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A SWITCH IN NODAL STRUCTURE IN COUPLED SYSTEMS OF NONLINEAR STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS

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ABSTRACT. It is well known that the bifurcating nontrivial solutions to nonlinear Sturm-Liouville boundary value problems may be globally distinguished via the nodal structure of solutions. We demonstrate in this article that such is not necessarily the case for appropriate coupled multiparameter systems of such problems. Specifically, we give a calculable condition for the existence of a continuum of nontrivial solutions to such a system where the nodal structure of solution components is not preserved.

1. Introduction. Nonlinear Sturm-Liouville boundary value problems and associated systems arise frequently in mathematical analysis and applications. Consequently, there has been substantial interest in a detailed understanding of the solution sets to these problems, and a great deal of information has been obtained in the case of a single equation. For instance, consider the problem

$$-(p(t)x'(t))' + q(t)x(t) = \lambda(r(t)x(t) + f(t, x(t))) \tag{1.1}$$

$$\begin{aligned} \alpha_1 x(a) + \alpha_1' x'(a) &= 0 \\ \alpha_2 x(b) + \alpha_2' x'(b) &= 0, \end{aligned} \tag{1.2}$$

where $t \in [a, b]$ and $(|\alpha_1| + |\alpha_1'|)(|\alpha_2| + |\alpha_2'|) > 0$. In addition, we require that $p \in C^1[a, b]$ with $p(t) > 0$ on $[a, b]$, that $q, r \in C[a, b]$ with $r(t) > 0$ on $[a, b]$, and that $f : [a, b] \times \mathbf{R} \to \mathbf{R}$ is continuous with $\lim_{s\to 0} \frac{f(t,s)}{s} = 0$ uniformly for $t \in [a, b]$. In this situation, as is well known, there is a sequence

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n \to +\infty$$

of simple eigenvalues for the problem

$$-(pw')' + qw = \lambda rw \tag{1.3}$$

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plus boundary conditions (1.2) such that if w_i is an eigenfunction corresponding to λ_i, then w_i has $i-1$ zeros in the interval (a,b), all of which are simple. Moreover, if E denotes the C^1 functions on $[a,b]$ satisfying (1.2), then E is a Banach space under the usual C^1 norm and from $(\lambda_i, 0)$ in $\mathbf{R}\times E$ there emanates a continuum $\mathcal{C}_i$ of solutions to (1.1–1.2). $\mathcal{C}_i$ is unbounded in $\mathbf{R}\times E$ and if $(\lambda, y)\in\mathcal{C}_i$, $y\neq 0$, then y has $i-1$ zeros in (a,b), all of which are simple. In particular, $\mathcal{C}_i\cap\mathcal{C}_j=\varnothing$ if $i\neq j$. These results, which can be found in any number of sources (e.g., [**3,5,9**]), illustrate the first alternative of the celebrated global bifurcation result of Rabinowitz [**7**].

Now, as regards system analogues to problems of the type (1.1)–(1.2), the level of understanding is considerably lower. A natural question to be addressed is whether there is any global analogue to the single equation phenomena described above. The answer seems to depend strongly on the coupling in the system. In certain instances of systems which arise in mathematical biology, the answer is yes. However, such is not always the case as we demonstrate in this article.

In [**2**] we considered the class of problems

$$\begin{aligned} Lu(x) &= \lambda f(u(x), v(x)) \\ Lv(x) &= \mu g(u(x), v(x)), \end{aligned} \tag{1.4}$$

where $x\in\Omega$, a smooth bounded domain in $\mathbf{R}^N$, L is a strongly uniformly elliptic second order linear differential operator and u and v are required to satisfy

$$u(x) = 0 = v(x) \tag{1.5}$$

for $x\in\partial\Omega$. We placed several requirements on the functions $f,g:\mathbf{R}^2\to\mathbf{R}$. First of all, f and g are required to be smooth with $f(0,0)=0=g(0,0)$. In addition, $\partial f(0,0)/\partial u$, $\partial f(0,0)/\partial v$, $\partial g(0,0)/\partial u$, and $\partial g(0,0)/\partial v$ are all assumed to be positive with

$$\frac{\partial f(0,0)}{\partial u}\frac{\partial g(0,0)}{\partial v} - \frac{\partial f(0,0)}{\partial v}\frac{\partial g(0,0)}{\partial u} > 0. \tag{1.6}$$

With these assumptions (1.4–1.5) has, as linearization about $(0,0)$:

$$\begin{aligned} Lw &= \lambda\left(\frac{\partial f(0,0)}{\partial u}w + \frac{\partial f(0,0)}{\partial v}z\right) \\ Lz &= \mu\left(\frac{\partial g(0,0)}{\partial u}w + \frac{\partial g(0,0)}{\partial v}z\right) \qquad \text{in } \Omega, \end{aligned} \tag{1.7}$$

$$w = 0 = z \qquad \text{on } \partial\Omega.$$

We showed in [**2**] that if the eigenvalue problem

$$\begin{aligned} Lw &= \lambda w \qquad \text{in } \Omega \\ w &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{1.8}$$

has eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots$, then the generalized spectrum [**6**] for (1.7) is given by the family of hyperbolae

$$\left\{ (\lambda, \mu) : \lambda = \frac{\lambda_n \left(\frac{\partial g(0,0)}{\partial v} \mu - \lambda_n \right)}{\left(\frac{\partial f(0,0)}{\partial u} \frac{\partial g(0,0)}{\partial v} - \frac{\partial f(0,0)}{\partial v} \frac{\partial g(0,0)}{\partial u} \right) \mu - \lambda_n \frac{\partial f(0,0)}{\partial u}} \right\},$$

$n = 1, 2, 3, \ldots$. The hyperbola corresponding to λ_n will intersect the one corresponding to λ_m, where $n < m$, if and only if

$$\frac{\lambda_n}{\lambda_m} \le \frac{\sqrt{\frac{\partial f(0,0)}{\partial u} \frac{\partial g(0,0)}{\partial v}} - \sqrt{\frac{\partial f(0,0)}{\partial v} \frac{\partial g(0,0)}{\partial u}}}{\sqrt{\frac{\partial f(0,0)}{\partial u} \frac{\partial g(0,0)}{\partial v}} + \sqrt{\frac{\partial f(0,0)}{\partial v} \frac{\partial g(0,0)}{\partial u}}}.$$

(Note that the intersection is transverse if the inequality is strict.)

Now suppose that $N = 1$, $\Omega = (a, b)$ and L is the Sturm-Liouville operator given by

$$Lw = -(pw')' + qw. \tag{1.9}$$

If (λ_0, μ_0) is a point on the hyperbola corresponding to λ_n in the generalized spectrum but not a point of intersection with any of the other hyperbolae in the collection, then 1 is an algebraically simple eigenvalue of

$$\begin{pmatrix} \lambda_0 \frac{\partial f(0,0)}{\partial u} L^{-1} & \lambda_0 \frac{\partial f(0,0)}{\partial v} L^{-1} \\ \mu_0 \frac{\partial g(0,0)}{\partial u} L^{-1} & \mu_0 \frac{\partial g(0,0)}{\partial v} L^{-1} \end{pmatrix}.$$

The corresponding eigenfunction is

$$\begin{pmatrix} x_n \\ \beta_n x_n \end{pmatrix},$$

where

$$\begin{aligned} Lx_n &= \lambda_n x_n \qquad \text{in } \Omega \\ x_n &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

and

$$\begin{pmatrix} \lambda_n - \lambda_0 \frac{\partial f(0,0)}{\partial u} & -\lambda_0 \frac{\partial f(0,0)}{\partial v} \\ -\mu_0 \frac{\partial g(0,0)}{\partial u} & \lambda_n - \mu_0 \frac{\partial g(0,0)}{\partial v} \end{pmatrix} \begin{pmatrix} 1 \\ \beta_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

On the other hand, if (λ_0, μ_0) is a point of intersection for the n^{th} and m^{th} hyperbolae, 1 is an algebraically double eigenvalue with eigenspace

$$\left\langle \begin{pmatrix} x_n \\ \beta_n x_n \end{pmatrix}, \begin{pmatrix} x_m \\ \beta_m x_m \end{pmatrix} \right\rangle.$$

Consequently, in the case of a simple eigenvalue the bifurcation theorem of Alexander and Antman [**1**] may be applied to assert the existence of a global multi-dimensional continuum of solutions to (1.4)–(1.5) emanating in $\mathbf{R}^2 \times E^2$ from $(\lambda_0, \mu_0, 0, 0)$. Moreover, if $\lambda_0 \neq 0$ and $\mu_0 \neq 0$, then if (λ, μ, u, v) is a solution to (1.4)–(1.5) with $||(\lambda, \mu, u, v) - (\lambda_0, \mu_0, 0, 0)||_{\mathbf{R}^2 \times E^2}$ sufficiently small and $(u, v) \neq (0, 0)$, then each of u and v has $n-1$ zeros in (a, b), all of which are simple. (We should note that these results, contained in [**2**], remain valid if u and v are required to satisfy the more general boundary condition (1.2).)

The purpose of this article is to show that, *in contrast to the single equation case* (1.1)–(1.2), the above mentioned local result will be the best that one can expect in general. In particular, we consider the case f and g both analytic with *not all of their second partial derivatives vanishing* at $(0, 0)$. We give a verifiable condition for the existence of a continuum in $\mathbf{R}^2 \times E^2$ *not containing any trivial solutions* for (1.4)–(1.5), between solutions to (1.4)–(1.5) with components having $n-1$ simple zeros in (a, b) and solutions whose components have $m-1$ simple zeros in (a, b), where $n \neq m$.

Let us be somewhat more explicit. To this end, let

$$\lambda^{(n)}(\mu) = \frac{\lambda_n \left(\frac{\partial g(0,0)}{\partial v} \mu - \lambda_n \right)}{\left(\frac{\partial f(0,0)}{\partial u} \frac{\partial g(0,0)}{\partial v} - \frac{\partial f(0,0)}{\partial v} \frac{\partial g(0,0)}{\partial u} \right) \mu - \lambda_n \frac{\partial f(0,0)}{\partial u}}.$$

Suppose that (λ^*, μ^*) is such that $\lambda^* = \lambda^{(n)}(\mu^*) = \lambda^{(m)}(\mu^*)$, $n \neq m$ and, moreover, that the graphs of $\lambda^{(n)}$ and $\lambda^{(m)}$ meet transversely at (λ^*, μ^*). Then, for $\delta > 0$ and sufficiently small, $\partial B((\lambda^*, \mu^*); \delta) = \{(\lambda, \mu) \in \mathbf{R}^2 : |(\lambda, \mu) - (\lambda^*, \mu^*)| = \delta\}$ meets the generalized spectrum for (1.7) in exactly four points, two of which lie on the graph of $\lambda^{(n)}$ and two on the graph of $\lambda^{(m)}$. Consider the problem (1.4)–(1.5) in $\partial B((\lambda^*, \mu^*); \delta) \times (C_0^1[a, b])^2$ and let $\mathcal{S}_\delta$ denote the closure of nontrivial solutions to (1.4)–(1.5) in $\delta B((\lambda^*, \mu^*); \delta) \times (C_0^1[a, b])^2$. Our main result is (Theorem 3.3 in the text)

Theorem. *Under appropriate technical assumptions* (*as described in* §3), *we may derive from* (1.4)–(1.5) *a sixth degree polynomial* P *with the following property. Namely, if* $P(\lambda^{(n)'}(\mu^*)) > 0$ *or if* $P(\lambda^{(m)'}(\mu^*)) > 0$, *then there is a* $\delta^* > 0$ *such that, for any* $\delta \in (0, \delta^*)$, *there is a continuum* $\mathcal{C}_\delta \subseteq \mathcal{S}_\delta$ *such that*

$$\mathcal{C}_\delta \cap (\{(\lambda^n(\mu), \mu) : \mu \in \mathbf{R}\} \times \{(0, 0)\}) \neq \varnothing$$

and

$$\mathcal{C}_\delta \cap (\{(\lambda^{(m)}(\mu), \mu) : \mu \in \mathbf{R}\} \times \{(0, 0)\}) \neq \varnothing.$$

For the remainder of this article, we refer to the existence of $\mathcal{C}_\delta$ as a "switch in nodal structure." Specifically, suppose that $(\lambda, \mu, u, v) \in \mathcal{C}_\delta$, $(u, v) \neq (0, 0)$ and $||(\lambda, \mu, u, v) - (\lambda^{(n)}(\bar{\mu}), \bar{\mu}, 0, 0)||_{\mathbf{R}^2 \times E^2}$ is sufficiently small, where $(\lambda^{(n)}(\bar{\mu}), \bar{\mu})$ is a point of the intersection of the generalized spectrum for (1.7) with $\delta B((\lambda^*, \mu^*); \delta)$. Then, as previously noted, u and v have nonzero derivatives at a and b and precisely $n - 1$ zeros in (a, b), each of which is simple. However, $\mathcal{C}_\delta$ also contains solutions whose components have nonzero derivatives of a and b and precisely $m-1$ zeros in (a, b), each of which is simple. Consequently, there can be no maintenance of a specific nodal structure along $\mathcal{C}_\delta$. Since prescribed nodal types form open sets in the C^1 topology (see, for example, [**3**]) and since the only trivial solutions in $\mathcal{C}_\delta$ occur for parameter values (λ, μ) in the intersection of $\partial B((\lambda^*, \mu^*); \delta)$ with the graphs of $\lambda^{(n)}$ and $\lambda^{(m)}$, $\mathcal{C}_\delta$ must contain a nontrivial solution in the boundary of a set of pairs of functions of a prescribed nodal type. Hence, $\mathcal{C}_\delta$ contains a nontrivial solution to (1.4)–(1.5) with a nontrivial component having a double zero.

The fact this linking phenomenon can occur in a class of nonlinear problems is not too surprising after some reflection on the special case when (1.4)–(1.5) is linear. Consider, for example, the linear system

$$\begin{aligned} -u''(x) &= \lambda(2u(x)+v(x)) \\ -v''(x) &= \mu(u(x)+v(x)) \end{aligned} \tag{1.10}$$

$$\begin{aligned} u(0) = 0 = u(\pi) \\ v(0) = 0 = v(\pi). \end{aligned} \tag{1.11}$$

It follows, from [**2**, §8] and a simple computation, that when $(\lambda,\mu) = (10-2\sqrt{7}, 20+4\sqrt{7})$, (1.10)–(1.11) has eigenfunctions

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} \sin 2x \\ \frac{\sqrt{7}-13}{9}\sin 2x \end{pmatrix}$$

and

$$\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} \sin 6x \\ (3+\sqrt{7})\sin 6x \end{pmatrix}.$$

Then, for $t \in [0,1]$,

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} t\sin 2x + (1-t)\sin 6x \\ t\left(\frac{\sqrt{7}-13}{9}\right)\sin 2x + (1-t)(3+\sqrt{7})\sin 6x \end{pmatrix}$$

is an eigenfunction for (1.10)–(1.11). Hence, a switch in nodal type occurs in $\mathbf{R}^2 \times E^2$. As previously noted, prescribed nodal types form open sets in the C^1 topology. Consequently, we must necessarily have passed through a boundary with our homotopy. In particular, we find that if $t_0 = 27(3+\sqrt{7})/(89+26\sqrt{7})$, then $v_{t_0}(\pi/2) = 0$ and $v'_{t_0}(\pi/2) = 0$. This phenomenon could not occur in the case of a single equation by uniqueness of initial value problems.

In the nonlinear case we analyze in some detail the structure of the solution set to (1.4)–(1.5) in a *neighborhood* of a point $(\lambda^*, \mu^*, 0, 0)$ where 1 is a double eigenvalue of

$$\begin{pmatrix} \lambda^* \frac{\partial f(0,0)}{\partial u} L^{-1} & \lambda^* \frac{\partial f(0,0)}{\partial v} L^{-1} \\ \mu^* \frac{\partial g(0,0)}{\partial u} L^{-1} & \mu^* \frac{\partial g(0,0)}{\partial v} L^{-1} \end{pmatrix}.$$

Upon reduction to $\mathbf{R}^4$ via a Lyapunov-Schmidt process (in §2), we obtain a local representation of the solution set as the intersection of two irreducible quadratic analytic varieties (the lowest order coefficients are calculable from the boundary value problem). This intersection is effectively quantified by the resultant [**10**] of two polynomials. In particular, since the varieties are quadratic, the resultant determines a quartic equation. However, since we always have the $(0,0)$ solution in E^2 for all $(\lambda,\mu) \in \mathbf{R}^2$, the resultant may be reduced to a cubic. Consequently, we can employ the cubic discriminant in order to count the number of nontrivial solutions "above" (λ,μ). Combining this information with Rabinowitz bifurcation theory, we obtain our main result in Theorem 3.3. Namely, the switch in nodal properties described above occurs if a certain polynomial P derived from the system of boundary value problems is positive when evaluated at the slope (at (λ^*,μ^*)) of either of the two intersecting hyperbolae of the generalized spectrum. Moreover, the coefficients of P are explicitly calculable provided the eigenfunctions of L are known. In §4, we demonstrate with a particular example.

A comment is in order at this point. The coefficients of P are determined through a succession of calculations (reflecting the Lyapunov-Schmidt procedure, the taking of a resultant, and so forth). Knowing how to make this succession of calculations is, of course, the heart of any application to a specific system of boundary value problems. However, the calculations *per se* are not crucial to an understanding of the proof of Theorem 3.3. Consequently, for the sake of clarity, we do not include the calculations in the body of the proof, but rather collect them in an Appendix.

Finally, we note that the computations for the example were made using the symbolic manipulation program MU-MATH. We wish to express our appreciation to Dr. Wagar Ali for his help in facilitating these computations.

2. Lyapunov-Schmidt reduction. Consider (1.4–1.5) which we write as

$$\begin{aligned} Lu &= \lambda(f_1 u + f_2 v + \tilde{f}(u,v)) \\ Lv &= \mu(g_1 u + g_2 v + \tilde{g}(u,v)), \end{aligned} \quad \text{in } \Omega \tag{2.1}$$

where $f_1 = \frac{\partial f(0,0)}{\partial u}$, $f_2 = \frac{\partial f(0,0)}{\partial v}$, $g_1 = \frac{\partial g(0,0)}{\partial u}$, $g_2 = \frac{\partial g(0,0)}{\partial v}$ and $\tilde{f}$ and $\tilde{g}$

are higher order in $||(u,v)||_{E^2}$ and

$$u|_{\partial\Omega} \equiv 0 \equiv v|_{\partial\Omega}. \tag{2.2}$$

If we let $G = L^{-1}$, (2.1)–(2.2) may be expressed as

$$\phi = \Lambda A \mathcal{G}\phi + \Lambda \mathcal{G} N(\phi), \tag{2.3}$$

with $\phi = \binom{u}{v}$, $\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, $A = \begin{pmatrix} f_1 & f_2 \\ g_1 & g_2 \end{pmatrix}$, $\mathcal{G} = \begin{pmatrix} G & 0 \\ 0 & G \end{pmatrix}$, and $N(\phi) = \begin{pmatrix} \tilde{f}(u,v) \\ \tilde{g}(u,v) \end{pmatrix}$. If now $\Lambda^* = \begin{pmatrix} \lambda^* & 0 \\ 0 & \mu^* \end{pmatrix}$ is such that 1 is a double eigenvalue of

$$\begin{pmatrix} \lambda^* f_1 G & \lambda^* f_2 G \\ \mu^* g_1 G & \mu^* g_2 G \end{pmatrix},$$

it follows from the Riesz theory of compact operators [**8**] that

$$E^2 = \left\langle \begin{pmatrix} x_n \\ \beta_n x_n \end{pmatrix}, \begin{pmatrix} x_m \\ \beta_m x_m \end{pmatrix} \right\rangle \oplus R\left(\mathbf{I} - \begin{pmatrix} \lambda^* f_1 G & \lambda^* f_2 G \\ \mu^* g_1 G & \mu^* g_2 G \end{pmatrix}\right),$$

where x_n and x_m are normalized in an appropriate manner. (Without any loss of generality, we assume $n < m$.) Then $T : E^2 \to E^2$, given by

$$T(\phi) = (I - \Lambda^* A \mathcal{G})\phi + \langle \phi, \gamma_1 \rangle \phi_1 + \langle \phi, \gamma_2 \rangle \phi_2,$$

is a linear homeomorphism, where $\phi_1 = \begin{pmatrix} x_n \\ \beta_n x_n \end{pmatrix}$, $\phi_2 = \begin{pmatrix} x_m \\ \beta_m x_m \end{pmatrix}$, and $\gamma_1, \gamma_2 \in (E^2)^*$ are such that $\gamma_i(\phi_j) = \delta_{ij}$ and

$$\gamma_i \mid R\left(\mathbf{I} - \begin{pmatrix} \lambda^* f_1 G & \lambda^* f_2 G \\ \mu^* g_1 G & \mu^* g_2 G \end{pmatrix}\right) \equiv 0.$$

Consequently, with $\tau_1 = \lambda - \lambda^*$ and $\tau_2 = \mu - \mu^*$ and $\tau = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$, we find that (2.3) is equivalent to the system of equations

$$\begin{cases} \phi = \alpha_1 \phi_1 + \alpha_2 \phi_2 + T^{-1}(\tau A \mathcal{G}\phi + \Lambda \mathcal{G} N(\phi)) \\ \alpha_1 = \langle \phi, \gamma_1 \rangle \\ \alpha_2 = \langle \phi, \gamma_2 \rangle. \end{cases} \tag{2.4}$$

As in [**2**], the right-hand side of the first equation of (2.4) (which we denote $S(\alpha_1, \alpha_2, \tau_1, \tau_2)$) is a contraction mapping of a neighborhood of the origin if $|(\alpha_1, \alpha_2, \tau_1, \tau_2)|$ is sufficiently small. If $\hat{\phi}(\alpha_1, \alpha_2, \tau_1, \tau_2)$ denotes the unique fixed point of $S(\alpha_1, \alpha_2, \tau_1, \tau_2)$, then $\hat{\phi}$ is analytic in $(\alpha_1, \alpha_2, \tau_1, \tau_2)$. Consequently, solvability of (2.4) is equivalent to

$$\alpha_i = \langle \hat{\phi}(\alpha_1, \alpha_2, \tau_1, \tau_2), \gamma_i \rangle,$$

$i = 1, 2$, which, in turn, simplifies to

$$0 = \langle A^{-1}\Lambda^{*-1}(\tau A\hat{\phi} + \Lambda N(\hat{\phi})), \gamma_i \rangle, \tag{2.5}$$

$i = 1, 2$ (see [**2**, pp. 272-273]). Since

$$\hat{\phi}(\alpha_1, \alpha_2, \tau_1, \tau_2) = \lim_{n\to\infty} (S(\alpha_1, \alpha_2, \tau_1, \tau_2))^n(0),$$

the lowest order terms of $\hat{\phi}$ are $\alpha_1\phi_1 + \alpha_2\phi_2$. Hence, since $\Lambda = \tau + \Lambda^*$, it is easy to see that the quadratic terms in (2.5) are given by

$$\langle A^{-1}\Lambda^{*-1}(\tau A(\alpha_1\phi_1 + \alpha_2\phi_2) + \Lambda^* \tilde{N}(\alpha_1\phi_1 + \alpha_2\phi_2)), \gamma_i \rangle, \tag{2.6}$$

where $\tilde{N}$ denotes the quadratic terms of N.

3. Main results. Suppose now that $f_2 = g_1$. Then (see [**2**]) γ_1 and γ_2 have the forms

$$\gamma_1 \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{k_1} \int_a^b u x_n + v\delta_1 x_n$$
$$\gamma_2 \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{k_2} \int_a^b u x_m + v\delta_2 x_m.$$

Consequently, since $\int_a^b x_n x_m = 0$, certain of the quadratic terms in (2.5) are automatically zero. It follows that (2.5) is equivalent to

$$\begin{aligned} &\text{(i)} \quad c_1\alpha_1^2 + c_2\tau_1\alpha_1 + c_3\tau_2\alpha_1 + c_4\alpha_2\alpha_1 + c_5\alpha_2^2 \\ &\qquad\qquad + \text{h.o.t.}\,(\alpha_1, \alpha_2, \tau_1, \tau_2) = 0, \\ &\text{(ii)} \quad d_1\alpha_1^2 + d_2\alpha_2\alpha_1 + d_3\tau_1\alpha_2 + d_4\tau_2\alpha_2 + d_5\alpha^2 \\ &\qquad\qquad + \text{h.o.t.}\,(\alpha_1, \alpha_2, \tau_1, \tau_2) = 0, \end{aligned} \tag{3.1}$$

where the c's and d's are as given in the Appendix and h.o.t. $(\alpha_1, \alpha_2, \tau_1, \tau_2)$ denotes higher order terms in $(\alpha_1, \alpha_2, \tau_1, \tau_2)$. Under the assumption that $c_1 \neq 0$ and $d_1 \neq 0$, the Weierstrass preparation theorem [**4**] is applicable with α_1 as distinguished variable. Moreover, the lowest order terms in (3.1)(i)-(ii) are known. Hence, (3.1)(i-ii) is equivalent to

$$(3.2)\quad \begin{aligned} &\text{(i)}\quad c_1\alpha_1^2 + (c_2\tau_1 + c_3\tau_2 + c_4\alpha_2 + \text{h.o.t. } (\alpha_2, \tau_1, \tau_2))\alpha_1 \\ &\qquad\quad + (c_5\alpha_2^2 + \text{h.o.t.}\,(\alpha_2, \tau_1, \tau_2)) = 0, \\ &\text{(ii)}\quad d_1\alpha_1^2 + (d_2\alpha_2 + \text{h.o.t.}\,(\alpha_2, \tau_1, \tau_2))\alpha_1 + (d_3\tau_1\alpha_2 + d_4\tau_2\alpha_2 \\ &\qquad\quad + d_5\alpha_2^2 + \text{h.o.t.}\,(\alpha_2, \tau_1, \tau_2)) = 0. \end{aligned}$$

Viewing (3.2) as two polynomials in α_1, (3.2)(i) and (3.2)(ii) are the Weierstrass polynomials [**4**, p. 68] corresponding to (3.1)(i) and (3.1)(ii), respectively. Equations (3.2)(i) and (3.2)(ii) have a common solution exactly when the resultant vanishes. From [**10**, p. 85], this condition may be expressed

$$(3.3)\quad \begin{aligned} &e_1\alpha_2^4 + e_2\tau_1\alpha_2^3 + e_3\tau_2\alpha_2^3 + e_4\tau_1^2\alpha_2^2 + e_5\tau_2^2\alpha_2^2 \\ &\quad + e_6\tau_1\tau_2\alpha_2^2 + e_7\tau_1^3\alpha_2 + e_8\tau_1^2\tau_2\alpha_2 + e_9\tau_1\tau_2^2\alpha_2 \\ &\quad + e_{10}\tau_2^3\alpha_2 + \text{h.o.t.}\,(\alpha_2, \tau_1, \tau_2) = 0, \end{aligned}$$

where the e's are expressed in terms of the c's and d's (see the Appendix). Since $(\alpha_1, \alpha_2) = (0, 0)$ solves (3.2) for any choice of (τ_1, τ_2) (by virtue of the fact that it corresponds to the trivial solution of (1.4)–(1.5) at $(\lambda^* + \tau_1, \mu^* + \tau_2)$), α_2 is necessarily a factor of (3.3). Consequently, if $e_1 \neq 0$, solutions to (3.2) are possible exactly when $\alpha_2 = 0$ or

$$(3.4)\quad \begin{aligned} &e_1\alpha_2^3 + (e_2\tau_1 + e_3\tau_2 + \text{h.o.t.}\,(\tau_1, \tau_2))\alpha_2^2 \\ &\quad + (e_4\tau_1^2 + e_5\tau_2^2 + c_6\tau_1\tau_2 + \text{h.o.t.}\,(\tau_1, \tau_2))\alpha_2 \\ &\quad + (e_7\tau_1^3 + e_8\tau_1^2\tau_2 + e_9\tau_1\tau_2^2 + e_{10}\tau_2^3 + \text{h.o.t.}\,(\tau_1, \tau_2)) = 0. \end{aligned}$$

Suppose now that $(\bar\tau_1, \bar\tau_2)$ is fixed for the moment and that $\bar\alpha_2 = 0$ or $\bar\alpha_2$ is a real root of (3.4). Then (3.2)(i) and (3.2)(ii) have a common root with $(\alpha_2, \tau_1, \tau_2) = (\bar\alpha_2, \bar\tau_1, \bar\tau_2)$. Observe that if there are two such roots, the polynomials in (3.2) at $(\bar\alpha_2, \bar\tau_1, \bar\tau_2)$ are the same up to a multiple.

Hence $(\bar{\alpha}_2, \bar{\tau}_1, \bar{\tau}_2)$ also satisfies

$$\text{(3.5)}\quad \begin{aligned} &\text{(i)}\ \ d_1(c_2\tau_1 + c_3\tau_2 + c_4\alpha_2 + \text{h.o.t.}\ (\alpha_2,\tau_1,\tau_2)) \\ &\qquad - c_1(d_2\alpha_2 + \text{h.o.t.}\,(\alpha_2,\tau_1,\tau_2)) = 0, \\ &\text{(ii)}\ \ d_1(c_5\alpha_2^2 + \text{h.o.t.}\,(\alpha_2,\tau_1,\tau_2)) \\ &\qquad - c_1(d_3\tau_1\alpha_2 + d_4\tau_2\alpha_2 + d_5\alpha_2^2 + \text{h.o.t.}\,(\alpha_2,\tau_1,\tau_2)) = 0. \end{aligned}$$

If now, $d_1c_4 - c_1d_2 \neq 0$ and $d_1c_5 - c_1d_5 \neq 0$, (3.5) is equivalent to

$$\text{(3.6)}\quad \begin{aligned} &\text{(i)}\ \ (d_1c_4 - c_1d_2)\alpha_2 + (d_1c_2\tau_1 + d_1c_3\tau_2 + \text{h.o.t.}\,(\tau_1,\tau_2)) = 0, \\ &\text{(ii)}\ \ (d_1c_5 - c_1d_5)\alpha_2^2 + (-c_1d_3\tau_1 - c_1d_4\tau_2 + \text{h.o.t.}\,(\tau_1,\tau_2))\alpha_2 \\ &\qquad + (\text{terms of order } \geq 3 \text{ in } (\tau_1,\tau_2)) = 0. \end{aligned}$$

We now have the following lemma.

Lemma 3.1. *Suppose that $c_1, d_1, e_1, d_1c_4 - c_1d_2, d_1c_5 - c_1d_5$ are all nonzero. If, in addition, either $(d_1c_5 - c_1d_5)d_1^2c_2^2 + (d_1c_4 - c_1d_2)(c_1c_2d_1d_3) \neq 0$ or $(d_1c_5 - c_1d_5)d_1^2c_3^2 + (d_1c_4 - c_1d_2)c_1c_3d_1d_4 \neq 0$, then there exist at most two curves in $\mathbf{R}^2$ passing through (λ^*, μ^*) such that if $(\lambda^* + \tau_1^0, \mu^* + \tau_2^0)$ is not on one of these curves and (τ_1^0, τ_2^0) is sufficiently near $(0,0)$ then the number of small norm nonzero solutions to (1.4–1.5) for $\lambda = \lambda^* + \tau_1^0$, $\mu = \mu^* + \tau_2^0$, is the number of distinct real roots of (3.4) at $(\tau_1, \tau_2) = (\tau_1^0, \tau_2^0)$.*

Proof. Since $d_1c_4 - c_1d_2 \neq 0$, we may solve for α_2 in terms of τ_1 and τ_2 in (3.6)(i). Substituting the result into (3.6)(ii) yields an analytic equation in τ_1 and τ_2 with lowest order pure power terms $((d_1c_5 - c_1d_5)d_1^2c_2^2 + (d_1c_4 - c_1d_2)(c_1c_2d_1d_3))\tau_1^2$ and $((d_1c_5 - c_1d_5)d_1^2c_3^2 + (d_1c_4 - c_1d_2)(c_1c_3d_1d_4))\tau_2^2$. The lemma now follows from the Weierstrass preparation theorem and the quadratic formula. ◻

Before establishing our main result, we need an additional observation. Assume that c_1, d_1, and e_1 are all nonzero. Then $(\lambda^*, \mu^*, 0, 0)$ is an isolated solution of (1.4)–(1.5) (in $\{(\lambda^*, \mu^*)\} \times E^2$). Consequently, there is an $\varepsilon_0 > 0$ with the property that if $0 < \varepsilon < \varepsilon_0$, there is a corresponding $\delta(\varepsilon)$ so that

$$\text{(3.7)}\quad \begin{aligned} &\{(\lambda,\mu,u,v) : (\lambda,\mu,u,v) \text{ solves (1.4)–(1.5)}, \\ &\qquad |(\lambda,\mu) - (\lambda^*,\mu^*)| \leq \delta(\varepsilon), \text{ and } ||(u,v)|| = \varepsilon\} = \varnothing. \end{aligned}$$

That such is the case follows from the compactness of $G = L^{-1}$.

Now let $\sigma > 0$ be small enough that

$$\{(\lambda,\mu) : (\lambda,\mu) \in B((\lambda^*,\mu^*);\sigma)$$

and

$$\dim\left\{N\left(\mathbf{I} - \begin{pmatrix} \lambda f_1 G & \lambda f_2 G \\ \mu g_1 G & \mu g_2 G \end{pmatrix}\right)\right\} = 2\} = \{(\lambda^*,\mu^*)\}.$$

Let $\tilde{\lambda}_{n_+}, \tilde{\lambda}_{n_-}, \tilde{\lambda}_{m_+}, \tilde{\lambda}_{m_-}$ denote the four components of the intersection of the generalized spectrum with $B((\lambda^*,\mu^*);\sigma)\backslash\{(\lambda^*,\mu^*)\}$.

Let $0 < \varepsilon < \varepsilon_0$ and $0 < \delta < \delta(\varepsilon)$ and consider (1.4)–(1.5) in $(\partial B(\lambda^*,\mu^*);\delta)) \times E^2$. Let $\mathcal{C}_{n_+}$ denote the maximal component of the closure of the set of nontrivial solutions to (1.4)–(1.5) (in $(\partial B(\lambda^*,\mu^*);\delta)) \times E^2$) which emanates from the intersection $\{(\lambda_{n_+},\mu_{n_+})\}$ of $\partial B((\lambda^*,\mu^*);\delta)$ and $\tilde{\lambda}_{n_+}$; similarly, for $\mathcal{C}_{n_-}, \mathcal{C}_{m_+}$, and $\mathcal{C}_{m_-}$. Then

$$\dim\left(\cup_{k\ge 1} N\left\{\left(\mathbf{I} - \begin{pmatrix} \lambda_{n_+} f_1 G & \lambda_{n_+} f_2 G \\ \mu_{n_+} g_1 G & \mu_{n_+} g_2 G \end{pmatrix}\right)^k\right\}\right) = 1.$$

Consequently, $\mathcal{C}_{n_+}$ satisfies the global bifurcation alternatives of Rabinowitz [**7**] relative to $(\partial B((\lambda^*,\mu^*);\delta)) \times E^2$, as do $\mathcal{C}_{n_-}, \mathcal{C}_{m_+}$, and $\mathcal{C}_{m_-}$. From (3.7), $\mathcal{C}_{n_+} \subseteq (\partial B((\lambda^*,\mu^*);\delta) \times B((0,0);\varepsilon) \subseteq (\partial B((\lambda^*,\mu^*);\delta)) \times E^2$, and similarly for $\mathcal{C}_{n_-}, \mathcal{C}_{m_+}$, and $\mathcal{C}_{m_-}$. Consequently, so long as $\lambda_n/\lambda_m < (\sqrt{f_1 g_2} - \sqrt{f_2 g_1})/(\sqrt{f_1 g_2} + \sqrt{f_2 g_1})$, at least one of the four components of $\partial B((\lambda^*,\mu^*);\delta)\backslash\{\lambda_{n_+},\mu_{n_+}),\ (\lambda_{n_-},\mu_{n_-}),\ (\lambda_{m_+},\mu_{m_+}),\ (\lambda_{m_-},\mu_{m_-})\}$ must be contained in the projection into $\partial B((\lambda^*,\mu^*);\delta)$ of both a $\mathcal{C}_n$ and a $\mathcal{C}_m$. Let $\mathcal{G}$ denote said component. If, for some $(\lambda,\mu) \in \mathcal{G}$, there is only one nontrivial solution to (1.4)–(1.5) of norm less than ε, the $\mathcal{C}_n$ and $\mathcal{C}_m$ intersect and by connectivity are the same. We have now proved the following lemma.

Lemma 3.2. *Assume c_1, d_1 and e_1 are all nonzero and that $\lambda_n/\lambda_m < (\sqrt{f_1 g_2} - \sqrt{f_2 g_1})/(\sqrt{f_1 g_2} + \sqrt{f_2 g_1})$. Let $\tilde{\lambda}_{n_+}$, $\tilde{\lambda}_{n_-}$, $\tilde{\lambda}_{m_+}$, and $\tilde{\lambda}_{m_-}$ be as above. Then a switch in nodal structure for the solutions of* (1.4)–(1.5) *occurs in a neighborhood of (λ^*,μ^*) provided that, for a sufficiently* *small* $\delta > 0$*, each component of* $\partial B((\lambda^*, \mu^*); \delta) \backslash \{\tilde{\lambda}_{n_+} \cup \tilde{\lambda}_{n_-} \cup \tilde{\lambda}_{m_+} \cup \tilde{\lambda}_{m_-}\}$ *contains a* (λ, μ) *with only one associated small norm nonzero solution.*

It is a consequence of Lemma 3.1 and Lemma 3.2 that a switch in nodal properties for the solutions of (1.4)–(1.5) will occur in a neighborhood of (λ^*, μ^*) provided (3.4) has exactly one real root for (τ_1, τ_2) in an open region of each component of $B((0,0); \delta) \backslash \{\tilde{\tau}_{n_+} \cup \tilde{\tau}_{n_-} \cup \tilde{\tau}_{m_+} \cup \tilde{\tau}_{m_-}\}$. (Here $\tilde{\tau}_{n_+}$ denotes the translate of $\tilde{\lambda}_{n_+}$ to $(0,0)$ in the τ_1-τ_2 plane.) Since (3.4) is cubic, we naturally use the cubic discriminant at this point. Recall that if we have cubic equation

$$y^3 + py^2 + qy + r = 0, \tag{3.8}$$

with p, q, and r real numbers, and if we let

$$a = \frac{1}{3}(3q - p^2)$$
$$b = \frac{1}{27}(2p^3 - 9pq + 27r),$$

then (3.8) has one real and two complex conjugate roots when

$$\frac{b^2}{4} + \frac{a^3}{27} > 0.$$

If we normalize (3.4) by letting $m_i = (ei + 1)/e_1$, then p, q and r have the forms

$$p = m_1\tau_1 + m_2\tau_2 + \text{h.o.t.}\,(\tau_1, \tau_2),$$
$$q = m_3\tau_1^2 + m_4\tau_2^2 + m_5\tau_1\tau_2 + \text{h.o.t.}\,(\tau_1, \tau_2),$$
$$r = m_6\tau_1^3 + m_7\tau^2\tau_2 + m_8\tau_1\tau_2^2 + m_9\tau_2^3 + \text{h.o.t.}\,(\tau_1, \tau_2).$$

Consequently, a and b have the forms

$$a = n_1\tau_1^2 + n_2\tau_1\tau_2 + n_3\tau_2^2 + \text{h.o.t.}\,(\tau_1, \tau_2),$$
$$b = n_4\tau_1^3 + n_5\tau_1^2\tau_2 + n_6\tau_1\tau_2^2 + n_7\tau_2^3 + \text{h.o.t.}\,(\tau_1, \tau_2),$$

where the n's are expressed in terms of the m's and are given in the Appendix. Finally,

$$\begin{aligned} \frac{b^2}{4} + \frac{a^3}{27} = {} & \gamma_1\tau_1^6 + \gamma_2\tau_1^5\tau_2 + \gamma_3\tau_1^4\tau_2^2 + \gamma_4\tau_1^3\tau_2^3 + \gamma_5\tau_1^2\tau_2^4 \\ & + \gamma_6\tau_1\tau_2^5 + \gamma_7\tau_2^6 + \text{h.o.t.}\,(\tau_1, \tau_2), \end{aligned} \tag{3.9}$$

where the γ's are expressed in terms of the n's (see the Appendix). We may now state our main result.

Theorem 3.3. *Suppose the hypotheses of Lemma* 3.1 *and Lemma* 3.2 *are satisfied and, in addition, that $\gamma_1 \neq 0$. Let*

$$\omega_n = \lambda^{(n)'}(\mu^*) = \frac{-\lambda_n^2 f_2 g_1}{((f_1 g_2 - f_2 g_1)\mu^* - f_1 \lambda_n)^2}$$

and

$$\omega_m = \lambda^{(m)'}(\mu^*) = \frac{-\lambda_m^2 f_2 g_1}{((f_1 g_2 - f_2 g_1)\mu^* - f_1 \lambda_m)^2}.$$

Let $P(x) = \gamma_1 x^6 + \gamma_2 x^5 + \gamma_3 x^4 + \gamma_4 x^3 + \gamma_5 x^2 + \gamma_6 x + \gamma_7$, where γ_i, $i = 1, 2, \ldots, 7$, are as in (3.9). *Then if $P(\omega_n) > 0$ or $P(\omega_m) > 0$, there is a switch of nodal types for solutions to* (1.4)–(1.5) *occurring for parameter values in any sufficiently small neighborhood of* (λ^*, μ^*).

Proof. Since $\gamma_1 \neq 0$, the Weierstrass preparation theorem [**4**] guarantees that the right-hand side of (3.9) may be written $\omega(\tau_1, \tau_2) \cdot E(\tau_1, \tau_2)$ where ω_1 is a Weierstrass polynomial of degree six in τ_1 and E is analytic with $E(0, 0) = 1$. The sign of (3.9) is then the same as $\omega(\tau_1, \tau_2)$ for $|(\tau_1, \tau_2)|$ sufficiently small. Then if $\tau_2 \neq 0$, $\omega(\tau_1, \tau_2) > 0$ if and only if $\omega(\tau_1, \tau_2)/\tau_2^6 > 0$. Hence, $\omega(\tau_1, \tau_2) > 0$ if and only if

$$P\left(\frac{\tau_1}{\tau_2}\right) + \left(\frac{\tau_1}{\tau_2}\right)^5 h_1(\tau_2) + \left(\frac{\tau_1}{\tau_2}\right)^4 h_2(\tau_2) + \left(\frac{\tau_1}{\tau_2}\right)^3 h_3(\tau_2) + \left(\frac{\tau_1}{\tau_2}\right)^2 h_4(\tau_2) + \left(\frac{\tau_1}{\tau_2}\right) h_5(\tau_2) + h_6(\tau_2) > 0,$$

where h_i is an analytic function of τ_2 such that $h_i(0) = 0$ and h_i is real-valued for $\tau_2 \in \mathbf{R}$.

Suppose now that $P(\omega_n) = c > 0$. Then there is an $\omega > 0$ so that $P(\tau_1/\tau_2) > c/2$ if $\tau_2 \neq 0$ and $(\tau_1/\tau_2) \in [\omega_n - \omega, \omega_n + \omega]$. Since $h_i(0) = 0$ for $i = 1, 2, \ldots, 6$, there is a number $\eta > 0$ such that

$$\sum_{i=1}^{6} \left|\frac{\tau_1}{\tau_2}\right|^{6-i} |h_i(\tau_2)| < \frac{c}{4}$$

if $\tau_1/\tau_2 \in [\omega_n - \omega, \omega_n + \omega]$ and $0 < |\tau_2| < \eta$. Consequently, $\omega(\tau_1, \tau_2) > 0$ if $\tau_1/\tau_2 \in [\omega_n - \omega, \omega_n + \omega]$ and $0 < |\tau_2| < \eta$, and the result follows. An analogous argument holds if $P(\omega_m) > 0$.

4. An example. Consider the problem

$$\begin{aligned} -u'' &= \lambda(2u + v + u^2) \\ -v'' &= \mu(u + v + v^2) \end{aligned} \tag{4.1}$$

$$\begin{aligned} u(0) &= 0 = u(\pi) \\ v(0) &= 0 = v(\pi). \end{aligned} \tag{4.2}$$

Since $-w'' = \alpha w$, $w(0) = 0 = w(\pi)$ has eigenvalues $\alpha = m^2$, $m = 1, 2, \dots$, with corresponding eigenfunctions $\sin mt$, $m = 1, 2, \dots$, it follows that the generalized spectrum for (4.1)–(4.2) is

$$\left\{ (\lambda, \mu) \in \mathbf{R}^2 : \lambda = \frac{n^2(\mu - n^2)}{\mu - 2n^2} \text{ for some } n \in Z^+ \right\}.$$

In particular, if $\lambda^{(1)}(\mu) = (\mu - 1)/(\mu - 2)$ and $\lambda^{(3)}(\mu) = 9(\mu - 9)/(\mu - 18)$, then

$$\frac{\lambda_1}{\lambda_3} = \frac{1}{9} < \frac{\sqrt{2 \cdot 1} - \sqrt{1 \cdot 1}}{\sqrt{2 \cdot 1} + \sqrt{1 \cdot 1}} = \frac{\sqrt{\frac{\partial f(0,0)}{\partial u} \frac{\partial g(0,0)}{\partial v}} - \sqrt{\frac{\partial f(0,0)}{\partial v} \frac{\partial g(0,0)}{\partial u}}}{\sqrt{\frac{\partial f(0,0)}{\partial u} \frac{\partial g(0,0)}{\partial v}} + \sqrt{\frac{\partial f(0,0)}{\partial v} \frac{\partial g(0,0)}{\partial u}}}.$$

Hence the hyperbolae $\lambda^{(1)}$ and $\lambda^{(3)}$ intersect. In fact, a simple computation shows that, at the points of intersection, $\mu = 5 + \sqrt{7}$ or $5 - \sqrt{7}$. We therefore take $(\lambda^*, \mu^*) = ((5 - \sqrt{7})/2, 5 + \sqrt{7})$.

We aim to show that Theorem 3.3 is applicable and that switching of nodal types occurs around $(\lambda^*, \mu^*) = ((5 - \sqrt{7})/2, 5 + \sqrt{7})$. Toward this end, we must calculate the coefficients for (3.2)(i) and (3.2)(ii). In order to do so, we must obtain ϕ_1, ϕ_2 and the functionals γ_1, γ_2. Notice that ϕ_1 and ϕ_2 may be taken as $\begin{pmatrix} \sin t \\ \beta_1 \sin t \end{pmatrix}$ and $\begin{pmatrix} \sin 3t \\ \beta_3 \sin 3t \end{pmatrix}$, respectively, where β_1 and β_3 are as given in Lemma 2.1. It is immediate that

$$\beta_1 = \frac{\lambda_1 - \lambda^* f_1}{\lambda^* f_2} = \frac{1 - \left(\frac{5 - \sqrt{7}}{2}\right) \cdot 2}{\left(\frac{5 - \sqrt{7}}{2}\right) \cdot 1} = \frac{\sqrt{7} - 13}{9}$$

and

$$\beta_3 = \frac{\lambda_3 - \lambda^* f_1}{\lambda^* f_2} = \frac{9 - \left(\frac{5-\sqrt{7}}{2}\right)\cdot 2}{\frac{5-\sqrt{7}}{2}} = 3 + \sqrt{7}.$$

From [**2**, §3], it follows that γ_1 and γ_2 may be taken as

$$\gamma_1 \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{k_1}\left(\int_0^\pi \left(u\frac{\sin t}{\lambda^*} + v\frac{\beta_1 \sin t}{\mu^*}\right) dt\right)$$

and

$$\gamma_2 \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{k_2}\left(\int_0^\pi \left(u\frac{\sin 3t}{\lambda^*} + v\frac{\beta_3 \sin 3t}{\mu^*}\right) dt\right)$$

where $k_1 = \int_0^\pi((\sin^2 t)/\lambda^* + (\beta_1^2 \sin^2 t)/\mu^*)\,dt$ and $k_2 = \int_0^\pi((\sin^2 3t)/\lambda^* + (\beta_3^2 \sin^2 3t)/\mu^*)\,dt$. A simple computation will show that $k_1 = ((52 - 4\sqrt{7})/81)\pi$ and $k_2 = ((12 + 4\sqrt{7})/9)\pi$. Consequently, γ_1 and γ_2 are given.

Observe now that, for (4.1)–(4.2), $N_{11} = N_{23} = 1$ (see the Appendix), while the other terms in the quadratic expansion of f and g are zero. The computations involved in determining the coefficients in (3.2)(i) and (3.2)(ii) for (4.1)–(4.2) are simplified somewhat by this fact. One will find that these coefficients are as follows:

$$c_1 = \frac{61\sqrt{7} - 235}{54\pi} \qquad d_1 = -\left(\frac{595 - 133\sqrt{7}}{(2430)\pi}\right)$$

$$c_2 = \frac{3 + \sqrt{7}}{8} \qquad d_2 = -\left(\frac{37 + 29\sqrt{7}}{35\pi}\right)$$

$$c_3 = \frac{3 - \sqrt{7}}{16} \qquad d_3 = \left(\frac{13 - \sqrt{7}}{72}\right)$$

$$c_4 = -\left(\frac{77 + 21\sqrt{7}}{15\pi}\right) \qquad d_4 = \frac{13 + \sqrt{7}}{144}$$

$$c_5 = -\frac{(275 + 107\sqrt{7})(9)}{70\pi} \qquad d_5 = \frac{35 + 11\sqrt{7}}{18\pi}.$$

It is easy to calculate that $\omega_1 = (3\sqrt{7} - 8)/2$ and $\omega_3 = (-88 + 13\sqrt{7})/166$. Employing the symbolic manipulation program MU-MATH, we may determine the coefficients in P in this case and evaluate $P(\omega_1)$ and $P(\omega_3)$. It turns out that

$$\begin{aligned}P(\omega_1) = {} & 135168820322265625/4655753851140470370 \\ & 35487855558745922088450164514619392 PI\hat{\ }6 \\ & (-1925753216968090061959996266798988436 74 \\ & + 727866299797168182096330666014268918 01\ 7\hat{\ }(1/2))\end{aligned}$$

and

$$\begin{aligned}P(\omega_3) = {} & 12981613503750390625/1396726155342141111064635666762 \\ & 3776626535049354385 8176\ PI\hat{\ }6(-674808543652011127476 3118 \\ & + 2549462243054411756659907\ 7\hat{\ }(1/2)).\end{aligned}$$

Observe now that $P(\omega_1)$ has the form $(p/q)(-a+b\sqrt{7})\pi^6$, where p, q, a, and b are positive integers and $P(\omega_1)$ is positive provided $7b^2 - a^2$ is positive. We find that this quantity is

$$154791369444175328081368584374993739121792599166400373469 31.$$

Consequently, $P(\omega_1) > 0$ and a switch in nodal structure from solutions to (4.1)–(4.2) whose components have no zeros in $(0, \pi)$ to solutions of (4.1)–(4.2) whose components have two simple zeros in $(0, \pi)$ occurs.

APPENDIX

The coefficients alluded to in the body of this paper are as follows:

$$\tilde{N}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \tilde{N}_1 & (u, v) \\ \tilde{N}_2 & (u, v) \end{pmatrix} = \begin{pmatrix} N_{11}u^2 + N_{12}uv + N_{13}v^2 \\ N_{21}u^2 + N_{22}uv + N_{23}v^2 \end{pmatrix}.$$

$$c_1 = \left\langle \begin{pmatrix} (g_2(N_{11}+N_{12}\beta_n+N_{13}\beta_n^2) - f_2(N_{21}+N_{22}\beta_n+N_{23}\beta_n^2))x_n^2 \\ (-g_1(N_{11}+N_{12}\beta_n+N_{13}\beta_n^2) + f_1(N_{21}+N_{22}\beta_n+N_{23}\beta_n^2))x_n^2 \end{pmatrix}, \gamma_1 \right\rangle$$

$$c_2 = \left\langle \begin{pmatrix} g_2(f_1+f_2\beta_n/\lambda^*)x_n \\ -g_1(f_1+f_2\beta_n/\lambda^*)x_n \end{pmatrix}, \gamma_1 \right\rangle$$

$$c_3 = \left\langle \begin{pmatrix} -f_2(g_1+g_2\beta_n/\mu^*)x_n \\ f_1(g_1+g_2\beta_n/\mu^*)x_n \end{pmatrix}, \gamma_1 \right\rangle$$

$$c_4 = \left\langle \begin{pmatrix} (g_2(2N_{11}+N_{12}(\beta_m+\beta_n)+2N_{13}\beta_m\beta_n) - f_2(2N_{21}+N_{22}(\beta_m+\beta_n) +2N_{23}\beta_m\beta_n))x_nx_m \\ (-g_1(2N_{11}+N_{12}(\beta_m+\beta_n)+2N_{13}\beta_m\beta_n) + f_1(2N_{21}+N_{22}(\beta_m+\beta_n^2) +2N_{23}\beta_m\beta_n))x_nx_m \end{pmatrix}, \gamma_1 \right\rangle$$

$$c_5 = \left\langle \begin{pmatrix} (g_2(N_{11}+N_{12}\beta_m+N_{13}\beta_m^2) - f_2(N_{21}+N_{22}\beta_m+N_{23}\beta_m^2))x_m^2 \\ (-g_1(N_{11}+N_{12}\beta_m+N_{13}\beta_m^2) + f_1(N_{21}+N_{22}\beta_m+N_{23}\beta_m^2))x_m^2 \end{pmatrix}, \gamma_1 \right\rangle$$

$$d_1 = \left\langle \begin{pmatrix} (g_2(N_{11}+N_{12}\beta_n+N_{13}\beta_n^2) - f_2(N_{21}+N_{22}\beta_n+N_{23}\beta_n^2))x_n^2 \\ (-g_1(N_{11}+N_{12}\beta_n+N_{13}\beta_n^2) + f_1(N_{21}+N_{22}\beta_n+N_{23}\beta_n^2))x_n^2 \end{pmatrix}, \gamma_2 \right\rangle$$

$$d_2 = \left\langle \begin{pmatrix} (g_2(2N_{11}+N_{12}(\beta_m+\beta_n)+2N_{13}\beta_m\beta_n) - f_2(2N_{21}+N_{22}(\beta_n+\beta_n) +2N_{23}\beta_m\beta_n))x_nx_m \\ (-g_1(2N_{11}+N_{12}(\beta_m+\beta_n)+2N_{13}\beta_m\beta_n) + f_1(2N_{21}+N_{22}(\beta_m+\beta_n^2) +2N_{23}\beta_m\beta_n))x_nx_m \end{pmatrix}, \gamma_2 \right\rangle$$

$$d_3 = \left\langle \begin{pmatrix} g_2(f_1+f_2\beta_m/\lambda^*)x_m \\ -g_1(f_1+f_2\beta_m/\lambda^*)x_m \end{pmatrix}, \gamma_2 \right\rangle$$

$$d_4 = \left\langle \begin{pmatrix} -f_2(g_1+g_2\beta_m/\mu^*)x_m \\ f_1(g_1+g_2\beta_m/\mu^*)x_m \end{pmatrix}, \gamma_2 \right\rangle$$

$$d_5 = \left\langle \begin{pmatrix} (g_2(N_{11}+N_{12}\beta_m+N_{13}\beta_m^2) - f_2(N_{21}+N_{22}\beta_m+N_{23}\beta_m^2))x_m^2 \\ (-g_1(N_{11}+N_{12}\beta_m+N_{13}\beta_m^2) + f_1(N_{21}+N_{22}\beta_m+N_{23}\beta_m^2))x_m^2 \end{pmatrix}, \gamma_2 \right\rangle$$

$$e_1 = (2c_1d_5 - c_4d_2 + 2c_5d_1)^2 - (4c_1c_5 - c_4^2)(4d_1d_5 - d_2^2)$$

$$e_2 = 2(2c_1d_3 - c_2d_2)(2c_1d_5 - c_4d_2 + 2c_5d_1) + 2c_2c_4(4d_1d_5 - d_2^2) - 4d_1d_3(4c_1c_5 - c_4^2)$$

$$e_3 = 2(2c_1d_4 - c_3d_2)(2c_1d_5 - c_4d_2 + 2c_5d_1) - 4d_1d_4(4c_1c_5 - c_4^2) + 2c_3c_4(4d_1d_5 - d_2^2)$$

$$e_4 = (2c_1d_3 - c_2d_2)^2 + c_2^2(4d_1d_5 - d_2^2) + 8c_2c_4d_1d_3$$

$$e_5 = (2c_1d_4 - c_3d_2)^2 + c_3^2(4d_1d_5 - d_2^2) + 8c_3c_4d_1d_4$$

$$e_6 = 2(2c_1d_3 - c_2d_2)(2c_1d_4 - c_3d_2) + 2c_2c_3(4d_1d_5 - d_2^2) + 8c_2c_4d_1d_4 + 8c_3c_4d_1d_3$$

$$e_7 = 4c_2^2d_1d_3$$

$$e_8 = 4c_2^2d_1d_4 + 8c_2c_3d_1d_3$$

$$e_9 = 4c_3^2d_1d_3 + 8c_2c_3d_1d_4$$

$$e_{10} = 4c_3^2d_1d_4$$

$$m_i = \frac{e_{i+1}}{e_1}, \qquad i = 1, \dots, 9$$

$$
\begin{aligned}
n_1 &= \frac{3m_3 - m_1^2}{3}\\
n_2 &= \frac{3m_5 - 2m_1m_2}{3}\\
n_3 &= \frac{3m_4 - m_2^2}{3}\\
n_4 &= \frac{2m_1^3 - 9m_1m_3 + 27m_6}{27}\\
n_5 &= \frac{6m_1^2m_2 - 9(m_2m_3 + m_1m_5) + 27m_7}{27}\\
n_6 &= \frac{6m_1m_2^2 - 9(m_1m_4 + m_2m_5) + 27m_8}{27}\\
n_7 &= \frac{2m_2^3 - 9m_2m_4 + 27m_9}{27}\\
\gamma_1 &= \left(\frac{n_1^3}{27} + \frac{n_4^2}{4}\right)\\
\gamma_2 &= \left(\frac{n_1^2n_2}{9} + \frac{n_4n_5}{2}\right)\\
\gamma_3 &= \left(\frac{n_1^2n_3 + n_1n_2^2}{9} + \frac{2n_4n_6 + n_5^2}{4}\right)\\
\gamma_4 &= \left(\frac{6n_1n_2n_3 + n_2^3}{27} + \frac{n_4n_7 + n_5n_6}{2}\right)\\
\gamma_5 &= \left(\frac{n_1n_3^2 + n_2^2n_3}{9} + \frac{2n_5n_7 + n_6^2}{4}\right)\\
\gamma_6 &= \left(\frac{n_2n_3^2}{9} + \frac{n_6n_7}{2}\right)\\
\gamma_7 &= \left(\frac{n_3^2}{27} + \frac{n_7^2}{4}\right).
\end{aligned}
$$

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