ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 21, Number 3, Summer 1991

A SIMPLE CONTINUED FRACTION TEST FOR THE IRRATIONALITY OF FUNCTIONS

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ABSTRACT. A theorem on irrationality of positive continued fractions with one variable is proved by introducing the behavior of tails.

Introduction. The subject for study in this paper is the continued fraction of the type

$$\underset{n=1}{\overset{\infty}{K}} \frac{a_n(x)}{b_n(x)} \tag{1}$$

where the elements $a_n(x)$ and $b_n(x)$ are nonzero polynomials of the variable $x > 0$ with nonnegative coefficients. The continued fractions

$$T_k(x) = \underset{n=k}{\overset{\infty}{K}} \frac{a_n(x)}{b_n(x)} \tag{2}$$

for $k \geq 1$ are called the tails of (1). As background to the problem of irrationality of functions by the aid of continued fraction theory we refer to [**1**; Corollary 5.3, p. 156 and p. 380] where further references are given. In the most known irrationality tests for functions where continued fractions are used, there is no involvement of convergence and the behavior of tails considered as functions. For an example, see the C-fraction test in [**1**, p. 156]. In the present paper we give an example of a theorem on irrationality where both convergence and the magnitude of the tails play a crucial role. The basic idea is quite general and flexible, but we apply it only on the continued fraction of type (1) for the sake of simplicity. An example of a continued fraction of type (1) is

$$x = \frac{x+1}{1} \,\underset{+}{}\, \frac{2}{x} \,\underset{+}{}\, \frac{x+1}{1} \,\underset{+}{}\, \cdots \tag{3}$$

Received by the editors on March, 2, 1987.

where $x > 0$. We will justify formula (3). First, notice that (3) converges because of [**2**; Satz 2.10, p. 46]. Second, since (3) is 2-periodic one can easily determine the value to be x. Thus, (3) is an example of a rational function of $x > 0$ of type (1). In the proof of the theorem we will give conditions imposed on (1) such that we can conclude (1) to be irrational. An example of this situation is given at the end. Before going further we note that the degree of a polynomial q is denoted by $\deg q$. Thus, $\deg q \geq 0$. We now turn to the proof of the theorem that contains the irrationality test.

The irrationality test.

Theorem. *Suppose that the continued fraction*

$$f(x) = \mathop{K}_{n=1}^{\infty} \frac{a_n(x)}{b_n(x)} \tag{4}$$

converges to a finite value. The elements in (4) *are nonzero polynomials in a variable* $x > 0$ *with nonnegative coefficients. Suppose further that* $\{p_{2k-1}\}_{k\geq 1}$ *is a sequence of integers such that*

$$\deg b_{2n} > p_{2n+1} \tag{5}$$

and

$$\deg b_{2n} > \deg a_{2n} + p_{2n-1} \tag{6}$$

for $n \geq 1$ *with the existence of*

$$\lim_{x\to\infty} T_{2n-1}(x) x^{-p_{2n-1}} \neq 0 \tag{7}$$

for $n \geq 1$. *Then* (4) *is an irrational function of* $x > 0$.

Proof. Since (4) is convergent we clearly have that the tails

$$T_k(x) = \mathop{K}_{n=k}^{\infty} \frac{a_n(x)}{b_n(x)}$$

for $k \geq 1$ also are convergent. Obviously then

$$T_k(x) = \frac{a_k(x)}{b_k(x) + T_{k+1}(x)} \tag{8}$$

for $k \geq 1$. Let, further, $g_0(x)$ be a given nonzero function of $x > 0$. Then define the sequence $\{g_k(x)\}_{k \geq 0}$ of functions by

$$g_k(x) = T_k(x) g_{k-1}(x) \tag{9}$$

for $k \geq 1$. Since, obviously, $T_k(x) \neq 0$ for $k \geq 1$ we have that $g_k(x) \neq 0$ for $k \geq 0$. Combining (8) and (9) one obtains the recurrence relation

$$a_{k-1}(x) g_{k-2}(x) = b_{k-1}(x) g_{k-1}(x) + g_k(x) \tag{10}$$

for $k \geq 2$. From (10) we notice that, if $g_0(x)$ and $g_1(x)$ both are polynomials in the variable x, we can conclude that $g_k(x)$ is a polynomial in x for all $k \geq 0$. Suppose now that (4) is a rational function of $x > 0$. Then $f(x)$ can be represented as a fraction where both the numerator and the denominator are polynomials in x such that the denominator is nonzero. We therefore choose $g_0(x)$ to be this denominator. Then, in the light of (9), clearly $g_1(x)$ is the numerator. Thus both $g_0(x)$ and $g_1(x)$ are polynomials. We can therefore conclude that $g_k(x)$ is a polynomial for $k \geq 0$. From (9), we obtain

$$T_k(x) = \frac{g_k(x)}{g_{k-1}(x)} \tag{11}$$

for $k \geq 1$. The condition (7) in the theorem makes us now able to conclude from (11) that

$$\deg g_{2k-1} - \deg g_{2k-2} = p_{2k-1} \tag{12}$$

for $k \geq 1$. Because of (5) we conclude from (12) that

$$\deg g_{2k+1} < \deg g_{2k} + \deg b_{2k} \tag{13}$$

for $k \geq 1$. Consider further that

$$a_{2k}(x) g_{2k-1}(x) = b_{2k}(x) g_{2k}(x) + g_{2k+1}(x) \tag{14}$$

for $k \geq 1$ which follows from (10). In light of (13) we then see from (14) that

$$\deg a_{2k} + \deg g_{2k-1} = \deg b_{2k} + \deg g_{2k} \tag{15}$$

for $k \geq 1$. From (15), we find further that

$$\begin{aligned} \deg g_{2k} &= \deg a_{2k} - \deg b_{2k} + \deg g_{2k-1} \\ &< \deg g_{2k-1} - p_{2k-1} = \deg g_{2k-2} \end{aligned} \tag{16}$$

for $k \geq 1$. We will justify this properly in the following. The first step in (16) is a rearrangement of (15). The second step uses the condition (6) in the theorem. The last step uses (12). From (16), we see that

$$\deg g_{2k} < \deg g_{2k-2} \tag{17}$$

for $k \geq 1$. This means that there is an infinite descending sequence of nonnegative integers which is impossible. Thus the function $f(x)$ in (4) cannot be rational, and the theorem is proved. □

It should be noted that the continued fraction in (3) does not fulfill the conditions in the theorem and therefore cannot be expected to be irrational in light of the theorem. As we have seen, (3) is, in fact, rational.

Example. Let $\alpha_{2n} > 0$ for $n \geq 1$. Consider the continued fraction

$$\frac{1}{x +} \, \frac{1 + \alpha_2 x}{x} \, \frac{1}{+ \, x +} \, \frac{1 + \alpha_4 x}{x} \, \frac{1}{+ \, x +} \cdots \tag{18}$$

for $x > 0$. We will show that the conditions in the theorem are fulfilled. Because of [**2**; Sats 2.10, p. 46] we see that (18) converges to a finite value. We notice further that $\deg a_{2n} = \deg b_{2n} = 1$ for $n \geq 1$. Also we see that

$$\frac{x}{x^2 + \alpha_{2n} x + 1} \leq T_{2n-1}(x) \leq \frac{1}{x} \tag{19}$$

for $n \geq 1$. From (19) we obtain

$$\lim_{x \to \infty} T_{2n-1}(x) \cdot x = 1 \tag{20}$$

for $n \geq 1$. Therefore, we must choose $p_{2k-1} = -1$ for $k \geq 1$. It is now easily checked that (5) and (6) are fulfilled and the irrationality test in the theorem can be applied. The conclusion is therefore that (18) is an irrational function of $x > 0$. If we choose $\alpha_{2n} = 1$ for $n \geq 1$ we obtain that

$$-\frac{1}{2}x - \frac{1}{2} + \frac{1}{2}\sqrt{x^2 + 2x + 5} = \frac{1}{x} {}_{+} \frac{1+x}{x} {}_{+} \frac{1}{x} {}_{+} \cdots \tag{21}$$

is an irrational function which is clearly the case. The formula (21) is easily checked since (21) is a 2-periodic continued fraction.

REFERENCES

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