \Delta v_{n+k}/\Delta v_n \leq \lambda^k$ and $-\rho^k \leq \Delta u_{n+k}/\Delta u_n \leq \rho^k$. Thus

$$\begin{aligned}0 < 1 - \frac{\lambda}{1-\lambda} &= 1 - \lambda - \lambda^2 - \cdots \\ &\leq 1 + \frac{\Delta v_{n+1}}{\Delta v_n} + \frac{\Delta v_{n+2}}{\Delta v_n} + \cdots \\ &\leq 1 + \lambda + \lambda^2 + \cdots = \frac{1}{1-\lambda}, \\ 0 < 1 - \frac{\rho}{1-\rho} &= 1 - \rho - \rho^2 - \cdots \leq 1 + \frac{\Delta u_{n+1}}{\Delta u_n} + \frac{\Delta u_{n+2}}{\Delta u_n} + \cdots \\ &\leq 1 + \rho + \rho^2 + \cdots = \frac{1}{1-\rho}.\end{aligned}$$

It follows that

$$0 < (1-2\rho)\frac{1-\lambda}{1-\rho} \le \frac{1+\frac{\Delta u_{n+1}}{\Delta u_n}+\frac{\Delta u_{n+2}}{\Delta u_n}+\cdots}{1+\frac{\Delta v_{n+1}}{\Delta v_n}+\frac{\Delta v_{n+2}}{\Delta v_n}+\cdots} \le \frac{1}{1-2\lambda}\frac{1-\lambda}{1-\rho} < +\infty,$$

which proves the result. □

This result shows that Theorem 6 is a necessary and sufficient condition when $\rho < 1/2$.

We shall now give a generalization of Theorem 4 when two sequences are involved.

Theorem 13. *Let* $\alpha, \beta, \alpha', \beta' \in \mathbf{R}$, $0 \le \alpha < \beta < 1$, $0 \le \alpha' < \beta' < 1$. *Suppose that* $\exists N$, *such that* $\forall n \ge N$, $\Delta u_{n+1}/\Delta u_n \in [\alpha, \beta]$ *and that* $\exists N'$, *such that* $\forall n \ge N'$, $\Delta v_{n+1}/\Delta v_n \in [\alpha', \beta']$. *If* $\exists a, b$, $0 \le a \le b$, $\exists M$, *such that* $\forall n \ge M$, $\Delta u_n/\Delta v_n \in [a, b]$, *then* $\forall n \ge \max(N, N', M)$, $(u_n - u)/(v_n - v) \in [a', b']$ *with* $a' = a(1-\beta')/(1-\alpha)$ *and* $b' = b(1-\alpha')/(1-\beta)$.

If $\exists a, b$, $0 \le a \le b$, $\exists M$, *such that* $\forall n \ge M$, $(u_n - u)/(v_n - v) \in [a, b]$, *then* $\forall n \ge \max(N, N', M)$, $\Delta u_n/\Delta v_n \in [a', b']$ *with* $a' = a(1-\beta)/(1-\alpha')$ *and* $b' = b(1-\alpha)/(1-\beta')$.

Proof. This is, again, based on the relation used in the proof of Theorem 11. We have, $\forall k \ge 0$, $\alpha^k \le \Delta u_{n+k}/\Delta u_n \le \beta^k$ and $\alpha'^k \le \Delta v_{n+k}/\Delta v_n \le \beta'^k$. Thus,

$$\begin{aligned}\frac{1}{1-\alpha'} &= 1+\alpha'+\alpha'^2+\cdots \le 1+\frac{\Delta v_{n+1}}{\Delta v_n}+\frac{\Delta v_{n+2}}{\Delta v_n}+\cdots\\ &\le 1+\beta'+\beta'^2+\cdots = \frac{1}{1-\beta'},\\ \frac{1}{1-\alpha} &= 1+\alpha+\alpha^2+\cdots \le 1+\frac{\Delta u_{n+1}}{\Delta u_n}+\frac{\Delta u_{n+2}}{\Delta u_n}+\cdots\\ &\le 1+\beta+\beta^2+\cdots = \frac{1}{1-\beta}\end{aligned}$$

and

$$\frac{1-\beta'}{1-\alpha} \le \frac{1+\frac{\Delta u_{n+1}}{\Delta u_n}+\frac{\Delta u_{n+2}}{\Delta u_n}+\cdots}{1+\frac{\Delta v_{n+1}}{\Delta v_n}+\frac{\Delta v_{n+2}}{\Delta v_n}+\cdots} \le \frac{1-\alpha'}{1-\beta},$$

and the results immediately follow. □

If $u_n = v_{n+1}$ then $\alpha = \alpha' = a$, $\beta = \beta' = b$ and the first part of Theorem 13 reduces to the first part of Theorem 4.

Applications to convergence acceleration. Let (S_n) be a sequence converging to S. Let $T : (S_n) \to (T_n)$ be a sequence transformation. The so-called θ-procedure consists in considering the new sequence transformation $\theta : (S_n) \to (\theta_n)$ given by [**4**]

$$\theta_n = S_n - \frac{T_n - S_n}{\Delta T_n - \Delta S_n} \Delta S_n, \quad n = 0, 1, \ldots .$$

We have

$$\frac{\theta_n - S}{S_n - S} = 1 - \frac{\frac{T_n - S}{S_n - S} - 1}{\frac{\Delta T_n}{\Delta S_n} - 1}$$

and, thus, immediately obtain

Theorem 14 [**14**]. *If* $\exists\, a \neq 1$, $\lim_{n\to\infty}(T_n - S)/(S_n - S) = \lim_{n\to\infty} \Delta T_n/\Delta S_n = a$, *then*

$$\lim_{n\to\infty} (\theta_n - S)/(S_n - S) = 0.$$

Very often, in practical applications, one can only prove that either $\lim_{n\to\infty}(T_n - S)/(S_n - S) = a$ or $\lim_{n\to\infty} \Delta T_n/\Delta S_n = a$. But, by the preceding theorem, convergence acceleration can only be achieved if both limits exist and are equal. Thus, the results given in the previous sections can be helpful in proving this equality.

Now, if $a = 0$, Theorem 14 still holds. However, in that case, the θ-procedure is only interesting if (θ_n) converges faster than (T_{n+1}). We also have

$$\theta_n = T_n - \frac{T_n - S_n}{\Delta T_n - \Delta S_n} \Delta T_n = T_{n+1} - \frac{T_{n+1} - S_{n+1}}{\Delta T_n - \Delta S_n} \Delta T_n.$$

Thus,

$$\begin{aligned} \frac{\theta_n - S}{T_{n+1} - S} &= 1 - \frac{1 - \frac{T_{n+1} - S}{S_{n+1} - S}}{1 - \frac{\Delta T_n}{\Delta S_n}} \frac{S_{n+1} - S}{T_{n+1} - S} \frac{\Delta T_n}{\Delta S_n} \\ &= 1 - \frac{1 - \frac{T_{n+1} - S}{S_{n+1} - S}}{1 - \frac{\Delta T_n}{\Delta S_n}} \frac{1 - \frac{T_n - S}{T_{n+1} - S}}{1 - \frac{S_n - S}{S_{n+1} - S}}, \end{aligned}$$

and, applying Theorem 1, yields

Theorem 15. *If* $\lim_{n\to\infty}(T_n - S)/(S_n - S) = 0$, *if* $\exists\, c \neq 0$ *and* $\neq 1$, *such that* $\lim_{n\to\infty}(S_{n+1} - S)/(S_n - S) = \lim_{n\to\infty}(T_{n+1} - S)/(T_n - S) = c$, *then* $\lim_{n\to\infty}(\theta_n - S)/(T_{n+1} - S) = 0$.

We shall now give a method for building, in some cases, a transformation T satisfying the assumptions of Theorem 14. It is based on a result, initially proved by Szász [**18**] (see also [**17**, pp. 39, 218]) but rediscovered and improved by van den Berg [**2**, pp. 54, 83] by means of nonstandard techniques.

Theorem 16. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *and* $g(z) = \sum_{n=0}^{\infty} b_n z^n$ *be two power series such that* $\lim_{n\to\infty} a_{n+1}/a_n = a \neq 0$ *and* g *has a convergence radius* $R > 1/|a|$. *Let* $h(z) = \sum_{n=0}^{\infty} c_n z^n = f(z)g(z)$. *Then*

(i) $lim_{n\to\infty} c_n/a_n = g(1/a)$

(ii) *if* $g(1/a) \neq 0$, *then* $\lim_{n\to\infty} c_{n+1}/c_n = a$.

(iii) *if* $a_{n+1}/a_n = a + o(n^{-1/2})$, *then* $c_n/a_n = g(1/a) + o(n^{-1/2})$. *If* $g(1/a) \neq 0$, *then* $c_{n+1}/c_n = a + o(n^{-1/2})$.

It is well known that the c_n's are given by $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0$ for $n = 0, 1, \ldots,$ that h converges in $|z| < 1/|a|$ and that $z = 1/a$ is a singular point of f. Of course, this theorem can be rewritten in terms of sequences instead of series. However, for the moment, we shall use it in this form.

We set $f_n(z) = \sum_{i=0}^{n} a_i z^i$, $g_n(z) = \sum_{i=0}^{n} b_i z^i$ and $h_n(z) = \sum_{i=0}^{n} c_i z^i$. Consider the transformation $(f_n(z)) \to (S_n(z) = h_n(z)/g(z))$. Such a transformation will be called a Cauchy-type series transformation and $S_n(z)$ a Cauchy-type approximant.

We have $\Delta S_n(z)/\Delta f_n(z) = c_{n+1}/a_{n+1} g(z)$ which, by Theorem 16, tends to $g(1/a)/g(z)$ if $\lim_{n\to\infty} a_{n+1}/a_n \neq 0$ and $g(z) \neq 0$. If $g(1/a) = 0$, and if the conditions of Theorems 5, 6 or 8 are satisfied, then $\lim_{n\to\infty}(S_n(z) - f(z))/(f_n(z) - f(z)) = 0\ \forall z$, $|z| < 1/|a|$ with $g(z) \neq 0$. We have thus obtained

Theorem 17. *Let $|z| < 1/|a|$ and be such that $g(z) \neq 0$. If $\lim_{n\to\infty} a_{n+1}/a_n = a \neq 0$ and if $g(1/a) = 0$, then*

$$\lim_{n\to\infty} \Delta S_n(z)/\Delta f_n(z) = 0.$$

Moreover, if the conditions of one of the Theorems 5, 6 or 8 are satisfied with $u_n = S_n(z)$ and $v_n = f_n(z)$ then

$$\lim_{n\to\infty} (S_n(z) - f(z))/(f_n(z) - f(z)) = 0.$$

Due to Theorem 11, a similar result holds if $g(1/a) \neq 0$ where the zero limit is replaced by $g(1/a)$.

Now if $g(1/a)$ is different from zero, the θ-procedure can be applied to obtain

$$\theta_n(z) = f_n(z) - \frac{S_n(z) - f_n(z)}{\Delta S_n(z) - \Delta f_n(z)} \Delta f_n(z), \quad n = 0, 1, \dots ,$$

and the preceding results immediately give

Theorem 18. *Let $|z| < 1/|a|$ and be such that $g(z) \neq 0$ and $g(z) \neq g(1/a)$, then $\lim_{n\to\infty}(\theta_n(z) - f(z))/(f_n(z) - f(z)) = 0$.*

Theorems 17 and 18 can be easily translated into the language of sequences. Of course, if a is known, the choice $g(z) = 1 - az$ forces itself. In that case, we have

$$\begin{aligned} h(z) &= a_0 + (a_1 - a_0 a)z + (a_2 - a_1 a)z^2 + \cdots \\ &= f(z) - azf(z). \end{aligned}$$

Thus,

$$h_n(z) = f_n(z) - azf_{n-1}(z)$$

and

$$S_n(z) = f_{n-1}(z) + a_n z^n/(1 - az) = f_n(z) + a a_n z^{n+1}/(1 - az).$$

S_n is a rational function with a numerator of degree n and a denominator of the first degree. We have

$$f(z) - f_{n-1}(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots$$
$$= a_n z^n \left(1 + \frac{a_{n+1}}{a_n} z + \frac{a_{n+2}}{a_n} z^2 + \cdots\right).$$

When n tends to infinity, the ratio $(f(z) - f_{n-1}(z))/a_n z^n$ tends to $1 + az + a^2 z^2 + \cdots = 1/(1 - az)$ since $|az| < 1$. This result shows that $S_n(z)$ is identical to the number one standard acceleration process given by Germain-Bonne [**15**, p. 6]. But, moreover,

$$\frac{S_n(z) - f(z)}{f_n(z) - f(z)} = 1 + a \frac{a_n}{a_{n+1}} \frac{a_{n+1} z^{n+1}}{(1 - az)(f_n(z) - f(z))},$$

and, thus, we obtain

Theorem 19. *If $\lim_{n\to\infty} a_{n+1}/a_n = a \neq 0$ and if $g(z) = 1 - az$, then, $\forall z$, $|z| < 1/|a|$, $(S_n(z))$ converges to $f(z)$ faster than $(f_n(z))$.*

This result is related to another result proved by van den Berg [**2**, Theorem 2.22, p. 66] by nonstandard analysis techniques but which can also be easily obtained by a classical proof. This result says that if $\lim_{n\to\infty} a_{(n+1)k+q}/a_{nk+q} = a \neq 0$, and, if $\exists N$, such that $\forall n \geq N$, $a_n = 0$ if $n \neq pk + q$, then, for $|z| < 1/|a|^{1/k}$,

$$f(z) - f_{n+1}(z) \sim a_n z^n / (1 - az^k).$$

Thus, $(a_n z^n/(1 - az^k)$ is a perfect estimation of the error of $(f_{n-1}(z))$ and, following [**5**], $(f_{n-1}(z) + a_n z^n/(1 - az^k))$ converges to $f(z)$ faster than $(f_{n-1}(z))$. Since $a \neq 0$, this sequence also converges faster than $(f_n(z))$, thus providing an extension to lacunary series of the result of Theorem 19. Van den Berg also gave an extension of Theorem 16 to lacunary series which can be used in the same way as above.

As previously seen, $S_n(z)$ is a rational function if $g(z) = 1 - az$. This result turns out, in fact, to be more general since we have

Theorem 20. *If g is a polynomial of degree k with $g(0) \neq 0$, then $S_n(z)$ is the Padé-type approximant $(n/k)_f$ whose generating polynomial is $v(z) = z^k g(z^{-1})$.*

Proof. We have

$$f_n(z)g(z) = h_n(z) + O(z^{n+1}).$$

Thus, since $g(0) \neq 0$,

$$h_n(z)/g(z) = f_n(z) + O(z^{n+1}) = f(z) + O(z^{n+1}),$$

which is the definition of the Padé-type approximant of f [**6**]. □

Thus, the Cauchy-type approximants S_n generalize the Padé-type ones. Now, instead of considering S_n as defined above, we can consider the approximants

$$V_{n/k}(z) = h_n(z)/g_k(z).$$

We have

$$f_n(z)g_k(z) = \begin{cases} h_k(z) + O(z^{k+1}), & \text{if } k \leq n, \\ h_n(z) + O(z^{n+1}), & \text{if } n \leq k, \end{cases}$$

and, thus, we obtain

Theorem 21. *If $g(0) \neq 0$, then, for $n \leq k$, $V_{n/k}$ is the Padé-type approximant $(n/k)_f$ whose generating polynomial is $v(z) = z^k g_k(z^{-1})$.*

We have

$$\frac{V_{n+1/k}(z) - V_{n/k}(z)}{f_{n+1}(z) - f_n(z)} = \frac{c_{n+1}}{a_{n+1} g_k(z)}$$

which tends to $g(1/a)/g_k(z)$ when n tends to infinity. Thus, for a fixed value of k and n tending to infinity, the results of Theorems 17 and 18 are still valid if the condition $g(z) \neq 0$ is now replaced by $g_k(z) \neq 0$, showing again the importance of the knowledge of the zeros of sections of power series [**13**].

Let us now give an illustrative example. Consider the series

$$f(z) = \ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots .$$

Since $\lim_{n\to\infty} a_{n+1}/a_n = -1$, we shall choose $g(z) = 1 + z$. It follows that

$$h(z) = z - z\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)} z^n.$$

Thus, the computation of $\ln 2$ with a precision of 10^{-k} will need 10^k terms with the series f and only $10^{k/2}$ terms with the Cauchy-type approximants.

We have $\ln 2 = 0.69314718$, and we obtain

n	$f_n(1)$	$S_n(1)$
8	0.634524	0.697024
9	0.745635	0.690079
42	0.681384	0.693289
43	0.704640	0.693012
73	0.699949	0.693100
74	0.686436	0.693193
100	0.688172	0.693172
101	0.698073	0.693122

Of course, the main practical problem is, as in Padé-type approximation, the choice of the denominator, that is, the choice of the series g. Since a series f satisfying the condition $\lim_{n\to\infty} a_{n+1}/a_n = a \neq 0$ can be accelerated by Aitken's Δ^2 process (that is, the Padé approximants $[n/1]$) the new series transformations proposed in this paper must be compared with it. They must also be compared with the definition of over-acceleration and the acceleration factor introduced by Lembarki [**16**] and studied by Benchiboun [**1**]. The cases when a is unknown or equal to zero, $az = 1$, or when $g(1/a) = 0$ or 1 also deserve further studies and the use of (iii) of Theorem 16, as well. The case of lacunary series and that of periodic-linear ones as defined by Delahaye [**12**] has to be treated. We hope to return to these questions in the future.

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Laboratoire d'Analyse Numérique et d'Optimisation, Université des Sciences et Techniques de Lille Flandres-Artois, 59655 Villeneuve d'Ascq Cedex, France