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FULL 2-CONVEXITY IN BOCHNER L^p-SPACES

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ABSTRACT. We prove that $L^p(\mu, X)$, $1 < p < \infty$, is 2R whenever X is 2R.

1. Introduction. Let $k \geq 2$ be an integer. A Banach space X is said to be *fully k-convex* (kR) if for every sequence $\{x_n\}$ in X, $\lim_{n_1,n_2,\ldots,n_k\to\infty} ||x_{n_1}+x_{n_2}+\cdots+x_{n_k}||/k = 1$ implies $\{x_n\}$ is a Cauchy sequence in X. X is said to be *fully convex* if X is kR for some $k \geq 2$. It is known [**1,4**] that if X is uniformly convex, then X is kR for all $k \geq 2$, and $L^p(\mu, \Omega, X)$, $1 < p < \infty$, is uniformly convex. One natural question arises as to whether $L^p(\mu, \Omega, X)$, $1 < p < \infty$, is kR when X is kR. The main result of this article is the following theorem.

Main Theorem. *$L^p(\mu, \Omega, X)$, $1 < p < \infty$, is 2R if X is 2R.*

The above question is still open if $k \geq 3$. (Note: the techniques in this paper do not work when $k \geq 3$.)

2. The ideas of the proof of Theorem. First, we need the following characterization of kR spaces.

Theorem. [**7**, also see **4, 5**]. *A Banach space X is* kR *if and only if X satisfies the following properties:*

(1) *X is reflexive and strictly convex*;

(2) *X has the Kadec-Klee property* (*known also as Property* H), *i.e., if $\{x_n\}$ converges weakly to x and $\lim_{n\to\infty} ||x_n|| = ||x||$, then $\{x_n\}$ converges to x in norm*, (3) *Let $\{x_n\}$ be a sequence in X. If*

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$$(*)\qquad \lim_{n_1,n_2,\dots,n_k\to\infty}\frac{||x_{n_1}+x_{n_2}+\cdots+x_{n_k}||}{k}=1$$

and $\{x_n\}$ *converges weakly to* x*, then* $||x||=1$.

It is known that if X is reflexive and strictly convex, then $L^p(\mu,\Omega,X)$, $1<p<\infty$, is reflexive and strictly convex. It is also known [**8**] that if X is a strictly convex Banach space with the Kadec-Klee property and X contains no copy ℓ_1, then $L^p(\mu,\Omega,X)$ has the Kadec-Klee property. Hence, if X is kR, then $L^p(\mu,\Omega,X)$ satisfies conditions (1) and (2) in the above Theorem. So we only need to verify whether $L^p(\mu,\Omega,X)$ satisfies condition (3) when X is kR. Let us assume that X is 2R and $\{f_n\}$ is a sequence in $L^p(\mu,\Omega,X)$ which satisfies $(*)$ for $k=2$ and which converges to f weakly. Since $L^p(\mu,\Omega)$, $1<p<\infty$, is uniformly convex, by a standard perturbation argument (see [**8**]) we may assume that $||f_n(t)||=||f_1(t)||$ for all $n\in\mathbf{N}$ and all $t\in\Omega$. (So we can assume that Ω is σ-finite.) It is easy to verify the following fact: for any measurable set A of Ω,

$$\limsup_{n_1,n_2\to\infty}\int_{\Omega\setminus A}||f_{n_1}(t)+f_{n_2}(t)||^p\,d\mu\le\int_{\Omega\setminus A}(2||f_1(t)||)^p\,d\mu,$$

and so

$$(**)\qquad \lim_{n_1,n_2\to\infty}\int_A||f_{n_1}(t)+f_{n_2}(t)||^p\,d\mu=\int_A(2||f_1(t)||)^p\,d\mu.$$

So, if $\{f_n\}$ satisfies condition $(*)$, then $\{f_n\cdot I_A\}$ satisfies condition $(*)$ where 1 is replaced by $||f_1\cdot I_A||$ (and $f_n\cdot I_A$ converges to $f\cdot I_A$ weakly).

Lemma 1. *Assume* $\{f_n\}$ *satisfies the above conditions. Moreover, assume there exists* $C>0$ *such that for any* $t\in\operatorname{supp}(||f_1(t)||)$,

$$C>||f_1(t)||>\frac{1}{C}.$$

Then there is a subsequence of $\{f_n\}$ *which converges to* f.

Assume Lemma 1 were proved. Then for any $m\in\mathbf{N}$, there is a subsequence of $\{f_n\}$ such that $\{f_n\cdot I_{\{t:\,1/m<||f_1(t)||<m\}}\}$ converges to

$\{f \cdot I_{\{t:\, 1/m<||f_1(t)||<m\}}\}$. So by the diagonal method, $\{f_n\}$ contains a subsequence which converges to f (therefore, $||f|| = 1$). Since for any $m \in \mathbf{N}$,

$$\mu\left\{t : \frac{1}{m} < ||f_1(t)|| < m\right\} < \infty,$$

we may assume that μ is a probability and $\mathrm{supp}(f) = \Omega$.

Remark 1. The above arguments except for the conclusion of Lemma 1 are still true when $k \geq 3$.

Remark 2. Using the above arguments, we can show that $\{f_n\}$ satisfies the assumption (respectively, conclusion) of Lemma 1 if and only if $\{f_n(t)/||f_1(t)||\}$ satisfies the assumption (respectively, conclusion) of Lemma 1. So we may assume that $||f_1(t)|| = 1$ for all $t \in \Omega$. But in Lemma 2, we give a proof without assuming that $||f_1(t)|| = 1$ for all $t \in \Omega$.

Assume that there is a subsequence $\{h_n\}$ of $\{f_n\}$ such that, for all $t \in \Omega$, $\lim_{n_1,n_2\to\infty} ||h_{n_1}(t) + h_{n_2}(t)|| = 2$. Since X is 2R, $\{h_n(t)\}$ is a Cauchy sequence, say it converges to $f(t)$. By the Lebesgue Dominated Theorem, $\{h_n\}$ converges to $\{f\}$ in $L^p(\mu, X)$. In this article we prove the fact indirectly. Up to now, we do not know any simple direct proof. Before giving the proofs, we explain our ideas.

It is known that if $\{f_n\}$ converges to f in $L^p(\mu, X)$, then there is a subsequence of $\{f_n\}$ which converges to f almost everywhere. Hence, the first step we need to show is that for almost all $t \in \Omega$, the set

$$F(t) = \{x : \text{ there is a subsequence of } \{f_n(t)\} \text{ which converges to } x\}$$

is nonempty. Assume we have done the first step. Then we may have a selection (for a definition, see Section 3) g of F. It is known that f is measurable. But we do not know g is measurable. So in the second step, we will use the function g to show that F has a measurable selection h. In the third step, we will show that there is a subsequence $\{h_n\}$ of $\{f_n\}$ such that for almost all $t \in \Omega$, $\lim_{n\to\infty} ||h_n(t) + h(t)|| = 2$. Finally, we show that $\{h_n\}$ converges to h. Clearly, to prove the above statements, we need to have the sequence $\{f_n\}$ satisfy the following strong property.

Lemma 2. *For any $N > 0$, there exists $n > N$ such that if $m > n$, then the set*

$$S_n = \left\{ t : ||h_n(t) + h_m(t)|| \le 2\left(1 - \frac{1}{2^N}\right)\right\}$$

$$\left(\textit{respectively, } S_n = \left\{ t : ||h_n(t) + h_m(t)|| \le 2\left(1 - \frac{1}{2^N}\right)||f_1(t)||\right\}\right)$$

has measure at most $1/2^N$. Hence, there is a subsequence $\{h_n\}$ which satisfies the property
$(***)$

$$\mu\left\{ t : ||h_n(t) + h_m(t)|| \le 2\left(1 - \frac{1}{2^m}\right)\right\} \le \frac{1}{2^{m+1}}$$

$$\left(\textit{respectively, } \mu\left\{ t : ||h_n(t) + h_m(t)|| \le 2\left(1 - \frac{1}{2^m}\right)||f_1(t)||\right\} \le \frac{1}{2^{m+1}}\right)$$

for all $m < n$.

Proof. Since $\lim_{n_1, n_2 \to \infty} ||f_{n_1} + f_{n_2}|| = 2$, there exists $M > N$ such that if $n, m > M$, then

$$2^p \ge \int ||f_n(t) + f_m(t)||^p \, d\mu \ge 2^p \left[\left(1 - \frac{1}{2^N}\right)^p \cdot \frac{1}{2^N} + \left(1 - \frac{1}{2^N}\right)\right].$$

(We leave it to the reader to figure out the right inequality when we do not assume that $||f_1(t)|| = 1$ for all $t \in \Omega$.) So the set $\{t : ||f_M(t) + f_m(t)|| < 2(1 - 1/2^N)\}$ has measure less than $1/2^N$ for any $m > M$, and there is a subsequence $\{h_n\}$ of $\{f_n\}$ which satisfies $(***)$. □

From now on, we will assume that $\{f_n\}$ satisfies $(***)$ of Lemma 2.

3. Some lemmas. Assume $\{f_n\}$ satisfies the assumptions in the last section and X is 2R.

Lemma 3. *For almost all $t \in \Omega$, $F(t) \neq \varnothing$.*

Proof. It is enough to show that for any $N > 0$ and for any measurable set E with $\mu(E) > 1/2^N > 0$, there is a measurable subset

E' of E such that $\mu(E') > 0$ and for all $t \in E'$, $\{f_n(t)\}$ contains a Cauchy subsequence. By Lemma 2, for any $n > m > N$, the set $\{t : ||f_n(t) + f_m(t)|| < 2(1 - 1/2^m)\}$ has measure less than $1/2^{m+1}$. So for any $n > N$, the set

$$S_n = \left\{ t \in E : \text{ for all } N < m < n ||f_n(t) + f_m(t)|| \geq 2\left(1 - \frac{1}{2^m}\right)\right\}$$

has measure at least $1/2^{N+1}$. Thus, there exists a subset $E' \subseteq E$ such that $\mu(E') > 0$ and for each $t \in E'$, $t \in S_n$ infinitely many times, say $t \in S_{n_k}$. (Note: $\{n_k\}$ depends on t.) Then we have

$$\lim_{k_1, k_2 \to \infty} ||f_{n_{k_1}}(t) + f_{n_{k_2}}(t)|| = 2.$$

By the assumption, X is 2R. So $\{f_{n_k}(t)\}$ is a Cauchy sequence and $F(t) \neq \varnothing$ for almost all $t \in \Omega$. □

Remark 3. We do not know whether there is a result similar to Lemma 3 when X is 3R.

Lemma 3 has shown that for almost all $t \in \Omega$, $F(t) \neq \varnothing$. So we may assume that $F(t) \neq \varnothing$ for all $t \in \Omega$. A function $g : \Omega \to X$ is said to be a *selection* of F if $g(t) \in F(t)$ for all $t \in \Omega$. If g is measurable, then we say g is a *measurable selection*.

Lemma 4. *There is a measurable selection of F.*

Proof. Let g be a selection of F. g is not necessarily measurable. Our idea to prove this Lemma is to construct a sequence of functions $\{\tilde{h}_n\}$ which satisfies the following conditions:

(1) for any $t \in \Omega$ and any n, there is an m such that $\tilde{h}_n(t) = f_m(t)$;

(2) there is a sequence of sets $\{S_n\}$ such that for all n

$$\mu^*(S_n) = 1 \text{ and } ||\tilde{h}_n(t) - g(t)|| < \frac{1}{2^n} \text{ for } t \in S_n.$$

So, if $m < n$ and $t \in S_m$, then

$$||\tilde{h}_n(t) - \tilde{h}_m(t)|| \leq ||\tilde{h}_n(t) - g(t)|| + ||g(t) - \tilde{h}_m(t)|| \leq \frac{1}{2^{m-1}}.$$

Since $\tilde{h}_n$ and $\tilde{h}_m$ are measurable and $\mu^*(S_m) = 1$,

$$||\tilde{h}_n(t) - \tilde{h}_m(t)|| \leq \frac{1}{2^{m-1}}$$

for almost all $t \in \Omega$. This implies $\{\tilde{h}_n\}$ is a Cauchy sequence in $L^p(\mu, X)$, say it converges to h. Then for almost all $t \in \Omega$, $h(t) \in F(t)$ and h is a measurable selection of F.

Let g be a selection of F. We claim that

(1) there exist $S_1 \subseteq S_2 \subseteq \cdots$ such that $\mu^*(S_k) = 1$ for all k;

(2) there exist an increasing sequence $\{N_k\}$ and a sequence of measurable sets $\{E_n\}$ such that

(i) $\{E_m\}_{N_k+1}^{N_{k+1}}$ are pairwise disjoint and $\mu\left(\Omega \setminus \cup_{m=N_k+1}^{N_{k+1}} E_m\right) < \frac{1}{2^k}$.

(ii) if $t \in E_m \cap S_k$ $(N_k < m \leq N_{k+1})$, then $||f_m(t) - g(t)|| < \frac{1}{2^k}$.

Suppose the claim was proved. Let

$$\tilde{h}_k = \sum_{m=N_k+1}^{N_{k+1}} f_m \Big|_{E_m}.$$

If $k' > k$ and $t \in (\cup_{m=N_{k'}+1}^{N_{k'+1}} E_m) \cap (\cup_{m=N_k+1}^{N_{k+1}} E_m) \cap S_{k'}$, then

$$||\tilde{h}_{k'}(t) - \tilde{h}_k(t)|| < \frac{1}{2^k} + \frac{1}{2^{k'}} \leq \frac{2}{2^k}.$$

Since the functions h_k are measurable and

$$\mu^*\left[\left(\bigcup_{m=N_{k'}+1}^{N_{k'+1}} E_m\right) \bigcap \left(\bigcup_{m=N_k+1}^{N_{k+1}} E_m\right) \bigcap S_{k'}\right] = \mu\left[\left(\bigcup_{m=N_{k'}+1}^{N_{k'+1}} E_m\right) \bigcap \left(\bigcup_{m=N_k+1}^{N_{k+1}} E_m\right)\right],$$

for almost all $t \in [(\cup_{m=N^{k'}+1}^{N_{k'+1}} E_m) \cap (\cup_{m=N_k+1}^{N_{k+1}} E_m)]$ we have

$$||\tilde{h}_{k'}(t) - \tilde{h}_k(t)|| \leq \frac{2}{2^k}.$$

Since $\mu[\cap_{k'=k}^{\infty}(\cup_{m=N_{k'}+1}^{N_{k'+1}} E_m)] \geq 1-2/2^k$, $\{\tilde{h}_k\}$ is a Cauchy sequence in $L^p(\mu, \Omega, X)$, say it converges to f. So $f(t) \in F(t)$ for almost all $t \in \Omega$ and F has a measurable selection.

Now, we prove our claim. Let S_1 be any subset with $\mu^*(S_1) = 1$ and suppose $S_k \subseteq S_1$ and $N_{k+1} > N_k$ are given. Since $S_k = \{t \in S_k : \exists\, n > N_{k+1} \text{ s.t. } ||f_n(t) - h(t)|| < \frac{1}{2^{k+1}}\}$, there exists $N_{k+2} > N_{k+1}$ such that the set S_k'

$$\left\{t \in S_k : \exists\, n\ N_{k+2} \geq n > N_{k+1},\ \text{s.t. } ||f_n(t) - h(t)|| < \frac{1}{2^{k+1}}\right\}$$

has outer measure greater than $1 - 1/2^{k+1}$. For $N_{k+1} < n \leq N_{k+2}$, let $F_n' = \{t \in S_k' : ||f_n(t) - h(t)|| < 1/2^{k+1}\}$, and let the F_n's be measurable sets such that $F_n' \subseteq F_n$ and $\mu^*(F_n') = \mu(F_n)$. Finally, let

$$E_n = F_n - \left(\bigcup_{m=N_{k+1}+1}^{n-1} F_m\right)$$

and

$$S_{k+1} = \left(S_k - \bigcup_{m=N_{k+1}+1}^{N_{k+2}} E_m\right) \bigcup \bigcup_{m=N_{k+1}+1}^{N_{k+2}} \left(E_m \bigcap F_m'\right).$$

The verification is left to the reader. □

Lemma 5. *Let h be a measurable selection of F. Then $\lim_{n\to\infty} ||h_n + h|| = 2$, and so there exists a subsequence $\{h_n\}$ of $\{f_n\}$ such that $||h_n(t) + h(t)||$ converges to 2 almost everywhere.*

Proof. Since for every $t \in \Omega$ there is a subsequence of $\{f_n(t)\}$ which converges to $h(t)$, for any $\varepsilon > 0$ and $N > 0$ there exist $n_k > n_{k-1} > \cdots > n_1 > N$ and $E_{n_j} \subseteq \Omega$, $1 \leq j \leq k$, such that

(1) the E_{n_j}'s are pairwise disjoint measurable sets,

(2) $\mu(\Omega \setminus \cup_{j=1}^{k} E_{n_j}) < \varepsilon$,

(3) if $t \in E_{n_j}$, then $||h(t) - f_{n_j}(t)|| < \varepsilon$.

If $n > n_k$, then for each $1 \leq j \leq k$, the set $\{t : ||f_n(t) + f_{n_j}(t)|| < 2(1 - 1/n_j)\}$ has measure less than $1/2^{n_j}$. Hence, there is a subset $E \subseteq \Omega$ such that

(1) $\mu(E) < \varepsilon + 1/2^N$,

(2) if $t \in \Omega \backslash E$, then $||h(t) + f_n(t)|| > 2(1 - \varepsilon - 1/2^N)$.

This implies $\lim_{n\to\infty} ||h + f_n|| = 2$.

For each $m \in \mathbf{N}$, there is N_m $(> N_{m-1})$ such that if $n > N_m$, then

$$\mu\left\{t : ||h(t) + f_n(t)|| < 2\left(1 - \frac{1}{m}\right)\right\} \leq \frac{1}{2^m}.$$

So for almost all $t \in \Omega$,

$$\lim_{m\to\infty} ||h_{N_m}(t) + h(t)|| = 2. \quad \square$$

Assume $\{h_n\}$ satisfies the conclusions of Lemma 5. Let $\overline{F}(t)$ denote the set

$$\{x : \text{ there is a subsequence of } h_n(t) \text{ which converges to } x\}.$$

Lemma 6. *Let $\tilde{h}$ be a measurable selection of $\overline{F}(t)$, then $\tilde{h} = h$ (a.e.).*

Proof. Clearly, for almost all $t \in \Omega$,

$$||\tilde{h}(t)|| = ||h(t)|| = 1 = \lim_{n\to\infty} ||f_n(t)||.$$

(So $||h|| = 1 = ||\tilde{h}||$.) Since $\tilde{h}$ is a selection of $\overline{F}$, for each $t \in \Omega$ there is a sequence $\{n_k\}$ such that $\lim_{k\to\infty} h_{n_k}(t) = \tilde{h}(t)$. By Lemma 5, for almost all $t \in \Omega$,

$$||\tilde{h}(t) + h(t)|| = 2.$$

By the Lebesgue Dominated Convergence Theorem, $||\tilde{h} + h|| = 2$. But since $L^p(\mu, X)$ is strictly convex, we must have $\tilde{h} = h$. $\square$

4. The proof of Main Theorem. Suppose $\{h_n\}$ satisfies the conclusion of Lemma 5. We claim that $\{h_n\}$ converges to h. If this is not true, by passing to a subsequence of $\{h_n\}$ we may assume that there is an $\varepsilon > 0$ such that for every n, the set

$$\{t : ||h_n(t) - f(t)|| > \varepsilon\}$$

has measure at least ε. By induction, we will construct a sequence $\{g_n\}$ which satisfies the following conditions:

(i) $\{g_n\}$ satisfies $(*)$ for $k = 2$,

(ii) for every n and $t \in \Omega$, $g_n(t) = h(t)$ or $h_m(t)$ for some m,

(iii) there is an $S \subseteq \Omega$, such that $\mu(S) \geq \varepsilon/2$ and if $t \in S$, then

$$||g_n(t) - h(t)|| > \varepsilon \quad \text{for all } n.$$

Suppose $\{g_n\}$ were constructed. Then by (iii), for every $t \in S$, any subsequences of $\{g_n(t)\}$ do not converge to $h(t)$. On the other hand, let

$$\tilde{F}(t) = \{x : x \text{ is a limit point of } \{g_n(t)\}\}.$$

By (i) and Lemma 4, there is a measurable selection $\tilde{h}$ of $\tilde{F}$. By (ii) and Lemma 6, $\tilde{F}(t) \subseteq \overline{F}(t)$ and $h = \tilde{h}$. So we get a contradiction.

Let N_1 be a natural number such that $\varepsilon > 2^{-N_1+3}$, and let

$$T_1 = \{t : \text{ there exists } n > N_1 \text{ such that } ||h_n(t) - h(t)|| > \varepsilon\}.$$

Since $\mu\{t : ||h_n(t) - h(t)|| > \varepsilon\} > \varepsilon$ for each n, $\mu(T_1) > \varepsilon$. We will use the induction to construct $\{N_k\}$, $\{E_m\}$, $\{S_k = \cup_{m=N_k+1}^{N_{k+1}} E_m\}$, $\{g_k\}$ and

$$\begin{aligned}\{T_{k+1} = \{t \in S_k : \exists\, n > N_{k+1} \text{ s.t. } &||h_n(t) - h(t)|| > \varepsilon \text{ and} \\ &||h_n(t) + g_l(t)|| > 2(1 - 1/N_l) \text{ for } l = 1, \dots, k\}\},\end{aligned}$$

such that

(iv) $\{E_m\}_{N_k+1}^{N_{k+1}}$ are pairwise disjoint measurable subsets of T_k and

$$\mu\bigg(T_k \Big\backslash \bigcup_{m=N_k+1}^{N_{k+1}} E_m\bigg) < \frac{\varepsilon}{2^{2+k}}.$$

(v) If $t \in E_m$, then $||h_m(t) - h(t)|| > \varepsilon$.

(vi) $\mu(S_k) \geq \varepsilon/2 + \varepsilon/2^{k+2} + \varepsilon/2^{k+1}$ (for $m > N_{k+1}$, let

$$F_{m,k} = \{t \in S_k : ||h_m(t) - h(t)|| > \varepsilon \text{ and } ||h_m(t) + g_l(t)|| > 2(1 - 1/n_l) \text{ for } l = 1, \ldots, k\}.$$

If $m > N_{k+1}$, then $F_{m,k} \subseteq T_{k+1}$ and $\mu(T_{k+1}) \geq \mu(F_{m,k}) \geq \frac{\varepsilon}{2} + \frac{\varepsilon}{2^{k+1}}$).

(vii) $\{g_n\}$ and $S = \cap_{k=1}^{\infty} S_k$ satisfy (i), (ii) and (iii).

Suppose that $\{N_n\}_1^{k+1}$, $\{S_n\}_1^k$, $\{g_n\}_1^k$, and $\{T_n\}_1^{k+1}$ have been constructed. By the definition of T_{k+1}, there exist $N_{k+2} > N_{k+1}$ and $N_{k+2} - N_{k+1}$ disjoint measurable sets $E_m \subseteq T_{k+1}$, $N_{k+1} < m \leq N_{k+2}$, such that

(a) if $t \in E_m$, then $||h_m(t) - h(t)|| > \varepsilon$ and $||h_m(t) + g_l(t)|| > 2(1 - 1/N_l)$ for $1 \leq l \leq k$,

(b) $\mu(T_{k+1} \backslash \cup_{m=N_{k+1}+1}^{N_{k+2}} E_m) < \varepsilon/2^{k+3}$.

So
$$\mu(S_{k+1}) \geq \mu(T_{k+1}) - \frac{\varepsilon}{2^{k+2}} \geq \frac{\varepsilon}{2} + \frac{\varepsilon}{2^{k+2}} + \frac{\varepsilon}{2^{k+3}}.$$

Let
$$g_{k+1} = h|_{\Omega \backslash S_{k+1}} + \sum_{m=N_{k+1}+1}^{N_{k+2}} h_m|_{E_m}.$$

By Lemma 2, if $m > N_{k+2} \geq j > N_{k+1}$, then

$$\mu\left\{t : ||f_j(t) + f_m(t)|| \leq 2\left(1 - \frac{1}{j}\right)\right\} \leq \frac{1}{2^{j+1}}.$$

So, if $m > N_{k+2}$, then

$$\begin{aligned}\mu(F_{m,k+1}) &\geq \mu(F_{m,k}) - \mu(T_{k+1} \backslash S_{k+1}) - \sum_{j=N_{k+1}+1}^{N_{k+2}} \frac{1}{2^{j+1}} \\ &\geq \frac{\varepsilon}{2} + \frac{\varepsilon}{2^{k+2}} + \frac{\varepsilon}{2^{k+3}} - \sum_{j=1}^{N_{k+2}-N_{k+1}} \frac{\varepsilon}{2^{j+k+3}} \\ &\geq \frac{\varepsilon}{2} + \frac{\varepsilon}{2^{k+2}}.\end{aligned}$$

Clearly, $\{g_n\}$ and S satisfy (ii) and (iii). By Lemma 6, there is a null set N such that $\forall\varepsilon > 0\ \forall t \notin N\, \exists\, M > 0$ such that

$$\text{if } n > M \text{ then } ||h_n(t) + h(t)|| \geq 2 - \varepsilon.$$

So by (a), if $n, m > M$, then

$$||g_n(t) + g_m(t)|| \geq \min\left(2 - \varepsilon, 2\left(1 - \frac{1}{n}\right)\right).$$

And so $\{g_n\}$ satisfies $(*)$, for $k = 2$. □

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