ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 21, Number 3, Summer 1991

PSEUDO-CONVERGENCE IN NORMED LINEAR SPACES

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A bounded sequence $\{x_n\}$ in a Banach space X is said to pseudo-converge to a point x_0, called the pseudo-limit, if x_0 minimizes the function $f_S(x) = \limsup ||y_m - x||$, $x \in X$, for every subsequence $S = \{y_m\}$ of $\{x_n\}$. If the pseudo-limit is unique, we call the sequence $\{x_n\}$ uniquely pseudo-convergent. This notion of convergence arose in connection with fixed point theory of multivalued nonexpansive mappings, see, e.g., [**1, 2, 4, 5, 7, 10**]. The basic idea is that if x_n is a bounded sequence of approximate fixed points of a multivalued nonexpansive mapping T, then there may exist a uniquely pseudo-convergent subsequence of x_n whose pseudo-limit is a fixed point of T.

In this paper we characterize pseudo-convergence in certain Banach spaces. A main result is that, in a space with a uniformly Gateaux differentiable norm, a sequence $\{x_n\}$ pseudo-converges to x if and only if $J(x_n - x)$ converges weak*ly to 0, where J is a duality map. We also consider other related types of convergence. Note that spaces with a uniformly Gateaux differentiable norm have appeared in several other contexts, including semigroups and approximations, see, e.g., Klee [**3**], Reich [**9, 11, 12**], Zizler [**13, 14**].

A space is said to satisfy (w^*)-Opial's condition if the condition $\{x_n\}$ converging weakly (weak*ly) to x_0 implies that

$$\limsup ||x_n - x_0|| < \limsup ||x_n - y||$$

for all $y \neq x_0$. Examples of spaces satisfying Opial's condition include Hilbert spaces and ℓ_p, $1 \leq p < \infty$. $L^p[0,1]$ for $p \neq 2$ do not satisfy the condition [**8**]. ℓ_1 satisfies w^*-Opial's condition.

Received by the editors on October 1, 1986, and in revised form on September 2, 1987.
Keywords and phrases. Pseudo-convergence, asymptotic center, nonexpansive mapping.

Proposition 1. *A reflexive (separable conjugate) space satisfies* (w^*)*-Opial's condition if and only if pseudo-convergence to a point is equivalent to weak (weak*) convergence to the same point.*

Proof. Suppose that X is a space satisfying Opial's condition and $\{x_n\}$ a pseudo-convergent sequence with pseudo-limit x_0. Then every weakly convergent subsequence of $\{x_n\}$ has weak limit x_0. By the reflexivity of X, this implies that $\{x_n\}$ converges weakly to x_0. The other parts of the proposition are obvious. The proof for conjugate space is similar. □

The following example of Opial [**8**] shows that a pseudo-convergent and weakly convergent sequence may have different pseudo-limit and weak limit.

Example 1. Let $f(t)$ be a periodic function of period 1 and

$$f(t) = \begin{cases} 1, & 0 < x \leq 3/4, \\ -3, & 3/4 < x \leq 1. \end{cases}$$

Define $\phi_n(t) = f(nt)$ for $0 \leq t \leq 1$. The sequence ϕ_n converges weakly to zero but pseudo-converges to the constant function $(3^{1/(p-1)}-3)/(1+3^{1/(p-1)})$ in $L^p[0,1]$.

Let $\phi(r)$ be a continuous strictly increasing function defined for $r \geq 0$ with $\phi(0) = 0$ and $\lim_{r\to\infty} \phi(r) = \infty$. A (generally multi-valued) mapping $J_\phi : X \to X^*$ is called a duality map with gauge function ϕ if $||J_\phi(x)|| = \phi(||x||)$ and $\langle J_\phi(x), x\rangle = \phi(||x||)||x||$. If $\phi(r) = r$, we shall write J_X, or simply J, instead of J_ϕ. Define $\phi(r) = \int_0^r \phi(s)\, ds$. Then $J_\phi(x)$ is the subdifferential of $\phi(||x||)$. X is said to have a weakly continuous duality mapping if there exists ϕ such that J_ϕ is continuous from X, with the weak topology, to X^*, with the weak* topology. A space having weakly continuous duality mapping satisfies Opial's condition.

Let X be a space with a Gateaux differentiable norm. X is said to be uniformly Gateaux differentiable if, for every bounded set K with

$0 \notin \overline{K}$ and every $y \in X$, the limit

$$\lim_{t\to 0} \frac{||x+ty|| - ||x||}{t} - \langle x^*, y\rangle$$

converges to 0 uniformly for $x \in K$. Here x^* denotes the (unique) element of X^* satisfying $x^*(x) = ||x||$ and $||x^*|| = 1$. It follows that, for any gauge function $\phi(r)$, the limit

$$\lim_{t\to 0} \frac{\phi(||x+ty||) - \phi(||x||)}{t} - \langle J_\phi(x), y\rangle$$

converges to 0 uniformly for $x \in K$.

We have the following general result which will be used in Theorems 1, 2 and 3.

Proposition 2. *Let X be a Banach space and ϕ a gauge function on X. Let $\{x_n\}$ be a bounded sequence in X. Suppose that there is a sequence $\{y_n\}$, a point $x_0 \in X$ and a sequence $z_n \in J_\phi(y_n)$ such that* $\lim ||x_n - x_0 - y_n|| = 0$ *and $z_n \overset{*}{\rightharpoonup} 0$. Then x_n pseudo-converges to x_0.*

Proof. Since a sequence $\{x_n\}$ pseudo-converges to x_0 if and only if $\{x_n - x_0\}$ pseudo-converges to 0, we may assume, without loss of generality, that $x_0 = 0$. Since $J_\phi(x)$ is the subdifferential of $\Phi(||x||)$, we have

$$\Phi(||y_n - z||) \geq \Phi(||y_n||) - \langle z_n, z\rangle$$

for every n and $z \in X$. Taking lim sup on both sides, we get

$$\limsup \Phi(||y_n - z||) \geq \limsup \Phi(||y_n||).$$

Since Φ is increasing, this implies that

$$\limsup ||y_n - z|| \geq \limsup ||y_n||.$$

The inequality is also valid for every subsequence of y_n. Hence, y_n pseudo-converges to 0. Since $\lim ||x_n - y_n|| = 0$, x_n also pseudo-converges to 0. □

Theorem 1. *Let X be a Banach space with a uniformly Gateaux differentiable norm. Let $\{x_n\}$ be a bounded sequence in X. The following are equivalent*:

(i) $\{x_n\}$ *pseudo-converges to* x_0;

(ii) $J_\phi(x_n - x_0) \overset{*}{\rightharpoonup} 0$ *for some gauge function* ϕ;

(iii) $J_\phi(x_n - x_0) \overset{*}{\rightharpoonup} 0$ *for every gauge function* ϕ.

Proof. Let $\{x_n\}$ be a pseudo-convergent sequence with pseudo-limit x_0. Let $\phi(r)$ be a gauge function and let $\Phi(r) = \int_0^r \phi(s)\,ds$. Let $y \in X$. Let $\{y_m\}$ be a subsequence of $\{x_n\}$ such that $\lim J_\phi(y_m - x_0)y$ exists. If $y_m - x_0$ has a subsequence converging to 0 in norm, then $\lim J_\phi(y_m - x_0)y = 0$. So we may assume that $\{y_m - x_0\}$ is bounded away from 0. We have

$$\Phi(||y_m - x_0 + ty||) - \Phi(||y_m - x_0||) = tJ_\phi(y_m - x_0)y + \xi_m(t),$$

where $\lim_{t\to 0} \xi_m(t)/t = 0$ uniformly for all m. Then

$$\begin{aligned} t\lim_m J_\phi(y_m - x_0)y &+ \limsup_m \xi_m(t) \\ &\geq \limsup_m \Phi(||y_m - x_0 + ty||) - \limsup_m \Phi(||y_m - x_0||) \\ &\geq 0. \end{aligned}$$

Thus $\lim_m J(y_m - x_0)y + \limsup_m \xi_m(t)/t \geq$ or ≤ 0, depending on whether $t >$ or < 0. By letting $t \to 0$, we obtain $\lim_m J_\phi(y_m - x_0)y = 0$. Thus, every subsequential limit of $J_\phi(x_n - x_0)y$ is 0. Since $J_\phi(x_n - x_0)y$ is bounded, this implies that $\lim_n J_\phi(x_n - x_0)y = 0$. Hence, $J_\phi(x_n - x_0) \overset{*}{\rightharpoonup} 0$. This proves that (i) implies (iii).

That (iii) implies (ii) is obvious. That (ii) implies (i) follows from Proposition 2. □

Theorem 1 has a rather interesting consequence when applied to L^p spaces. Since every pseudo-convergent sequence is a translation of a pseudo-convergent sequence with zero pseudo-limit and vice versa, it suffices to characterize pseudo-convergent sequences with zero pseudo-limit.

Corollary 1. *Let $1 < p < \infty$ and $1/p + 1/q = 1$. For a bounded sequence $\{x_n(t)\}$ in L^p, the following are equivalent*:

(i) $\{x_n(t)\}$ *pseudo-converges to* 0;

(ii) $|x_n(t)|^{p-1}\operatorname{sgn} x_n(t)$ *converges weakly to* 0 *in* L^q;

(iii) $x_n(t) = |f_n(t)|^{q-1}\operatorname{sgn} f_n(t)$ *for some weak null sequence* $\{f_n(t)\}$ *in* L^q.

Proof. For the gauge function $\phi(r) = r^{p-1}$, one has $J_\phi(x(t)) = |x(t)|^{p-1}\operatorname{sgn} x(t)$. Thus, (i) and (ii) are equivalent by Theorem 1. If J_ψ is the duality map from L^q to L^p with gauge function $\psi(r) = r^{q-1}$, then $J_\psi \circ J_\phi = \mathrm{id}_{L^p}$ and $J_\phi \circ J_\psi = \mathrm{id}_{L^q}$. It is then obvious that (ii) and (iii) are equivalent. □

Corollary 2. *Let X be a reflexive space with a uniformly Gateaux differentiable norm and $\{x_n\}$ a bounded sequence in X. The following are equivalent*:

(i) $\{x_n\}$ *pseudo-converges to* 0;

(ii) $J_X(x_n)$ *converges weakly to* 0;

(iii) *there exists a sequence $\{y_n\}$ in X^* such that $w-\lim y_n = 0$ and $x_n \in J_{X^*}(y_n)$.*

Proof. (ii) and (iii) are equivalent because $J_X(J_{X^*}(y)) = \{y\}$ and $J_{X^*}(J_X(x)) \supseteq \{x\}$ for any $y \in X^*$ and $x \in X$. □

For a nonzero vector $x = (x_1, x_2, \dots, x_n, \dots)$ in c_0, let $I(x) = \{i : |x_i| = ||x||\}$. Clearly, $I(x)$ is finite. Let $S(x) = \mathrm{co}\{\operatorname{sgn}(x_i)e_i : i \in I(x)\}$, where e_i is the unit vector in ℓ_1 whose ith coordinate is 1 and whose other coordinates are 0. Then $J(x) = ||x||S(x)$. Similarly, for a nonzero vector $x = (x_1, x_2, \dots, x_n, \dots)$ in c, let $x_0 = \lim x_n$ and $I(x) = \{i : i \geq 0, |x_i| = ||x||\}$. Let $S(x) = \overline{\mathrm{co}}\{\operatorname{sgn}(x_i)e_i : i \in I(x)\}$, where e_i are defined as above and the closure is taken in the topology of ℓ_1. Then $J(x) = ||x||S(x)$. Note that a sequence $x_n = (x_n^{(m)} : m \geq 0)$ in ℓ_1, as a dual of c, converges to zero in the weak* topology if and only if it is bounded, $\lim_n x_n^{(m)} = 0$, for $m \geq 0$, and $\lim_n \sum_{m=0}^\infty x_n^{(m)} = 0$.

Theorem 2. *Let $\{x_n\}$ be a bounded sequence in c_0. Then the following are equivalent*:

(i) *$\{x_n\}$ pseudo-converges to x_0;*

(ii) *there exists a sequence $\{y_n\}$ such that $\lim ||x_n - x_0 - y_n|| = 0$ and such that $z_n \overset{*}{\rightharpoonup} 0$ for every sequence $\{z_n\}$ with $z_n \in J(y_n)$;*

(iii) *there exist sequences $\{y_n\}$ and $\{z_n\}$ such that $\lim ||x_n - x_0 - y_n|| = 0$, $z_n \in J(y_n)$ and $z_n \overset{*}{\rightharpoonup} 0$.*

Proof. As remarked earlier, we may assume that $x_0 = 0$.

Suppose that x_n is a bounded sequence in c_0 pseudo-converging to 0. First we show that, for every $\varepsilon > 0$, and every integer $M \geq 1$, there exists an integer N such that

$$\max\{|x_n^{(m)}| : m > M\} > \max\{|x_n^{(m)}| : m = 1, \dots, M\} - \varepsilon$$

for every $n \geq N$. For, if not, there would exist an $\varepsilon > 0$, an integer $M \geq 1$ and a subsequence x_{n_i} such that

$$\max\{|x_{n_i}^{(m)}| : m = 1, \dots, M\} \geq \max\{|x_{n_i}^{(m)}| : m > M\} + \varepsilon$$

for all $i \geq 1$. Let P_M be the natural projection of X onto the finite dimensional subspace generated by $e_1, \dots, e_M$. The bounded sequence $P_M(x_{n_i})$ has a convergent subsequence which we denote by the same notation and whose limit we call z. Then

$$\begin{aligned}\limsup_i ||x_{n_i}|| &\geq \limsup_i \max\{|x_{n_i}^{(m)}| : m = 1, \dots, M\} \\ &\geq \limsup_i \max\{|x_{n_i}^{(m)}| : m > M\} + \varepsilon \\ &= \limsup_i ||x_{n_i} - z|| + \varepsilon,\end{aligned}$$

contradicting the assumption that x_n pseudo-converges to 0.

It is then clear that, by perturbing $\{x_n\}$, we get a sequence $\{y_n\}$ such that $\lim ||x_n - y_n|| = 0$ and $\lim k_n = \infty$, where $k_n = \min\{m : |y_n^{(m)}| = ||y_n||\}$. Thus, for every $z_n \in J(y_n)$, $z_n^{(m)} = 0$ for $m = 1, \dots, k_n$. Since $\{z_n\}$ is bounded, it follows that $\{z_n\}$ converges weak*ly to 0. Hence, (i) implies (ii).

That (ii) implies (iii) is obvious. That (iii) implies (i) follows from Proposition 2. □

Theorem 3. *Let* $\{x_n\}$ *be a bounded sequence in* c. *Then the following are equivalent*:

(i) $\{x_n\}$ *pseudo-converges to* x_0;

(ii) *there exist a sequence* $\{y_n\}$ *and a sequence* $\{z_n\}$ *such that* $\lim ||x_n - x_0 - y_n|| = 0$, $z_n \in J(y_n)$ *and* $z_n \xrightarrow{*} 0$.

Proof. As before, we may assume that $x_0 = 0$. Let $\{x_n\}$ be a sequence pseudo-converging to 0. We claim that, for every $\varepsilon > 0$, and every integer $M \geq 1$, there exists an integer $N \geq 1$ such that

$$\sup\{|x_n^{(m)}| : m > M\} > \max\{|x_n^{(m)}| : m = 1, \ldots, M\} - \varepsilon$$

and

$$|\sup_{m>M} x_n^{(m)} + \inf_{m>M} x_n^{(m)}| < \varepsilon$$

for all $n \geq N$. The first inequality follows from the proof of Theorem 2. Assume that the second inequality was false. Then there would exist an $\varepsilon > 0$, an integer $M \geq 1$ and a subsequence x_{n_i} such that

$$|\sup_{m>M} x_{n_i}^{(m)} + \inf_{m>M} x_{n_i}^{(m)}| \geq \varepsilon$$

for all $i \geq 1$. Let $\lambda_i = \sup_{m>M} x_{n_i}^{(m)}$, $\mu_i = \inf_{m>M} x_{n_i}^{(m)}$ and $\xi_i = (\lambda_i + \mu_i)/2$. Then $|\lambda_i - \xi_i| \leq \max(|\lambda_i|, |\mu_i|) - \varepsilon/2$. By taking a subsequence, if necessary, we may assume that $P_M x_{n_i}$ converges to z and ξ_i converges to a number ξ, where P_M is defined as in the proof of Theorem 2. Let $w \in c$ be defined by $w^{(m)} = z^{(m)}$ for $m = 1, \ldots, M$

and $w^{(m)} = \xi$ for $m > M$. Then

$$\begin{aligned}\limsup_i ||w - x_{n_i}|| &= \limsup_i \sup_{m>M} |\xi - x_{n_i}^{(m)}| \\ &\le \limsup_i |\xi - \xi_i| + \limsup_i \sup_{m>M} |\xi_i - x_{n_i}^{(m)}| \\ &\le \limsup_i |\xi_i - \lambda_i| \\ &\le \limsup_i \max(|\lambda_i|, |\mu_i|) - \varepsilon/2 \\ &= \limsup_i \sup_{m>M} |x_{n_i}^{(m)}| - \varepsilon/2 \\ &\le \limsup_i ||x_{n_i}|| - \varepsilon/2,\end{aligned}$$

contradicting the assumption that x_n pseudo-converges to 0.

It is then clear that one can perturbate x_n to obtain a sequence y_n such that (i) $\lim ||x_n - y_n|| = 0$; (ii) $|y_n^{(m)}| = ||y_n||$ for exactly two values of m: one $m_1 = m_1(n)$ for which $y_n^{(m_1)} = ||y_n||$ and another $m_2 = m_2(n)$ for which $y_n^{(m_2)} = -||y_n||$; and (iii) $\lim_n m_1(n) = \lim_n m_2(n) = \infty$. Define $z_n \in \ell_1$ by

$$z_n = \frac{1}{2}||y_n||(e_{m_1(n)} - e_{m_2(n)}).$$

Then $z_n \in J(y_n)$ and $z_n \overset{*}{\rightharpoonup} 0$. Hence, (i) implies (ii).

That (ii) implies (i) follows from Proposition 2. □

Proposition 3. *Theorem* 3 *is also valid for the space* ℓ_1.

Proof. Let $x_n = (x_n^{(m)} : m \ge 1)$, $n = 1, 2, \dots$, be a pseudo-convergent sequence in ℓ_1 with zero pseudo-limit. By Proposition 1, $x_n \overset{*}{\to} 0$. Thus, x_n, $n = 1, 2, \dots$, is bounded and $\lim_n x_n^{(m)} = 0$ for each m. We can define $y_n \in \ell_1$ such that $||x_n|| = ||y_n||$, $\lim_n ||x_n - y_n|| = 0$ and $\min\{m : y_n^{(m)} \neq 0\} \to \infty$ as $n \to \infty$. For $y \in \ell_1$, define $\operatorname{sgn} y$ to be the vector in ℓ_∞ whose m-component is $\operatorname{sgn} y^{(m)}$. Then $||x_n|| \operatorname{sgn} y_n$ belongs to $J(y_n)$ and $||x_n|| \operatorname{sgn} y_n \overset{*}{\to} 0$. Therefore, Theorem 3 is also valid for the space ℓ_1. □

We now turn to some variants of pseudo-convergence.

Let $\{x_n\}$ be a bounded sequence in a Banach space X. The number

$$r(\{x_n\}) = \inf\{\limsup_n ||x_n - x|| : x \in X\}$$

and the set

$$A(\{x_n\}) = \{x \in X : \limsup_n ||x_n - x|| = r(\{x_n\})\}$$

are called, respectively, the asymptotic radius and the asymptotic center of $\{x_n\}$. The sequence $\{x_n\}$ is said to be regular if $r(\{y_n\}) = r(\{x_n\})$ for every subsequence y_n of x_n and A-regular if $A(\{y_n\}) = A(\{x_n\})$ for such y_n. Following Kirk [**2**], a regular and A-regular sequence will be called asymptotically uniform. Every bounded sequence has a regular subsequence [**1, 4**] and, in case that X is separable, an asymptotically uniform subsequence [**2**]. The following simple example shows that the condition of separability cannot be removed.

Example 2. Let $\{e_n\}$ be the standard unit vectors in ℓ^∞. For every subsequence $\{e_{n_i}\}$, we have $r(\{e_{n_i}\}) = 1/2$ and $A(\{e_{n_i}\}) = \{x : -1/2 \le x^{(m)} \le 1/2, m = 1, 2, \dots \text{ and } \lim_i x^{(n_i)} = 1/2\}$. So no two subsequences have the same asymptotic center unless they are essentially the same. It follows that no asymptotically uniform subsequence exists.

We need the following two propositions taken from [**6**].

Proposition A. *Let $\{x_n\}$ be a bounded sequence in c_0. Then*

(i) *$A(\{x_n\})$ is nonempty;*

(ii) $r(\{x_n\}) = \max\{\frac{1}{2}\lim_k \sup_{m\ge 1}(\sup_{n\ge k} x_n^{(m)} - \inf_{n\ge k} x_n^{(m)},$ $\lim_k \limsup_m \sup_{n\ge k} |x_n^{(m)}|\}$;

(iii) *If $\{x_n\}$ converges coordinatewise to $x_0 \in \ell^\infty$, then $r(\{x_n\}) = \lim_k \limsup_m \sup_{n\ge k} |x_n^{(m)}|$ and $A(\{x_n\}) = \{x \in c_0 : ||x - x_0||_\infty \le r(\{x_n\})\}$.*

Proposition B. *Let $\{x_n\}$ be a bounded sequence in c. Then*

(i) *$A(\{x_n\})$ is nonempty*;

(ii) $r(\{x_n\}) = \max\{\frac{1}{2}\lim_k \sup_{m\geq 1}(\sup_{n\geq k} x_n^{(m)} - \inf_{n\geq k} x_n^{(m)}),$ $\frac{1}{2}\lim_k(\limsup_m \sup_{n\geq k} x_n^{(m)} - \liminf_m \inf_{n\geq k} x_n^{(m)})\}$;

(iii) *If $\{x_n\}$ converges coordinatewise to $x_0 \in \ell^\infty$, then*

$$r(\{x_n\}) = \frac{1}{2}\lim_k(\limsup_m \sup_{n\geq k} x_n^{(m)} - \liminf_m \inf_{n\geq k} x_n^{(m)})$$

and $A(\{x_n\}) = \{x \in c : ||x - x_0||_\infty \leq r(\{x_n\}),$

$$\lim_m x^{(m)} = \frac{1}{2}\lim_k(\limsup_m \sup_{n\geq k} x_n^{(m)} + \liminf_m \inf_{n\geq k} x_n^{(m)})\}.$$

Lemma 1. *For a sequence $\{a_{mn}\}$ of numbers with double index, one has*

(i) $\lim_k \limsup_n \sup_{m\geq k} a_{mn} = \lim_k \limsup_m \sup_{n\geq k} a_{mn}$ *and*

(ii) $\lim_k \liminf_n \inf_{m\geq k} a_{mn} = \lim_k \liminf_m \inf_{n\geq k} a_{mn}$.

Proof. Let $L = \lim_k \limsup_n \sup_{m\geq k} a_{mn}$. Choose a subsequence $a_{m_p n_p} : p = 1, 2, \dots$ such that $m_p, n_p \to \infty$ and $\lim_p a_{m_p n_p} = L$. For every k, $a_{m_p n_p} \leq \sup_{n\geq k} a_{m_p n}$ for sufficiently large p. Thus, $L = \lim_p a_{m_p n_p} \leq \limsup_p \sup_{n\geq k} a_{m_p n} \leq \limsup_m \sup_{n\geq k} a_{mn}$. The reverse inequality is proved by interchanging the roles of m and n. (ii) follows from (i) by replacing a_{mn} by $-a_{mn}$. □

Corollary 3. *Let $\{a_{mn}\}$ be as in Lemma* 1. *Then*

(i) $\lim_k \liminf_n \sup_{m\geq k} a_{mn} \leq \lim_k \limsup_m \sup_{n\geq k} a_{mn}$ *and*

(ii) $\lim_k \limsup_n \inf_{m\geq k} a_{mn} \geq \lim_k \liminf_m \inf_{n\geq k} a_{mn}$.

Theorem 4. *A bounded sequence $\{x_n\}$ in c_0 is regular if and only if*

(i) $\lim_k \sup_{m\geq 1}(\sup_{n\geq k} x_n^{(m)} - \inf_{n\geq k} x_n^{(m)}) \leq 2\lim_k \limsup_m \sup_{n\geq k} |x_n^{(m)}|$ *and*

(ii) $\lim_k \liminf_n \sup_{m\geq k} |x_n^{(m)}| = \lim_k \limsup_m \sup_{n\geq k} |x_n^{(m)}|$.

In this case $r(\{x_n\})$ is equal to the number in (ii).

Proof. Let $\{x_n\}$ be a regular sequence and let r_1 be its asymptotic radius. By Proposition A(ii), $r_1 \geq \lim_k \limsup_m \sup_{n\geq k} |x_n^{(m)}|$. Extract a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\lim_i x_{n_i}^{(m)}$ exists for each m. Let $r' = r(\{x_n\})$. By Proposition A(iii), $r' = \lim_k \limsup_m \sup_{i\geq k} |x_{n_i}^{(m)}|$. Since $r_1 = r'$ and, obviously, $\lim_k \limsup_m \sup_{i\geq k} |x_{n_i}^{(m)}| \leq$

$\lim_k \limsup_m \sup_{n\geq k} |x_n^{(m)}|$, one has $r_1 = \lim_k \limsup_m \sup_{n\geq k} |x_n^{(m)}|$. (i) then follows from Proposition A(ii). To prove (ii), let $r_2 = \lim_k \liminf_n \sup_{m\geq k} |x_n^{(m)}|$. By Corollary 3, $r_2 \leq r_1$. If $r_2 < r_1$, then there would exist $\varepsilon > 0$ and an integer M such that $\liminf_n \sup_{m\geq M} |x_n^{(m)}| < r_1 - \varepsilon$. Choose a subsequence n_i such that $\sup |x_{n_i}^{(m)}| < r_1 - \varepsilon$, $i = 1, 2, \dots$. Taking a subsequence, if necessary, we may assume that $\lim_i x_{n_i}^{(m)}$ exists for each m. then, by regularity, Proposition A and Lemma 1,

$$r_1 = r(\{x_n\}) = \lim_k \limsup_m \sup_{i\geq k} |x_{n_i}^{(m)}| = \lim_k \limsup_i \sup_{m\geq k} |x_{n_i}^{(m)}| \leq r_1 - \varepsilon,$$

a contradiction. Hence, $r_1 = r_2$.

Suppose now that the sequence $\{x_n\}$ satisfies (i) and (ii). From (i) and Proposition A(ii), we have $r(\{x_n\}) = \lim_k \limsup_m \sup_{n\geq k} |x_n^{(m)}|$. Let $\{y_i\} = \{x_{n_i}\}$ be a subsequence and let r' be its asymptotic radius. Then

$$\begin{aligned}
\lim_k \limsup_m \sup_{i\geq k} |y_i^{(m)}| &\leq \lim_k \limsup_m \sup_{n\geq k} |x_n^{(m)}| \\
&= \lim_k \liminf_n \sup_{m\geq k} |x_n^{(m)}| \\
&\leq \lim_k \liminf_i \sup_{m\geq k} |y_i^{(m)}| \\
&\leq \lim_k \limsup_i \sup_{m\geq k} |y_i^{(m)}| \\
&= \lim_k \limsup_m \sup_{i\geq k} |y_i^{(m)}|,
\end{aligned}$$

showing that $\lim_k \limsup_m \sup_{i\geq k} |y_i^{(m)}| = r(\{x_n\})$. Also,

$$\begin{aligned}\lim_k \limsup_m \sup_{i\geq k} |y_i^{(m)}| &= \lim_k \limsup_m \sup_{n\geq k} |x_n^{(m)}| \\ &\geq \frac{1}{2} \lim_k \sup_{m\geq 1} (\sup_{n\geq k} x_n^{(m)} - \inf_{n\geq k} x_n^{(m)}) \\ &\geq \frac{1}{2} \lim_k \sup_{m\geq 1} (\sup_{i\geq k} y_i^{(m)} - \inf_{i\geq k} y_i^{(m)}).\end{aligned}$$

It follows that $r(\{y_i\}) = \lim_k \limsup_m \sup_{i\geq k} |y_i^{(m)}| = r(\{x_n\})$, completing the proof. □

Theorem 5. *For a bounded sequence* $\{x_n\}$ *in* c_0, *the following are equivalent*:

(i) $\{x_n\}$ *is* A*-regular*;

(ii) $\{x_n\}$ *is asymptotically uniform*;

(iii) $\{x_n\}$ *converges coordinatewise and*

$$\lim_k \limsup_m \sup_{n\geq k} |x_n^{(m)}| = \lim_k \liminf_n \sup_{m\geq k} |x_n^{(m)}|.$$

In this case, the asymptotic radius and center of $\{x_n\}$ *are given as in Proposition* A(iii).

Proof. Let $\{x_n\}$ be an A-regular sequence. If $\{x_n\}$ were not coordinatewise convergent, then there would exist two coordinatewise convergent subsequences $\{u_i\}$ and $\{v_i\}$ with different coordinatewise limits. Then it is obvious from Proposition A(iii) that $A(\{u_i\}) \neq A(\{v_i\})$. Hence, $\{x_n\}$ must be coordinatewise convergent. It is then also obvious from Proposition A(iii) that asymptotic radii of all subsequences of $\{x_n\}$ are the same. Hence, (i) implies (ii). The above proof together with Theorem 4 proves that (ii) implies (iii). The remainder of the theorem follows from Theorem 4 and Proposition A; we omit the details. □

The next two theorems follow from Proposition B and similar arguments given above. We omit the proofs.

Theorem 6. *A bounded sequence $\{x_n\}$ in c is regular if and only if*

(i)

$$\lim_k \lim_m \sup \sup_{n\geq k} x_n^{(m)} - \lim_k \lim_m \inf \inf_{n\geq k} x_n^{(m)} \geq \lim_k \sup_{m\geq 1} (\sup_{n\geq k} x_n^{(m)} - \inf_{n\geq k} x_n^{(m)}),$$

(ii)

$$\lim_k \lim_m \sup \sup_{n\geq k} x_n^{(m)} = \lim_k \lim_n \inf \sup_{m\geq k} x_n^{(m)},$$

(iii)

$$\lim_k \lim_m \inf \inf_{n\geq k} x_n^{(m)} = \lim_k \lim_n \sup \inf_{m\geq k} x_n^{(m)}.$$

In this case,

$$r(\{x_n\}) = \frac{1}{2}(\lim_k \lim_m \sup \sup_{n\geq k} x_n^{(m)} - \lim_k \lim_m \inf \inf_{n\geq k} x_n^{(m)}).$$

Theorem 7. *For a bounded sequence $\{x_n\}$ in c, the following are equivalent*:

(i) $\{x_n\}$ *is A-regular*,

(ii) $\{x_n\}$ *is asymptotically uniform*,

(iii) $\{x_n\}$ *is coordinatewise convergent*,

$$\lim_k \lim_m \sup \sup_{n\geq k} x_n^{(m)} = \lim_k \lim_n \inf \sup_{m\geq k} x_n^{(m)}$$

and

$$\lim_k \lim_m \inf \inf_{n\geq k} x_n^{(m)} = \lim_k \lim_n \sup \inf_{m\geq k} x_n^{(m)}.$$

In this case the asymptotic radius and center of $\{x_n\}$ are as in Proposition B(iii).

Open Question. Is it true that, in a general Banach space X, if x_n, $n = 1, 2, \ldots$, pseudo-converges to zero, then there exists a sequence y_n in X and a sequence $z_n \in J(y_n)$ such that $\lim ||x_n - y_n|| = 0$ and $z_n \overset{*}{\to} 0$? In other words, is the converse of Proposition 2 true? The results of this paper show that it is true in spaces with uniform Gateaux differentiable norm, c_0, c and ℓ_1.

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