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GLOBAL EXISTENCE FOR SEMILINEAR PARABOLIC SYSTEMS ON ONE-DIMENSIONAL BOUNDED DOMAINS

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ABSTRACT. We consider semilinear parabolic systems of partial differential equations of the form

$$u_t(x,t) = Du_{xx}(x,t) + Cu_x(x,t) + f(u(x,t)) \quad 0 < x < 1,\ t > 0$$

with bounded initial data and homogeneous Dirichlet boundary conditions, where D is an m by m diagonal positive definite matrix, C is an m by m diagonal matrix and $f : \mathbf{R}^m \to \mathbf{R}^m$ is locally Lipschitz. We prove that if the vector field f satisfies a generalized Lyapunov type condition, then solutions of (1) exist for all $t > 0$. Our result begins an extension of recent results in Morgan [**7**].

1. Introduction and notation. Until recent years, most of the work on semilinear parabolic systems of partial differential equations has fallen into one of two groups; one either assumes that sufficient a priori bounds can be obtained for solutions of the system or assumes a bounded invariant region exists for the system. Of these two approaches, generally only the second considers the vector field involved as anything more than an algebraic expression. Consequently, since invariant regions do not exist for many systems, the geometry of the vector field involved is often ignored. Recently, however, Alikakos [**1**], Groger [**3**], Hollis, Martin, and Pierre [**4**], Masuda [**6**], and others have begun to exploit this geometry via Lyapunov type structures. Some of their results are extended in [**7**]. In this work we extend the results in [**7**] to include systems containing linear convection terms. For simplicity, we restrict our attention to the one-dimensional setting. A complete extension of these results in arbitrary space dimensions, including an interesting treatment of a class of systems with nonlinear diffusion coefficients, will be given in a forthcoming paper of Fitzgibbon, Morgan and Waggoner.

Specifically, we consider systems of the form

(1.1)

$$\begin{aligned} u_t(x,t) &= Du_{xx}(x,t) + Cu_x(x,t) + f(u(x,t)) && 0 < x < 1,\ t > 0 \\ u(x,t) &= 0 && x = 0, 1,\ t > 0 \\ u(x,0) &= u_0(x) && 0 < x < 1 \end{aligned}$$

where

(A1) D is a diagonal m by m matrix with entries $d_i > 0$ on the diagonal,

(A2) C is a diagonal m by m matrix with entries c_i on the diagonal,

(A3) $f : \mathbf{R}^m \to \mathbf{R}^m$ is locally Lipschitz,

(A4) $u_0 : (0,1) \to \mathbf{R}^m$ is bounded and measurable.

Furthermore, it is assumed there exists some *unbounded* invariant m-rectangle $M = M_1 \times \cdots \times M_m$ for (1.1) with faces parallel to the coordinate hyperplanes and a smooth function $H : M \to \mathbf{R}^+$ which satisfies:

(H1) there exists $z \in M$ such that $H(z) = 0$, and if $y \in M$, $y \neq z$, then $H(y) > 0$,

(H2) $H(z) \to \infty$ as $|z| \to \infty$ in M,

(H3) $\partial^2 H(z)$ is nonnegative definite for all $z \in M$,

(H4) there exist $L_1 \geq 0$ such that for all $z \in M$, $\partial H(z) f(z) \leq L_1 H(u)$.

That is, H is a nonnegative convex coercive functional and the vector field f has a restricted growth rate across level curves of H (this is the geometric exploitation of f). In addition, if $L_1 = 0$, then (H1) and (H4) imply that H is a Lyapunov function for the ordinary differential equation

$$v' = f(v) \qquad t \geq 0.$$

Hence, we refer to this H-structure as a generalized Lyapunov structure.

2. Statements of the main results.

Before stating and proving our main results, we state the following well known result (cf. [**4**]).

Theorem 2.1. *Suppose that* (A1)–(A4) *are satisfied. Then there exists* $T_{\max} > 0$ *and* $N = (N_i) \in C([0, T_{\max}), \mathbf{R}^m)$ *such that*

(i) (1.1) *has a unique, classical, noncontinuable solution* $u(x,t)$ *on* $[0,1] \times [0, T_{\max})$*, and*

(ii) $|u_i(\cdot,t)|_{\infty,[0,1]} \le N_i(t)$ *for all* $1 \le i \le m$, $0 \le t < T_{\max}$.

Moreover, if $T_{\max} < \infty$*, then* $|u_i(\cdot,t)|_{\infty,[0,1]} \to \infty$ *as* $t \to T_{\max^-}$ *for some* $1 \le i \le m$.

We state our first result.

Theorem 2.2. *Assume that* (A1)–(A4) *and* (H1)–(H4) *are satisfied and* $u_0 : (0,1) \to M$*. In addition, assume that*

(H5) *there exist* $h_i : M_i \to \mathbf{R}^+$ *for all* $1 \le i \le m$ *such that*

$$H(z) = \sum_{i=1}^{m} h_i(z_i) \quad \text{for all } z \in M.$$

Then there exists $S \in C([0,\infty))$ *such that if* $0 < T < T_{\max}$*, then* $||H(u)||_{2,(0,1)\times(0,T)} \le S(T)$.

This a priori bound actually guarantees global existence for a large class of systems. We state this as our main result.

Theorem 2.3. *Assume that* (A1)–(A4) *and* (H1)–(H5) *are satisfied and* $u_0 : (0,1) \to M$*. In addition, assume that*

(H6) *there exists an* m *by* m *lower triangular matrix* $A = (a_{ij})$ *with positive entries on the diagonal and* $K_1, K_2 \ge 0$ *such that, for all* $1 \le k \le m$,

$$\sum_{j=1}^{k} a_{kj} h_j'(z_j) f_j(z) \le K_1 [H(z)]^2 + K_2 \quad \text{for all } z \in M$$

and

(H7) *there exist* $r, K_3, K_4 \geq 0$ *such that, for all* $1 \leq i \leq m$,

$$h_i'(z_i)f_i(z) \leq K_3[H(z)]^r + K_4 \quad \textit{for all } z \in M.$$

Then $T_{\max} = \infty$.

We note that condition (H6) *does not* imply that the nonlinearities present can be no more than quadratic. It simply states that there is a "balancing" of higher order terms between components of f. We have found many model systems in the literature that satisfy (H1)–(H6).

3. Proofs of Theorems 2.2 and 2.3. Throughout this section we will denote $(0,1)$ by Ω. Furthermore, *all norms given in this section will be taken over* $\Omega \times (\tau, T)$ *unless otherwise stated*. That is, $||v||_p$ will denote $||v||_{p,\Omega\times(\tau,T)}$. The values of τ and T will be given in the context.

In order to prove Theorems 2.2 and 2.3, we will need the following results concerning the scalar equation

$$\begin{aligned} &v_t = d(v_{xx} - \varepsilon v) + cv_x + g && x \in \Omega,\ \tau < t < T \\ &v = 0 && x = 0,1,\ \tau < t < T \\ &v = 0 && x \in \Omega,\ t = \tau \end{aligned} \tag{3.1}$$

where $d, \varepsilon > 0$, $c \geq 0$ and $0 \leq \tau < T$.

Lemma 3.1. *Suppose that* $1 < p < \infty$, $g \in LP(\Omega \times (\tau, T))$ *such that* $g \geq 0$ *and* $c \geq 0$. *Then* (3.1) *has a unique nonnegative solution* v *which satisfies the following.*

(i) *If* $d = 1$ *and* $\varepsilon \geq 0$, *then there exists* $C_p > 0$ *independent of* g *such that*

$$||v(\cdot,T)||_{p,\Omega}, ||v||_p, ||v_t||_p, ||v_x||_p, ||v_{xx}||_p \leq C_p||g||_p.$$

(ii) *If* $1 < p < 3/2$ *and* $p' = 3p/(3-2p)$, *then there exists* $C_{p,(T-\tau)} > 0$ *independent of* g *such that*

$$||v||_{p'} \leq C_{p,(T-\tau)}||g||_p.$$

(iii) *If* $1 < p < 3$ *and* $p' = 3p/(3-p)$, *then there exists* $K_{p,(T-\tau)} > 0$ *independent of* g *such that*

$$||v_x||_{p'} \le K_{p,(T-\tau)}||g||_p.$$

Proof. Existence, uniqueness and part (i) are well known for parabolic equations [**5**]. The solution is nonnegative from basic maximum principles. Parts (ii) and (iii) are a special case [**5**] of Lemma 3.3 combined with part (i). □

Remark. If $\varepsilon > 0$ in part (i), then C_p can be given independent of τ and T. One can obtain much sharper estimates when $p = 2$.

Lemma 3.2. *Suppose* $p = 2$, $d > 0$, $c = 0$ *and* $\varepsilon = 1$ *in the statement of Lemma* 3.1. *If* $||g||_2 = 1$, *then*

(i) $||v_{xx}||_2 \le 1/d$

(ii) $||v||_2 \le [1 - \exp(-d(t-\tau))]/d$ and

(iii) $||v_x||_2 \le \operatorname{sqr}([1 - \exp(-d(t-\tau))]/d)$.

Proof. To obtain part (i) we rewrite the first equation in (3.1) as $v_t - dv_{xx} + dv = g$. Then we square both sides and integrate over $\Omega \times (\tau, T)$. For part (ii) we multiply the first equation in (3.1) by v on both sides and integrate over $\Omega \times (\tau, t)$ for all $\tau \le t \le T$ to obtain

(3.2)
$$\frac{1}{2}\int_\Omega (v(x,t))^2\,dx + d\int_\tau^t \int_\Omega |v_x(x,s)|^2\,dx\,ds + d\int_\tau^t \int_\Omega (v(x,s))^2\,dx\,ds \le \left(\int_\tau^t \int_\Omega (v(x,s))^2\,dx\,ds\right)^{1/2}.$$

Thus, if we denote the third term on the LHS in (3.2) by $w(t)$, then we obtain

$$w'(t) + 2dw(t) \le 2(w(t))^{1/2} \quad \text{for all } \tau \le t \le T$$

and

$$w(0) = 0.$$

Thus, a simple calculation yields part (ii). Substituting this quantity into (3.2) yields part (iii). □

Proof of Theorem 2.2. Let $0 \leq \tau < T < T_{\max}$. Suppose that g is given as in Lemma 3.2, $c = 0$, $\varepsilon = 1$, $d > \max\{d_i\}$ and let v be the solution of (3.1). Set $w(x,t) = v(x, \tau + T - t)$ and $q(x,t) = g(x, \tau + T - t)$ for all $(x,t) \in \Omega \times (\tau, T)$, and note that w satisfies parts (i)–(iii) in Lemma 3.2. Furthermore, let $b = \min\{d_i\}$ and $e = \max\{|c_i|\}$. Then we can easily show that

$$\begin{aligned} &\int_\tau^T \int_\Omega H(u) q \, dx \, dt \\ &\quad \leq \int_t^T \int_\Omega H(u)[(L_1 + d)w + (d-b)|w_{xx}| + e|w_x|] \, dx \, dt \\ &\qquad + \int_\Omega H(u(x,\tau)) w(x,\tau) \, dx. \end{aligned} \tag{3.3}$$

Also, from Lemma 3.2 and Holder's inequality, the first term on the RHS of (3.3) is bounded by

$$||H(u)||_2 \left[\frac{L_1 + d}{d}(1 - \exp(-d(T-\tau))) + \left(1 - \frac{b}{d}\right) + e\left(\frac{1}{d}(1 - \exp(-d(T-\tau)))\right)^{1/2} \right] \tag{3.4}$$

which is bounded by

$$||H(u)||_2[1 - b/(2d)] \quad \text{for } \tau \text{ sufficiently close to } T_{\max}. \tag{3.5}$$

In addition, the second term on the RHS of (3.3) is bounded by

$$||H(u(\cdot,\tau))||_{\infty,\Omega}[|\Omega|(T-\tau)]^{1/2}. \tag{3.6}$$

Consequently, if we combine (3.3) with the bounds obtained in (3.4)–(3.6), then we obtain

$$||H(u)||_2 \leq \frac{2d}{b}||H(u(\cdot,\tau))||_{\infty,\Omega}[|\Omega|(T-\tau)]^{1/2} \tag{3.7}$$

for all τ sufficiently close to T. The result follows. $\square$

In order to prove Theorem 2.3, we will need the following lemma.

Lemma 3.3. *Suppose* $\omega \geq 1$ *and for all* $\omega \leq p < \infty$ *there exist* $0 < \delta_p < 1$ *and* $M_p, N_p \in C([0,\infty))$ *such that, for all* $1 \leq j \leq m$ *and* $0 = \tau < T < T_{\max}$,

$$||h_j(u_j)||_p \leq M_p(T) + N_p(T)||H(u)||_p^{\delta_p}. \tag{3.8}$$

Then $T_{\max} = \infty$.

Proof. Suppose (by way of contradiction) that $T_{\max} < \infty$. Then (3.8) holds when $T = T_{\max}$. Consequently,

$$||H(u)||_p \leq M_p(T_{\max}) + N_p(T_{\max})||H(u)||_p^{\delta_p} \tag{3.9}$$

for all $\omega \leq p < \infty$, which implies that $||H(u)||_p < \infty$ for all $1 \leq p < \infty$. It then follows that

$$||K_3[H(u)]^r + K_4||_p < \infty \tag{3.10}$$

for all $1 \leq p < \infty$, where K_3, K_4 and r are given in (H7). Now let $1 \leq j \leq m$, set $w = h_j(u_j)$ and suppose $v_0 = || \, |u_0| \, ||_{\infty,\Omega}$. Then from the convexity of H and standard maximum principles we have $w \leq v$ where v solves

$$\begin{aligned} &v_t = d_j v_{xx} + c_j v_x + K_3[H(u)]^r + K_4 && \text{on } \Omega \times (0, T_{\max}) \\ &v = 0 && \text{on } \{0,1\} \times (0, T_{\max}) \\ &v = v_0 && \text{on } \Omega \times \{0\}. \end{aligned} \tag{3.11}$$

Furthermore, from Lemma 3.1 and the Sobolev imbedding theorem we have $||w||_\infty \leq ||v||_\infty < \infty$. That is, $||h_i(u_i)||_\infty < \infty$ for all $1 \leq i \leq m$. Hence, from the coercivity of H, we have $||u_i||_\infty < \infty$ for all $1 \leq i \leq m$. This contradicts (via Theorem 2.1) our assumption that $T_{\max} < \infty$, and, therefore, $T_{\max} = \infty$. $\square$

Proof of Theorem 2.3. Let $1 \leq k \leq m$ be given such that the hypotheses of Lemma 3.3 hold with the restriction $j < k$ (this holds trivially if $k = 1$). We will show that these hypotheses also hold for $j = k$.

One easily shows that if we choose $1 < p < 3/2$, with p sufficiently close to 1, and set

$$b_1 = \frac{6p - 2p^2}{(2-p)(5p-3)} \quad \text{and} \quad b_2 = \frac{2p^2}{4p-3}$$

then

$$\frac{2}{b_1} > 1, \qquad \left(\frac{6p}{5p-3} - b_1\right)\frac{2}{2-b_1} = \frac{p}{p-1} \quad \text{and}$$
$$\left(\frac{p-1}{p} + \frac{2}{3}\right)\frac{2-b_1}{2} < \frac{p-1}{p}$$

and

$$\frac{2}{b_2} > 1, \qquad \left(\frac{3p}{4p-3} - b_2\right)\frac{2}{2-b_2} = \frac{p}{p-1} \quad \text{and}$$
$$\left(\frac{4p-3}{3p}\right)\frac{2-b_2}{2} < \frac{p-1}{p}.$$

Now, let $\varepsilon = 1$, $c = 0$ and $d = d_k$ in (3.1), $p' = p/(p-1)$ and $0 \le \tau < T < T_{\max}$. Suppose that g is given as in Lemma 3.1, with $||g||_p = 1$, and let v be the solution of (3.1). Set $w(x,t) = v(x, \tau + T - t)$ and $q(x,t) = g(x, \tau + T - t)$ for all $(x,t) \in \Omega \times (\tau, T)$, and note that w satisfies parts (i)–(iii) in Lemma 3.1. Applying (H1)–(H6), we obtain

(3.12)
$$\begin{aligned}\int_\tau^T \int_\Omega h_i(u_i) q \, dx \, dt = &\int_t^T \int_\Omega h_i(u_i)[d_k w + (d_i - d_k) w_{xx} + c_i w_x] \, dx \, dt \\ &+ \int_\Omega h_i(u_i(x,\tau)) w(x,\tau) \, dx \\ &+ \int_\tau^T \int_\Omega w h_i'(u_i) f_i(u) \, dx \, dt\end{aligned}$$

and, thus,

(3.13)

$$\int_\tau^T \int_\Omega \sum_{j=1}^{k} a_{kj} h_j(u_j) q \, dx \, dt$$

$$\leq \int_\tau^T \int_\Omega \sum_{j=1}^{k} a_{kj} h_j(u_j)[d_k w + (d_j - d_k) w_{xx} + c_j w_x] \, dx \, dt$$

$$+ \int_\Omega \sum_{j=1}^{k} a_{kj} h_j(u_j(x,\tau)) w(x,\tau) \, dx$$

$$+ \int_\tau^T \int_\Omega w[K_1 (H(u))^2 + K_2] \, dx \, dt.$$

Consequently,

(3.14)

$$\int_\tau^T \int_\Omega a_{kk} h_k(u_k) q \, dx \, dt \leq \text{ RHS above } - \sum_{j=1}^{k-1} a_{kj} \int_\tau^T \int_\Omega h_j(u_j) q \, dx \, dt.$$

We proceed to develop some bounds on the terms appearing on the right-hand side of (3.14). First, note that part (ii) of Lemma 3.1 and Holder's inequality yield

(3.15)

$$\int_\tau^T \int_\Omega w(H(u))^2 \, dx \, dt \leq C_{p,(T-\tau)} \left[\int_\tau^T \int_\Omega (H(u))^{6p/(5p-3)} \, dx \, dt \right]^{(p-1)/p+2/3}.$$

Hence, from the choice of p above, there exists $0 < \varepsilon_{p'} < 1$ such that

(3.16)

$$\left[\int_\tau^T \int_\Omega (H(u))^{6p/(5p-3)} \, dx \, dt \right]^{(p-1)/p+2/3}$$

$$= \left[\int_\tau^T \int_\Omega (H(u))^{6p/(5p-3)-b_1} (H(u))^{b_1} \, dx \, dt \right]^{(p-1)/p+2/3}$$

$$\leq ||H(u)||_{p'}^{\varepsilon_{p'}} ||H(u)||_2^{b_1[(5p-3)/6p]}.$$

Thus, (3.15) and (3.16) imply

$$\text{(3.17)} \quad \int_\tau^T \int_\Omega w[K_1 (H(u))^2 + K_2] \, dx \, dt$$

$$\leq K_1 C_{p,(T-\tau)} ||H(u)||_2^{b_1[(5p-3)/6p]} ||H(u)||_{p'}^{\varepsilon_{p'}} + K_2 C_p [|\Omega| (T-\tau)]^{1/p'}.$$

In a similar manner, we apply part (iii) of Lemma 3.1 to obtain $0 < \varepsilon'_{p'} < 1$ such that

$$\int_\tau^T \int_\Omega H(u)|w_x|\,dx\,dt \le K_{p,(T-\tau)}||H(u)||_2^{b_2[(4p-3)/6p]}||H(u)||_{p'}^{\varepsilon'_{p'}}. \tag{3.18}$$

Now, from part (ii) of Lemma 3.1 and Holder's inequality,

$$\int_\tau^T \int_\Omega H(u)w\,dx\,dt \le C_{p,(T-\tau)}||H(u)||_2[|\Omega|(T-\tau)]^{(3-2p)/3p}. \tag{3.19}$$

Furthermore,

$$\sum_{j=1}^k |a_{kj}| \int_\Omega H(u(x,\tau))w(x,\tau)\,dx \le \sum_{j=1}^k |a_{kj}|||H(u(\cdot,\tau))||_{\infty,\Omega} C_p|\Omega|^{1/p'} \tag{3.20}$$

and from our hypothesis and (3.8), for all $1 \le j \le k-1$, we have

$$\int_\tau^T \int_\Omega h_j(u_j)|w_{xx}|\,dx\,dt \le C_p[|\Omega|(T-\tau)]^{1/p}[M_{p'}(T) + N_{p'}(T)||H(u)||_{p'}^{\delta_{p'}}]. \tag{3.21}$$

Finally, from our hypothesis and (3.8), for all $1 \le j \le k-1$, we have

$$\int_\tau^T \int_\Omega h_j(u_j)q\,dx\,dt \le M_{p'}(T) + N_{p'}(T)||H(u)||_{p'}^{\delta_{p'}}. \tag{3.22}$$

Therefore, if we apply (3.17)–(3.22) to the right-hand side of (3.14), then we see that the hypotheses of Lemma 3.3 hold with the restriction $1 \le j \le k$, and, hence, they hold without restriction, i.e., $T_{\max} = \infty$. □

4. An application. In this section we consider an extension of the Schlogl model [**8**] due to Gray and Scott [**2**]:

$$A + 2B \rightleftarrows 3B$$
$$B \rightleftarrows C.$$

We consider the Gray-Scott model in a one-dimensional reaction-diffusion system in which chemicals are fed perpendicular to a thin reacting layer of length one. The equations in nondimensionalized form are:

$$(4.1)\qquad \begin{aligned} a_t &= d_1 a_{xx} + k_1 a_x - ab^2 + \eta_1 b^3 + \alpha(1-a) \\ b_t &= d_2 b_{xx} + k_2 b_x + ab^2 - \eta_1 b^3 - \beta(b - \eta_2 c) - \alpha b \\ c_t &= d_3 c_{xx} + k_3 c_x + \beta(b - \eta_2 c) - \alpha c \end{aligned}$$

with zero Dirichlet boundary conditions at $x = 0$ and $x = 1$ and nonnegative initial data. Here k_1, k_2, and k_3 are real numbers and $d_1, d_2, d_3, \alpha, \beta, \eta_1, \eta_2 > 0$. Setting $H(a,b,c) = a^2 + \eta_1 b^2 + \eta_1\eta_2 c^2 + \alpha$, we see that (H1)–(H7) are clearly satisfied. Therefore, from Theorem 2.3, solutions to (4.1) exist for all $t > 0$.

REFERENCES

1. N.D. Alikakos, *L_p-bounds of solutions of reaction-diffusion equations*, Comm. Partial Differential Equations **4**, No. 8 (1979), 827–868.

2. P. Gray and S.K. Scott, Chem. Engn. Sci. **38** (1983) 29; Chem. Engn. Sci. **39** (1984) 1087; J. Phys. Chem. **89** (1985) 22.

3. K. Groger, *On the existence of steady states of certain reaction-diffusion systems*, Arch. Rational Mech. Anal. **1**, 1986.

4. S. Hollis, R. Martin, and M. Pierre, *Global existence and boundedness in reaction-diffusion systems*, SIAM J. Math. Anal. **18**, No. 3 (1987), 744–761.

5. O.A. Ladyzenskaja, V.A. Solonnikov, and N.N. Uralceva, *Linear and quasilinear equations of parabolic type*, Transl. Math. Monographs, **33**, Amer. Math. Soc., Providence, Rhode Island, 1968.

6. K. Masuda, *On the global existence and asymptotic behavior of solutions of reaction-diffusion equations*, Hokkaido Math. J., **XII** (1982), 360–370.

7. J. Morgan, *Global existence for semilinear parabolic systems*, SIAM J. Math. Anal. **20** (1989), 1128–1144.

8. F. Schlogl, Z. Phys. **253** (1972), 147.

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