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FOCAL SUBFUNCTIONS AND SECOND ORDER DIFFERENTIAL INEQUALITIES

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1. Introduction. In this paper we are interested in the differential equation

$$y'' = f(x, y, y'), \tag{1.1}$$

along with the "right focal" and "conjugate" boundary conditions (BC's), denoted respectively by

$$y(x_1) = y_1, \qquad y'(x_2) = y_2 \tag{1.2R}$$

and

$$y(x_1) = y_1, \qquad y(x_2) = y_2, \tag{1.2C}$$

where $x_1 < x_2, x_1, x_2 \in I$, an interval in $\mathbf{R}$, and $y_1, y_2 \in \mathbf{R}$ are arbitrary.

The BC's of the above type for equation (1.1) have been considered by several authors and for a variety of results concerning these problems, reference may be made to the papers [**1**–**10**] and to some of the other references contained therein.

For the sake of convenience we label the hypotheses that we use as follows:

A. f is continuous on $I \times \mathbf{R}^2$.

UC. Solutions of conjugate boundary value problems (BVP's) of (1.1), if they exist, are unique on I (that is, $y(x)$, $z(x)$ are solutions of the BVP (1.1), (1.2C) for arbitrary x_1, x_2 in I, $x_1 < x_2$ and $y_1, y_2 \in \mathbf{R}$ implies $y(x) \equiv z(x)$ on $[x_1, x_2]$).

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E. All solutions of (1.1) exist on I.

It is well known [**9,** Theorem 1] that, under hypotheses A, UC and E, if $u(x)$ is a "lower solution" equation (1.1) on I (that is, $u(x) \in C^2(I)$ and $u''(x) \geq f(x, u(x), u'(x))$ for all x in I) then $u(x)$ is a conjugate subfunction with respect to solutions of (1.1) on I (that is, $u(x)$ satisfies the inequality

$$u(x) \leq y(x) \tag{1.3}$$

on $[x_1, x_2]$ whenever

$$u(x_1) \leq y_1, \qquad u(x_2) \leq y_2 \tag{1.4C}$$

holds and $y(x)$ is a solution of the BVP (1.1), (1.2C) for arbitrary $x_1, x_2 \in I$, $x_1 < x_2$ and $y_1, y_2 \in \mathbf{R}$).

In order to motivate further discussion and state our main result we need the following definitions:

Definition 1.1. $u(x) \in C^1(I)$ is a "right focal subfunction" with respect to solutions of (1.1) on I if the inequality (1.3) holds on $[x_1, x_2]$ whenever

$$u(x_1) \leq y_1, \qquad u'(x_2) \leq y_2 \tag{1.4R}$$

holds and $y(x)$ is a solution of the BVP (1.1), (1.2R) for arbitrary $x_1, x_2 \in I$, $x_1 < x_2$ and $y_1, y_2 \in \mathbf{R}$.

Hereafter, we shall simply use the term lower solution (right focal subfunction) omitting the words with respect to (with respect to solutions of) equation (1.1) since it is always meant that way.

It follows (Lemma 2.2) from the above definition that a right focal subfunction on an interval I is necessarily a conjugate subfunction on I, and hence a C^2-right focal subfunction on I is a lower solution on I by [**5**, Theorem 3.2]. However, the example given in Section 2 shows that, even in the case of a linear differential equation with hypotheses, A, UC, E, a lower solution need not be a right focal subfunction.

It also follows easily from the definitions of lower solution and right focal subfunction that if lower solutions on an interval I are right focal subfunctions on I, then UR holds on I where UR stands for

UR. Solutions of right focal BVP's of (1.1) if they exist are unique on I (that is, if $y(x), z(x)$ are solutions of the BVP (1.1), (1.2R), then $y(x) \equiv z(x)$ on $[x_1, x_2]$ for arbitrary x_1, x_2 in I, $x_1 < x_2$ and $y_1, y_2 \in \mathbf{R}$).

So now one can raise the question whether, under the stronger hypotheses A, UR (for the result UR implies UC refer to Lemma 2.6), and E lower solutions on an interval I, are right focal subfunctions on I. We answer this question in the affirmative in our main theorem (Theorem 4.1).

This theorem is proved by means of a "local existence" theorem (Theorem 3.1) for focal BVP's, the result (Corollary 3.2) that lower solutions under the hypothesis UR are right focal subfunctions in the "small," and an induction argument similar to that used in the proof of Theorem 1 of [**9**] but using hypothesis UR rather than UC.

We also show in Theorem 5.1 that if I is an interval which is open at the left end point, then, under hypotheses A, UR and E, the BVP (1.1), (1.2R) has a solution for $x_1, x_2 \in I$, $x_1 < x_2$ and $y_1, y_2 \in \mathbf{R}$ arbitrary. This theorem removes the restrictions of I being open at the right end point and the uniqueness of solutions of initial value problems but yields the same conclusion as that of the theorem in [**8**] and the theorem with $n = 2$ of [**2**].

Finally we show, by means of an example, that the above stated theorem is not true if I is a closed interval. However, it remains an open question whether the theorem is true or not if I is closed at the left end point, but open at the right end point.

It will be assumed that hypothesis A holds through the remaining sections of this paper.

2. Preliminary results.

Lemma 2.1. *Suppose* $u \in C^1[c, d]$, $u(c) \geq 0$, $u(d) \geq 0$ *and* $u(x) < 0$ *for some* x, $c < x < d$. *Then there exists an interval* $[x_1, x_2] \subset [c, d)$ *such that* $u(x_1) = 0$, $u'(x_2) = 0$ *and* $u(x) < 0$ *on* (x_1, x_2).

Proof. By hypotheses, $u(x)$ attains its negative minimum at some point, say $x_2 \in (c, d)$. Let $x_1 = \text{Sup}\{c < x < x_2 : u(x) = 0\}$. Then $[x_1, x_2] \subset [c, d)$ and $u(x)$ satisfies $u(x_1) = 0$, $u'(x_2) = 0$ and $u(x) < 0$ on (x_1, x_2). □

Lemma 2.2. *If $u \in C^1(I)$ is a right focal subfunction on I, then u is a conjugate subfunction on I.*

Proof. Suppose $u(x)$ satisfies the condition (1.4C) for some $x_1, x_2 \in I$, $x_1 < x_2$ and $y_1, y_2 \in \mathbf{R}$ and $y(x)$ is a solution of the BVP (1.1), (1.2C). If $u(x) \leq y(x)$ does not hold on $[x_1, x_2]$, then there exists x', $x_1 < x' < x_2$ such that $u(x') > y(x')$ and consequently, by Lemma 2.1, an interval $[x_3, x_4] \subset [x_1, x_2)$ such that $u(x_3) = y(x_3)$, $u'(x_4) = y'(x_4)$ and $u(x) > y(x)$ on (x_3, x_4). This contradicts the hypothesis that $u(x)$ is a right focal subfunction on I. □

Corollary 2.3. *If $u \in C^2(I)$ is a right focal subfunction on I, then u is a lower solution on I.*

Proof. This is a consequence of Lemma 2.2 above and Theorem 3.2 [**5**]. □

However, the converse of Corollary 2.3 need not be true even in the case of linear differential equations satisfying hypotheses A, UC and E as shown by the following example.

Example. Consider the equation $y'' + y = 0$, $0 \leq x \leq 3\pi/4$. Then $y(x) \equiv \text{Sin}\, x$ and $z(x) \equiv 0$ are both solutions satisfying the same right focal BC's, $y(0) = 0$, $y'(\pi/2) = 0$, and hence not all lower solutions are right focal subfunctions on $[0, 3\pi/4]$.

Lemma 2.4. *If $u \in C^1[x_1, x_2]$ and attains a minimum at a point x_0, $x_1 < x_0 \leq x_2$, then* (i) *$u'(x_0) = 0$ if $x_1 < x_0 < x_2$ and* (ii) *$u'(x_0) \leq 0$ if $x_0 = x_2$.*

Lemma 2.5. *Suppose* $u \in C^1[c,d]$, $u(c) \geq 0$ *and* $u(x) < 0$ *for some* x *in* (c,d). *Then there exists a subinterval* $[c_1, d_1] \subset [c,d]$ *such that* $u(c_1) = 0$, $u'(d_1) \leq 0$ *and* $u(x) < 0$ *on* (c_1, d_1).

Proof. To see this, let $c < d_1 \leq d$ be such that $u(x)$ attains its negative minimum on $[c,d]$ at d_1 and $c_1 = \mathrm{Sup}\{c \leq x < d_1 : u(x) = 0\}$. Now the conclusion is obvious by virtue of Lemma 2.4. □

Lemma 2.6. UR *implies* UC.

Proof. This is a consequence of Rolle's theorem. □

In the following three lemmas, too, we assume hypothesis UR holds. In addition, assume hypothesis E holds in Lemmas 2.7 and 2.8. These lemmas are easy consequences of these hypotheses and other assumptions made therein. Hence their proofs are omitted.

Lemma 2.7. *Suppose* $y(x), z(x)$ *are solutions of* (1.1) *satisfying* $y(x_1) = z(x_1)$, $y'(x_2) > z'(x_2)$ *for some* x_1, x_2 *in* I *and* $x_1 \leq x_2$. *Then* $y'(x) > z'(x)$ *for all* $x \geq x_2$, $x \in I$.

Lemma 2.8. *Suppose* $y(x), z(x)$ *are solutions of* (1.1) *such that* $y(x_1) = z(x_1)$ *and* $y(x_2) > z(x_2)$ *for some* x_1, x_2 *in* I *and* $x_1 < x_2$. *Then* $y'(x) > z'(x)$ *for all* $x \geq x_2, x \in I$.

Lemma 2.9. *Suppose* $y(x), z(x)$ *are solutions of* (1.1) *satisfying* $y(x_1) = z(x_1)$, $y(x_2) \geq z(x_2)$ *and* $y'(x_2) \leq z'(x_2)$ *for some* x_1, x_2 *in* I *and* $x_1 < x_2$. *Then* $y(x_2) = z(x_2)$ *holds, and hence* $y(x) \equiv z(x)$ *on* $[x_1, x_2]$.

3. Local existence theorem and consequences for focal BVP's.

Theorem 3.1. *Let* $M > 0$, $N > 0$ *be given. Let* q *be the maximum of* $|f(x,y,y')|$ *on the compact set* $\{(x,y,y') : a \leq x \leq b,\ |y| \leq 2M,\ |y'| \leq 2N\}$. *Assume* $q > 0$ *and* $\delta = \mathrm{Min}\{\sqrt{(2M/q)}, N/q\}$. *Then*

(i) *The BVP* (1.1), (1.2R) *with* $[x_1, x_2] \subset [a, b]$, $x_2 - x_1 \le \delta$, $|y_1| \le M$, $|y_2| \le N$, $|y_1 + y_2(x_2 - x_1)| \le M$ *has a solution* $y(x)$.

(ii) *If* $\varepsilon > 0$ *is given,* $\delta^* = \mathrm{Min}\{\delta, \varepsilon/q, \sqrt{(2\varepsilon/q)}\}$, $[x_1, x_2] \subset [a, b]$, $x_2 - x_1 \le \delta^*$ *and* $\omega(x)$ *is the unique linear function satisfying* $\omega(x_1) = y_1$, $\omega'(x_2) = y_2$, *then the* BVP (1.1), (1.2R) *has a solution* $y(x)$ *satisfying* $|y(x) - \omega(x)| < \varepsilon$ *and* $|y'(x) - \omega'(x)| < \varepsilon$ *on* $[x_1, x_2]$.

Remark. The proof of this theorem is a standard application of Schauder's fixed point theorem and, hence, is not given here. However, we remark that the estimate for δ, given in the theorem, is arrived at by using the following estimates of Green's function and integrals involving the Green's function, namely,

$$|G(x,t)| \le (x_2 - x_1), \int_{x_1}^{x_2} |G(x,t)|\, dt \le (x_2 - x_1)^2/2$$

and

$$\int_{x_1}^{x_2} |G_x(x,t)|\, dt \le (x_2 - x_1).$$

Note. If, in the above theorem, $q = 0$, that is, $f(x, y, y') \equiv 0$ for $a \le x \le b$, $|y| \le 2M$, $|y'| \le 2N$, then $y(x) = \omega(x)$ is the solution of the BVP (1.1) and (1.2R) for arbitrary x_1, x_2, y_1, y_2.

The following corollary is the analogue to right focal BVP's of a similar result for conjugate BVP's which is contained in the proof of Lemma 1 in [**9**]. This corollary which implies that lower solutions under hypothesis UR are right focal subfunctions "in the small" is used frequently in the proof of Theorem 4.1.

Corollary 3.2. *Let* $u \in C^1[a, b]$ *be given. Then*

(i) *There exists a* $\delta > 0$ *such that, for* $[x_1, x_2] \subset [a, b]$ *and* $x_2 - x_1 \le \delta$, *there exists a solution* $y(x)$ *of the BVP* (1.1) *and* $y(x_1) = u(x_1)$, $y'(x_2) = u'(x_2)$.

(ii) *Given* $\varepsilon > 0$, $[x_1, x_2] \subset [a, b]$, *and* $\omega(x)$ *the unique linear function satisfying* $\omega(x_1) = u(x_1)$, $\omega'(x_2) = u'(x_2)$, *there exists* δ^*, $0 < \delta^* \le \delta$ *such that* $x_2 - x_1 \le \delta^*$ *implies the BVP* (1.1) *and* $y(x_1) = u(x_1)$, $y'(x_2) = u'(x_2)$ *has a solution* $y(x)$ *satisfying* $|y^{(i)}(x) - \omega^{(i)}(x)| < \varepsilon$ *on* $[x_1, x_2]$ *for* $i = 0, 1$.

(iii) *If, in addition,* $u(x)$ *is a lower solution of* (1.1) *on* $[a,b]$*, then there exists a* δ^{**}*,* $0 < \delta^{**} \leq \delta$ *such that the BVP in* (i) *has a solution* $y(x)$ *satisfying* $u(x) \leq y(x)$ *on* $[x_1, x_2]$*, provided* $x_2 - x_1 \leq \delta^{**}$.

Proof. (i) and (ii) follow from Theorem 3.1 if we choose

$$M = \mathrm{Max}\{|u(x)| : a \leq x \leq b\} + (b-a)\mathrm{Max}\{|u'(x)| : a \leq x \leq b\},$$
$$N = \mathrm{Max}\{|u'(x)| : a \leq x \leq b\},$$
$$u(x_1) = y_1 \quad \text{and} \quad u'(x_2) = y_2.$$

(iii). If we define

$$F(x,y,y') = \begin{cases} f(x,y,y'), & y \geq u(x) \\ & a \leq x \leq b \\ f(x,u(x),y') - (u(x)-y), & y \leq u(x) \\ & a \leq x \leq b \end{cases},$$

then the proof is the same as that of [**9**, Lemma 1], except, in showing $y(x) \geq u(x)$ on $[x_1, x_2]$, we have to note that, when we assume the opposite inequality in the proof of our theorem, the positive maximum of $u(x) - y(x)$ can occur at the end point x_2; if it does we use the boundary condition $y'(x_2) = u'(x_2)$ to continue the proof along the same lines as that of [**9**, Lemma 1]. □

Corollary 3.3. (Global existence theorem for right focal BVP's). *Let* $f(x,y,y')$ *be such that* $|f(x,y,y')| \leq h + k|y|^{\alpha}$ *for some constants* $h > 0$*,* $k > 0$ *and* $0 \leq \alpha < 1$*. Then the BVP* (1.1), (1.2R) *for arbitrary* $x_1, x_2 \in I$*,* $x_1 < x_2$ *and* $y_1, y_2 \in \mathbf{R}$ *has a solution* $y(x)$.

Proof. This is along the same lines as that of [**6**, Lemma 2.2]. □

It may be remarked that the above corollary is the same as Corollary 3.2 of [**1**] with $n = 2$, $a_0 = \alpha$, and $\alpha_1 = 0$, but in [**1**] its proof was based on the topological transversality method.

4. Focal subfunctions.

Theorem 4.1. *Assume hypotheses* A, UR, E. *Then lower solutions on I are right focal subfunctions on I.*

Proof. Suppose $u(x)$ is a lower solution on I, but not a right focal subfunction on I. Then, by Lemmas 2.4, 2.5, and Definition 1.1, there exists an interval $[c,d] \subset I$ and a solution $y_1(x)$ of (1.1) such that $y_1(c) = u(c)$, $y_1'(d) = u'(d)$ and $y_1(x) < u(x)$ on (c,d).

Since $u \in C^1[c,d]$ is a lower solution on $[c,d]$, by Corollary 3.2(iii) there exists a δ, $0 < \delta < d-c$ such that, for $[x_1,x_2] \subset [c,d]$ and $x_2 - x_1 \le \delta$, the BVP (1.1) and $y(x_1) = u(x_1)$, $y'(x_2) = u'(x_2)$ has a solution $y(x)$ satisfying $u(x) \le y(x)$ on $[x_1,x_2]$ (that is, $u(x)$ is a right focal subfunction "in the small").

For each positive integer n, let $P(n)$ be the proposition that there exists an interval $[c_n,d_n] \subset [c,d]$ with $0 < d_n - c_n \le d - c - (n-1)\delta$ and a solution $y_n(x)$ with $y_n(c_n) = u(c_n)$, $y_n'(d_n) = u'(d_n)$ and $y_n(x) < u(x)$ on (c_n,d_n). Obviously $P(n)$ cannot be true for all $n \ge 1$. However, assuming that $u(x)$ is not a right focal subfunction on I, we will show by an induction argument that $P(n)$ is true for all n, thereby proving that $u(x)$ must be a right local subfunction on I.

$P(1)$ is true since we can choose $[c_1,d_1] = [c,d]$ with $y_1(x)$ same as above.

Assume $P(k)$ is true, that is, there exists an interval $[c_k,d_k] \subset [c,d]$, with $0 < d_k - c_k \le d - c - (k-1)\delta$ and a solution $y_k(x)$, with $y_k(c_k) = u(c_k)$, $y_k'(d_k) = u'(d_k)$ and $y_k(x) < u(x)$ on (c_k,d_k). Now $d_k - c_k > \delta$ since, otherwise, by Corollary 3.2(iii), there exists a solution distinct from $y_k(x)$ for the BVP (1.1) and $y(c_k) = u(c_k)$, $y'(d_k) = u'(d_k)$.

Let $z_1(x)$ be the solution of the BVP (1.1) and $y(c_k) = u(c_k)$, $y'(c_k+\delta) = u'(c_k+\delta)$ so that, by Corollary 3.2(iii), $u(x) \le z_1(x)$ on $[c_k,c_k+\delta]$ and, hence, $y_k(x) < z_1(x)$ on $(c_k,c_k+\delta]$. By Lemma 2.8, hypotheses UR and E, we have $z_1'(x) > y_k'(x)$ for all $x > c_k$ and, hence, $z_1'(d_k) > y_k'(d_k) = u'(d_k)$.

Now, assuming $P(k+1)$ not true, we first prove

Claim (i). $z_1(x) \ge u(x)$ *on* $[c_k+\delta, d_k]$.

For, if not, there exists x', $c_k + \delta < x' \le d_k$ such that $\mathrm{Min}\{z_1(x) - u(x) : c_k + \delta \le x \le d_k\} = z_1(x') - u(x') < 0$. However, if $x' = d_k$, then $z_1'(d_k) - u'(d_k) \le 0$ by (ii) of Lemma 2.4, a contradiction. Hence $c_k + \delta < x' < d_k$, which in turn implies, by (i) of Lemma 2.4, that $P(k+1)$ is true with $y_{k+1}(x) \equiv z_1(x)$ and $[c_{k+1}, d_{k+1}] \subsetneq [c_k + \delta, d_k]$, an interval of length $d_k - c_k - \delta \le d - c - k\delta$. Hence the claim is true. ■

Claim (ii). $d_k - (c_k + \delta) > \delta$.

For, if otherwise, the focal BVP (1.1) and $y(c_k + \delta) = u(c_k + \delta)$, $y'(d_k) = u'(d_k)$, by Corollary 3.2(iii) will have a solution $z_2(x)$ with $u(x) \le z_2(x)$ on $[c_k + \delta, d_k]$, and consequently $u'(c_k + \delta) \le z_2'(c_k + \delta)$. By hypothesis E, $z_2(x)$ exists on I, and hence either (a) there exists x', $c_k \le x' < c_k + \delta$ such that $z_2(x') = y_k(x')$ or (b) there exists a largest x'', $c_k < x'' \le c_k + \delta$ such that $z_2(x'') = z_1(x'')$.

If case (a) occurs, then, by Lemmas 2.7 and 2.8, we must have $z_2'(d_k) > y_k'(d_k) = u'(d_k)$, a contradiction.

On the other hand, if case (b) occurs again there are two possibilities, namely, $x'' < c_k + \delta$ and $x'' = c_k + \delta$. If $x'' < c_k + \delta$, then, since $z_1(c_k + \delta) \ge z_2(c_k + \delta)$ and $z_1'(c_k + \delta) = u'(c_k + \delta) \le z_2'(c_k + \delta)$, we have, by Lemma 2.9, that

$$w_0(x) \equiv \begin{cases} z_1(x), & c_k \le x \le c_k + \delta, \\ z_2(x), & c_k + \delta \le x \le d_k, \end{cases}$$

and $y_k(x)$ are two distinct solutions of the same right focal BVP, a contradiction to UR. If $x'' = c_k + \delta$, then $z_1'(c_k + \delta) = u'(c_k + \delta) \le z_2'(c_k + \delta)$. However, the strict inequality in the above statement is ruled out by virtue of the fact that $z_1'(d_k) > z_2'(d_k)$ and by Lemma 2.7. Hence, $z_1'(c_k + \delta) = z_2'(c_k + \delta)$, and consequently,

$$z(x) \equiv \begin{cases} z_1(x), & c_k \le x \le c_k + \delta, \\ z_2(x), & c_k + \delta \le x \le d_k, \end{cases}$$

and $y_k(x)$ are two distinct (distinct since $y_k(x) < u(x) \le z_1(x)$ on $(c_k, c_k + \delta)$) solutions of the same right focal BVP, a contradiction to UR. Hence, claim (ii) is true. ■

Now there exists, by Corollary 3.2(iii), a solution $z_2(x)$ of (1.1) such that $z_2(c_k+\delta) = u(c_k+\delta)$, $z_2'(c_k+2\delta) = u'(c_k+2\delta)$ and $u(x) \le z_2(x)$ on $[c_k+\delta, c_k+2\delta]$. Consequently, $z_2'(c_k+\delta) \ge u'(c_k+\delta) = z_1'(c_k+\delta)$. By hypothesis E, $z_2(x)$ exists on I and hence either (a) there exists an x', $c_k \le x' \le c_k+\delta$ such that $z_1(x') = z_2(x')$ or (b) there exists a largest x'', $c_k < x'' < c_k+\delta$ such that $z_2(x'') = y_k(x'')$.

If case (a) occurs let

$$w_1(x) = \begin{cases} z_1(x), & x \le c_k+\delta, \\ z_2(x), & x \ge c_k+\delta, \end{cases}$$

so that, by Lemma 2.9, $w_1(x)$ is a solution of (1.1) satisfying $w_1(c_k) = u(c_k)$ and $w_1'(c_k+2\delta) = u'(c_k+2\delta)$. If case (b) occurs let $w_1(x) \equiv z_2(x)$, for $x \ge x''$.

Claim (iii). $d_k - (c_k+2\delta) > \delta$.

Suppose $d_k - (c_k+2\delta) \le \delta$. Then, by Corollary 3.2(iii), there exists a solution $z_3(x)$ of equation (1.1) satisfying $z_3(c_k+2\delta) = u(c_k+2\delta)$, $z_3'(d_k) = u'(d_k)$ and $z_3(x) \ge u(x)$ on $[c_k+2\delta, d_k]$, and, consequently, $z_3'(c_k+2\delta) \ge u'(c_k+2\delta) = w_1'(c_k+2\delta)$ and $z_3(c_k+2\delta) > y_k(c_k+2\delta)$. By hypothesis E, $z_3(x)$ exists on I, and, hence, either (a) there exists an x', $c_k \le x' \le c_k+2\delta$ such that $z_3(x') = w_1(x')$ or (b) there exists a largest x''', $c_k < x''' < c_k+2\delta$ such that $z_3(x''') = y_k(x''')$.

If (a) occurs, we can consider, as in the proof of Claim (ii), the two possibilities $x' < c_k+2\delta$ and $x' = c_k+2\delta$. In either case, by using an argument identical to that in Claim (ii), we can arrive at a contradiction to UR.

If (b) occurs, choose $w_2(x) \equiv z_3(x)$ for $x \ge x'''$. Then, by Lemma 2.8, $w_2'(d_k) > y_k'(d_k) = u'(d_k) = w_2'(d_k)$, a contradiction. ∎

Let $j \ge 0$ be the unique integer such that $c_k+j\delta < d_k \le c_k+(j+1)\delta$. Now, repeating the above steps a finite number of times, we arrive at a solution $w_j(x)$ of equation (1.1) satisfying $w_j(c_k+j\delta) = u(c_k+j\delta)$, $w_j'(d_k) = u'(d_k)$ and $w_j(x) \ge u(x)$ on $[c_k+j\delta, d_k]$. Consequently, $w_j'(c_k+j\delta) \ge u'(c_k+j\delta) = w_{j-1}'(c_k+j\delta)$, where $w_{j-1}(x)$ is the solution obtained in the previous step of the proof. (Note that $w_{j-1}(x)$

satisfies $w_{j-1}(x^*) = y_k(x^*)$ for some $c_k \le x^* < c_k + (j-1)\delta$, $w_{j-1}(c_k + (j-1)\delta) = u(c_k + (j-1)\delta)$, $w'_{j-1}(c_k + j\delta) = u'(c_k + j\delta)$ and $w_{j-1}(x) \ge u(x)$ on $[c_k + (j-1)\delta, c_k + j\delta]$.) Since $w_j(x)$ extends to I by E, either (a) there exists x', $c_k \le x' \le c_k + j\delta$ such that $w_j(x') = w_{j-1}(x')$, or (b) there exists a largest x^{**}, $x^* < x^{**} < c_k + (j-1)\delta$ such that $w_j(x^{**}) = y_k(x^{**})$.

If case (a) occurs, considering the two possibilities $x' < c_k + j\delta$ and $x' = c_k + j\delta$, one can arrive at a contradiction to UR by using Lemmas 2.7 and 2.9 as in the proof of Claim (ii).

If case (b) occurs, then the function $w_{j+1}(x)$ defined by $w_{j+1}(x) \equiv w_j(x)$, $x \ge x^{**}$ must satisfy, by Lemmas 2.7, 2.8, $w'_{j+1}(d_k) > y'_k(d_k) = u'(d_k) = w'_{j+1}(d_k)$, a contradiction. This contradiction shows that $P(k+1)$ must be true, and hence $P(n)$ is true for all n. This completes the proof of the theorem. □

5. An existence theorem.

Theorem 5.1. *Let I be an interval open at the left end point. Assume hypotheses* A, UR, E *hold on I. Then the* BVP (1.1), (1.2R) *has a solution where $x_1 < x_2$, $x_1, x_2 \in I$ and $y_1, y_2 \in \mathbf{R}$ are arbitrary.*

Proof. Let $x_1, x_2 \in I$ and $y_1 \in \mathbf{R}$ be arbitrary but fixed. Let

$$S = \{\gamma \in \mathbf{R} : y(x_1) = y_1, y'(x_2) = \gamma \text{ and } y(x) \text{ is a solution of (1.1)}\}.$$

Clearly S is nonempty by hypothesis E. Now, to prove the theorem it suffices to show $S = \mathbf{R}$. We do this through the following claims.

Claim 1. S is connected.

For suppose $\gamma_1, \gamma_2 \in S$, $\gamma_1 < \gamma_2$ and $\gamma_1 < \gamma' < \gamma_2$ is arbitrary. Let $z_i(x)$ be the solution of (1.1), (1.2R) with $y_2 = \gamma_i$, $i = 1, 2$. Then, by Lemma 2.8, we have $z_2(x) \ge z_1(x)$ for all $x > x_1$, $x \in I$. Applying Theorem 5 of [**7**] with $\Phi = z_1$ and $\Psi = z_2$, we obtain that there exists a solution of the BVP (1.1), (1.2R) with $y_2 = \gamma'$, so $\gamma' \in S$ and, hence, the claim. ■

Let $\beta_0 = \operatorname{Sup} S$ and $\gamma_0 = \operatorname{Inf} S$. To show $S = \mathbf{R}$ it suffices to show $\beta_0 = +\infty$ and $\gamma_0 = -\infty$. We will only prove $\beta_0 = +\infty$, since the other proof is similar.

Claim 2. $\beta_0 \notin S$.

For, if not, suppose $y_0(x)$ is the solution of (1.1) and (1.2R) with $y_2 = \beta_0$. Then the solution $z(x)$ of the IVP (1.1) and

$$y(x_1) = y_1,\ y'(x_1) = y_0'(x_1) + 1$$

satisfies, by Lemma 2.7, that $z'(x_2) > y_0'(x_2) = \beta_0$, a contradiction and, hence, the claim. ■

Now let $y_0(x)$ be the solution of the IVP (1.1) and

$$y(x_1) = y_1, \qquad y'(x_1) = 1,$$

and let $z_0(x)$ be the solution of the IVP (1.1) and

$$y(x_2) = y_0(x_2) + 1, \qquad y'(x_2) = \beta_0.$$

Now $y_0(x)$ and $z_0(x)$ exist on I, $z_0(x_1) \neq y_1$ and $y_0'(x_2) < \beta_0$.

Claim 3. $z_0(x_1) < y_1$.

Suppose, if possible, $z_0(x_1) > y_1$. This implies $z_0(x) \geq y_0(x)$ for $x \geq x_1$. If, otherwise, there exists x', $x_1 < x' < x_2$ such that $z_0(x') < y_0(x')$, and then $y_0(x) - z_0(x)$ will have at least one zero on each of the intervals (x_1, x') and (x', x_2), a contradiction to the conclusion of Lemma 2.6. Now $y_0(x) \leq z_0(x)$ on $[x_1, x_2]$, with $y_0(x_1) = y_1 < z_0(x_1)$ and $y_0'(x_2) < \beta_0 = z_0'(x_2)$. So choosing $\Phi = y_0$ and $\Psi = z_0$ in Theorem 5 of [**7**], we obtain that there exists a solution $w(x)$ of the BVP (1.1), (1.2R), with $y_2 = \beta_0$ satisfying $y_0(x) \leq w(x) \leq z_0(x)$ on $[x_1, x_2]$. This implies $\beta_0 \in S$, a contradiction to Claim 2. ■

By Claim 3, the fact that $z_0(x_2) > y_0(x_2)$ and hypothesis UC, it follows that there exists x', $x_1 < x' < x_2$ such that $z_0(x') = y_0(x')$,

$z_0(x) \le y_0(x)$ for $x < x'$, $x \in I$, with $z_0(x) < y_0(x)$ for $x_1 - \varepsilon \le x \le x_1$ where $\varepsilon > 0$ is sufficiently small. By Theorem 3.1 of [**10**] and Lemma 2.6, the BVP (1.1) and

$$y(x_1 - \varepsilon) = z_0(x_1 - \varepsilon)$$
$$y(x_1) = y_1$$

has a solution $w(x)$ satisfying $z_0(x) \le w(x) \le y_0(x)$ on $[x_1 - \varepsilon, x_1]$, and hence, by Lemma 2.8, $w'(x) > z_0'(x)$ for all $x \ge x_1$, $x \in I$. In particular, $w'(x_2) > z_0'(x_2) = \beta_0$. This contradicts the definition of β_0 since $w'(x_2) \in S$. This completes the proof of the theorem. $\square$

The conclusion of Theorem 5.1 need not hold if I is a closed interval, as shown by the following example. We now state a lemma which is used in the example.

Lemma 5.2. *The equation $y'' = -y$ is right disfocal on the interval $[0, \pi/2)$ $((0, \pi/2])$ (that is, $y(x_1) = 0$, $y'(x_2) = 0$, $0 \le x_1 < x_2 < \pi/2$ $(0 < x_1 < x_2 \le \pi/2)$, and $y(x)$ is a solution of the above equation imply $y(x) \equiv 0$), and hence UR holds for the above equation on $[0, \pi/2)$ $((0, \pi/2])$.*

Example. Consider the differential equation

$$y'' = -y + \arctan y \tag{5.1}$$

with $-\pi/2 < \arctan y < \pi/2$ and $I = [0, \pi/2]$.

The hypotheses A, UC and E hold for equation (5.1) on $[0, \pi]$ as shown on page 347 of [**5**].

We first claim that UR holds for (5.1) on $[0, \pi/2)$ and $(0, \pi/2]$. If it does not hold on $[0, \pi/2)$, suppose $y(x)$, $z(x)$ are two distinct solutions of (5.1) satisfying $y(x_1) = z(x_1)$, $y'(x_2) = z'(x_2)$ for some x_1, x_2, $0 \le x_1 < x_2 < \pi/2$. Since UC holds for (5.1) on $[0, \pi/2]$, we can assume without loss of generality that $y(x) > z(x)$ on $(x_1, \pi/2]$. Now let $w(x) = y(x) - z(x)$ so that we have $w(x_1) = 0$, $w'(x_2) = 0$, $w(x) > 0$ and $w''(x) > -w(x)$ on $(x_1, \pi/2]$ as shown on page 347 of [**5**]. Consequently, by Theorem 4.1 and Lemma 5.2, $w(x)$ is a right focal subfunction with respect to solutions of the equation

$$y'' = -y \tag{5.2}$$

on $[x_1, \pi/2]$ and hence $w(x) \le 0$ on $[x_1, x_2]$, a contradiction. This proves that UR holds for (5.1) on $[0, \pi/2)$. The proof for $(0, \pi/2]$ is analogous.

Now, to show that UR holds for (5.1) on $[0, \pi/2]$, we only need to show that if $y(x)$, $z(x)$ are solutions of (5.1) satisfying $y(0) = z(0)$, $y'(\pi/2) = z'(\pi/2)$, then $y(x) \equiv z(x)$ on $[0, \pi/2]$. If the above assertion is not true, then, since UR holds on $[0, \pi/2)$, we can suppose without loss of generality that $y'(x) > z'(x)$ for $0 < x < \pi/2$ and hence $w(x) \equiv y(x) - z(x) > 0$ on $(0, \pi/2]$. Thus, $w(x)$ attains its positive maximum on $[0, \pi/2]$ only at $x = \pi/2$, yielding $w(\pi/2) > 0$ and $w'(\pi/2) = 0$. Further, $w''(x) > -w(x)$ on $(0, \pi/2]$.

Now let $u(x)$ be the solution of the IVP (5.2) and $y(\pi/2) = w(\pi/2)$, $y'(\pi/2) = w'(\pi/2) = 0$. Since $w''(\pi/2) > -w(\pi/2) = -u(\pi/2) = u''(\pi/2)$, it follows that $(w - u)$ has a relative minimum at $x = \pi/2$, that is, $(w - u)(x) > 0$ for x, $0 < \pi/2 - x < \pi/2$, sufficiently small. Since, by Theorem 1 of [**9**], $w(x)$ is a conjugate subfunction with respect to solutions of (5.2) on $[0, \pi/2]$, we must have $(w - u)(x) \ne 0$ for $0 \le x < \pi/2$, and hence $(w - u)(x) > 0$ for $0 \le x < \pi/2$. In particular, $u(0) < w(0) = 0$, whereas $u(\pi/2) = w(\pi/2) > 0$. Therefore, $u(x') = 0$ for some x', $0 < x' < \pi/2$. This, together with $u'(\pi/2) = 0$, implies, by Lemma 5.2, that $u(x) \equiv 0$ on $[x', \pi/2]$. In particular, $u(\pi/2) = 0$, a contradiction. This shows that UR holds on $[0, \pi/2]$.

Claim. The BVP (5.1) *and* $y(0) = 0$, $y'(\pi/2) = 3\pi$ *has no solution.*

Suppose, on the contrary, that the above BVP has a solution $y(x)$ and $y'(0) = m$. Let $v(x)$ be the solution of the IVP

$$\begin{gathered} v'' = -v + \pi \\ v(0) = 0, \qquad v'(0) = m + 1. \end{gathered}$$

As shown on page 347 of [**5**], $v(x)$ is a lower solution of equation (5.1) on $[0, \pi/2]$, satisfying $v(0) = y(0)$, $v(x) > y(x)$ for $0 < x$ sufficiently small. Consequently, by Theorem 4.1, $v(x)$ is a right focal subfunction of (5.1) on $[0, \pi/2]$. This, together with the fact $v(0) = y(0)$, must imply $v'(x) > y'(x)$ on $(0, \pi/2]$ and, in particular, $v'(\pi/2) > y'(\pi/2) = 3\pi$.

However, an easy computation yields $v(x) = (m+1)\sin x - \pi \cos x + \pi$, whereby we get $3\pi < v'(\pi/2) = \pi$, a contradiction. ■

REFERENCES

1. P.W. Eloe and Johnny Henderson, *Nonlinear boundary value problems and a priori bounds on solutions*, SIAM J. Math. Anal. **15** (1984), 642–647.

2. Johnny Henderson, *Existence of solutions of right focal boundary value problems for ordinary differential equations*, Nonlinear Anal. **5** (1981), 989–1002.

3. ———, *Uniqueness of solutions of right focal boundary value problems for ordinary differential equations*, J. Differential Equations **41** (1981), 218–227.

4. ———, *Existence and uniqueness of solutions of right focal point boundary value problems for third and fourth order equations*, Rocky Mountain J. Math. **14** (1984), 487–497.

5. Lloyd K. Jackson, *Subfunctions and second order differential inequalities*, Adv. Math. **2** (1968), 307–363.

6. ——— and Keith W. Schrader, *Comparison theorems for nonlinear differential equations*, J. Differential Equations **3** (1967), 248–255.

7. Gene A. Klaasen, *Differential inequalities and existence theorems for second and third order boundary value problems*, J. Differential Equations **10** (1971), 529–537.

8. A. Lasota and M.A. Luczynski, *A note on the uniqueness of two point boundary value problems*, I Zesyty Naukowe UJ, Prace Mathematyezna **12**, (1968), 27–29.

9. Keith W. Schrader, *A note on second order differential inequalities*, Proc. Amer. Math. Soc. **19** (1968), 1007–1012.

10. ———, *Existence theorems for second order boundary value problems*, J. Differential Equations **5** (1969), 572–584.

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