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STEADY-STATE TURBULENT FLOW WITH REACTION

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ABSTRACT. Existence and uniqueness of nonnegative solutions of the two-point boundary value problem $\psi(\phi(u)')' = f(x,u,u'), u(-1) = a, u(1) = b$ are established for appropriate functions ϕ, ψ, and f. Included in this formulation are the one-dimensional steady-state equations for turbulent or diffusive flow in a porous catalytic pellet, irreversible reaction with change of volume, etc. Also examined is the possibility that the concentration u might vanish on some nontrivial subset of $[-1,1]$, the dead core.

Introduction. A mathematical description for one-dimensional turbulent flow of a polytropic gas in a porous medium has been given by Leibenson [**8**]; cf. Esteban and Vazquez [**5**]. If the gas is being consumed in the medium through undergoing an irreversible reaction, then the steady-state concentration u is described by the nonlinear differential equation

$$\frac{d}{dx}\Big(\frac{du^q}{dx}\Big|\frac{du^q}{dx}\Big|^{p-1}\Big) = \lambda f(u).$$

Here the constants p and q satisfy $1/2 \le p \le 1$ and $q \ge 2$ in the physical problem, the Thiele modulus λ is a positive constant (essentially reaction rate divided by diffusion rate), and $f > 0$ specifies the nature of the reaction. If we assume that the porous catalyst occupies the region $-1 \le x \le 1$, then a reasonable problem arises on specifying Dirichlet boundary conditions

$$u(-1) = a \ge 0, \qquad u(1) = b \ge 0.$$

For physical reasons we are interested only in nonnegative u.

This problem can be recast in the more general form

$$\frac{d}{dx}\psi\Big(\frac{d}{dx}\phi(u)\Big) = \lambda f(u), \qquad u(-1) = 1, u(1) = b,$$

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for suitable functions ψ and ϕ generalizing $\psi(s) = s|s|^{p-1}$ and $\phi(u) = u^q$. In this formulation other problems can be included as well: for example, the choice

$$\psi(s) = s, \qquad \phi(u) = u^q$$

yields the usual problem for diffusion in a porous medium;

$$\psi(s) = s, \qquad \phi(u) = \frac{1}{\theta}\ln(1+\theta u),$$

where θ, which depends on the order of the reaction and on the diffusion coefficient, is the parameter defined in [**2**, p. 132], yields the equation for change of volume with irreversible reaction [**9**, **2**]; and

$$\psi(s) = s|s|^{p-1}, \qquad \phi(u) = u$$

appears in the study of non-Newtonian fluids [**7**]. For the steady-state problem examined here we may assume without loss of generality that $\phi(u) \equiv u$, since otherwise the change of variable $z = \phi(u)$ reduces the problem to this case provided ϕ is invertible.

In the following section we consider existence and uniqueness of solutions to a suitable generalized problem of this sort. We approach the question of existence by establishing suitable *a priori* bounds that any solution must satisfy and then using the topological transversality theorem of Granas (see Granas, Guenther, and Lee [**6**]). Ours may be the first application of this convenient formalization to a problem that is nonlinear in the highest derivatives of the solution.

In the final section we examine the possible occurrence of a *dead core*, a set with nonempty interior on which the solution vanishes. Our results here generalize some of those of Bandle, Sperb, and Stakgold [**3**] and are closely related to some of the results of Diaz [**4**].

Existence and uniqueness. We shall write $\psi(u')'$ for $\frac{d}{dx}\big(\psi(u'(x))\big)$. By a solution of the problem

$$\psi(u')' = f(x,u,u'), \qquad u(-1) = a, u(1) = b, \tag{1}$$

we shall mean a function $u \in C^2[-1,1]$ such that $\psi(u')'$ exists and (1) is satisfied; this notion will be relaxed somewhat in the second theorem below, which deals with the equation $\psi(u')' = f(x,u)$. Without loss of generality we assume throughout that $a \geq 0, b \geq a$ and if $a = 0$ then $b > 0$. Guided by the physically important example $\psi(s) = |s|^{p-1}s$ for $1/2 \leq p \leq 1$, we make the following hypotheses on ψ and f:

(H_1) ψ is a continuous and increasing map from $(-\infty, \infty)$ onto $(-\infty, \infty)$.

(H_2) ψ is continuously differentiable on $(-\infty, 0) \cup (0, \infty)$ and, for each positive number M,

$$\inf_{[-M,0)\cup(0,M]} \psi'(s) > 0;$$

moreover, ψ^{-1} is continuously differentiable on $(-\infty, \infty)$.

(H_3) f is continuous on $[-1,1] \times [0,\infty) \times (-\infty, \infty)$, and $f(x,u,\zeta) > 0$ for $u > 0, f(x,0,\zeta) = 0$ for $x \in [-1,1]$ and $\zeta \in (-\infty, \infty)$.

(H_4) There exists a continuous positive function θ such that

$$f(x,u,\zeta) \leq \theta(\zeta)$$

on $[-1,1] \times [0,b] \times (-\infty, \infty)$, and θ satisfies

$$\int_{\psi(\frac{b-a}{2})}^{\pm\infty} \frac{d\zeta}{\theta(\psi^{-1}(\zeta))} > 2.$$

H_4 requires that f satisfy a suitable generalized Nagumo condition; some such condition is known to be necessary even for $\psi(s) \equiv s$. H_3 is natural for right-hand sides representing a chemical reaction. We shall assume that f is extended to all u as an odd function; then f remains continuous. We shall denote the extension by f also.

Note that no solution of (1) can ever be negative. For, whenever a solution of $\psi(u')' = f(t,u,u')$ is negative, $\psi(u')$ is a decreasing function of x, and thus u' is decreasing. But this contradicts the boundary conditions.

In the next lemma we gather the *a priori* bounds needed for the topological transversality theorem. Instead of (1) we consider the family of problems

$$(1)_\rho \qquad \psi(u')' = \rho f(t,u,u'), \qquad u(-1) = a, u(1) = b,$$

$\rho \in [0,1]$. The dependence of u on the parameter ρ will generally be suppressed in the notation.

Lemma 1. *Let* $\mathrm{H}_1 - \mathrm{H}_4$ *hold. There exist constants (independent of* $\rho \in [0,1]$*)* M_1 *and* M_2 *such that, if* u *is any solution of* $(1)_\rho$*,*

$$|u(x)| \le b, \qquad |u'(x)| \le M_1, \qquad |u''(x)| \le M_2.$$

Proof. Since $u \ge 0$, we have $f(x,u,u') \ge 0$, and thus $\psi(u')$ is a nondecreasing function of x. Hence u' is nondecreasing. It follows that $|u(x)| \le b$ throughout $[-1,1]$. Using H_4 on the differential equation, we see that

$$\frac{\psi(u')'}{\theta(u')} = \frac{\psi(u')'}{\theta(\psi^{-1}(\psi(u')))} \le \rho. \tag{2}$$

There exists $x_0 \in [-1,1]$ such that $u'(x_0) = (b-a)/2$ by the mean value theorem. Integrating (2) from x_0 to $x \ge x_0$ yields

$$\int_{\psi(\frac{b-a}{2})}^{\psi(u'(x))} \frac{d\zeta}{\theta(\psi^{-1}(\zeta))} \le \rho(x - x_0) \le 2.$$

It follows from H_4 that $\psi(u'(x))$ is bounded uniformly in $\rho \in [0,1]$ for $x_0 \le x \le 1$. A similar argument shows that $\psi(u'(x))$ is also bounded for $-1 \le x \le x_0$. But, by H_1, there is then a constant M_1 such that $|u'(x)| \le M_1$, uniformly for $\rho \in [0,1]$.

Let

$$M = \max_{[-1,1]\times[0,b]\times[-M_1,M_1]} f(x,u,v).$$

Then we have that $0 \le \psi(u'(x))' \le M$. At any x such that $u'(x) \ne 0$, we get that $0 \le \psi'(u'(x))u''(x) \le M$. By H_2, let $\delta = \inf \psi' > 0$, where the infimum is taken over $[-M_1, 0) \cup (0, M_1]$. Then

$$\sup_{\{x : u'(x) \ne 0\}} |u''(x)| \le M/\delta.$$

Since $u'' = 0$ at any nonisolated zero of u', it follows that $|u''(x)| \le M_2 \equiv M/\delta$ for all x. □

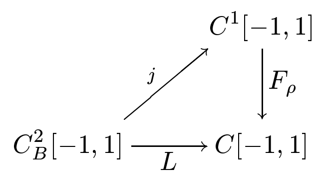
Theorem 1. *Let* $\mathrm{H}_1 - \mathrm{H}_4$ *hold. Then* (1) *has a solution in* $C^2[-1,1]$.

Proof. We introduce the following notation.

$$
\begin{aligned}
||u||_0 &= \max_{[-1,1]} |u(t)|, \\
||u||_1 &= ||u||_0 + ||u'||_0, \\
||u||_2 &= ||u||_1 + ||u''||_0, \\
C_B^2[-1,1] &= \{u \in C^1[-1,1] : u(-1) = a, u(1) = b\}.
\end{aligned}
$$

Then $(C[-1,1], ||\cdot||_0), (C^1[-1,1], ||\cdot||_1)$ and $(C^2[-1,1], ||\cdot||_2)$ are Banach spaces, and $C_B^2[-1,1]$ is a convex subset of $C^2[-1,1]$.

Consider the triangle of maps



where j is the embedding $ju = u, (F_\rho u)(t) = \rho f(t, u(t), u'(t))$, and $Lu = \psi(u')'$ on its domain.

Lemma 2. (i) j *is completely continuous*;

(ii) F_ρ *is continuous*;

(iii) L^{-1} *is well-defined and continuous.*

Proof. (1). Standard.

(2) is obvious from the continuity of f.

(3). To show that L^{-1} exists, let $g \in C[-1,1]$. If a solution u of

$$\psi(u')' = g, \qquad u(-1) = a, u(1) = b, \tag{3}$$

exists, there is $\xi \in [-1,1]$ such that $u'(\xi) = (b-a)/2$. Therefore we can write

$$u'(x) = \psi^{-1}\Big[\psi\Big(\frac{b-a}{2}\Big) + \int_\xi^x g(s)\, ds\Big], \tag{4}$$

from which it follows that

$$u(x) = a + \int_{-1}^{x} \psi^{-1}\Big[\psi\Big(\frac{b-a}{2}\Big) + \int_{\xi}^{t} g(s)\,ds\Big]\,dt. \tag{5}$$

If this is to satisfy the boundary condition at 1, we must choose ξ to satisfy

$$H_g(\xi) \equiv \int_{-1}^{1} \psi^{-1}\Big[\psi\Big(\frac{b-a}{2}\Big) + \int_{\xi}^{t} g(s)\,ds\Big]\,dt = b-a. \tag{6}$$

Suppose this has been done. Then u defined by (5) is indeed a solution in $C_B^2[-1,1]$ of (3), for we have that u' is given by (4), so $u' \in C^1[-1,1]$ by the chain rule and the assumed differentiability of ψ^{-1}.

It therefore remains to show that ξ can be chosen so that (6) is satisfied. Set

$$G(x) = \int_0^x g(s)\,ds$$

so that

$$H_g(\xi) = \int_{-1}^{1} \psi^{-1}\Big[\psi\Big(\frac{b-a}{2}\Big) + G(t) - G(\xi)\Big]\,dt.$$

Let G assume its minimum on $[-1,1]$ at ξ_1, its maximum at ξ_2. Since ψ and ψ^{-1} are increasing functions, we have

$$H_g(\xi_1) \geq \int_{-1}^{1} \psi^{-1}\Big[\psi\Big(\frac{b-a}{2}\Big)\Big]\,dt = b-a;$$

similarly, $H_g(\xi_2) \leq b-a$. By the intermediate value theorem, there exists $\xi \in [-1,1]$ such that $H_g(\xi) = b-a$. Thus a solution of (3) exists.

In order to show that L^{-1} is well-defined, we must show that the solution of (3) is unique, even though the parameter ξ found above may not be unique. Indeed, if ξ is a solution of (6), then a necessary and sufficient condition that $\overline{\xi}$ also be a solution is that

$$\int_{\xi}^{\overline{\xi}} g(s)\,ds = 0,$$

as is readily seen. To see that the solution of (3) is nevertheless unique, let u and v be two solutions such that $u - v$ has an extremum at $\eta \in (-1,1)$, so $u'(\eta) = v'(\eta)$. From $[\psi(u') - \psi(v')]' = 0$ it follows that

$$\psi(u'(x)) - \psi(v'(x)) = \psi(u'(\eta)) - \psi(v'(\eta)) = 0$$

and thus that $u'(x) = v'(x)$ on $[-1,1]$. That $u \equiv v$ is a consequence.

It remains to show that L^{-1} is continuous. We show first continuity in the norm $||\cdot||_0$. Denote the solution of (3) by $u_g(x)$ to indicate its dependence on g, and similarly denote ξ by ξ_g. Let $g, h \in C[-1,1]$; from the definition of ξ_g and ξ_h we have

$$0 = H_g(\xi_g) - H_h(\xi_h) = \int_{-1}^{1} \Bigg\{ \psi^{-1}\Big(\psi\Big(\frac{b-a}{2}\Big) + \int_{\xi_g}^{t} g(s)\,ds\Big) - \psi^{-1}\Big(\psi\Big(\frac{b-a}{2}\Big) + \int_{\xi_h}^{t} h(s)\,ds\Big)\Bigg\}\,dt.$$

By the mean value theorem there must be a $T \in [-1,1]$ such that

$$\psi^{-1}\Big(\psi\Big(\frac{b-a}{2}\Big) + \int_{\xi_g}^{T} g(s)\,ds\Big) = \psi^{-1}\Big(\psi\Big(\frac{b-a}{2}\Big) + \int_{\xi_h}^{T} h(s)\,ds\Big)$$

and therefore such that

$$\int_{\xi_g}^{T} g(s)\,ds = \int_{\xi_g}^{T} h(s)\,ds + \int_{\xi_h}^{\xi_g} h(s)\,ds.$$

If $||h-g||_0 \to 0$, it follows that

$$\int_{\xi_h}^{\xi_g} h(s)\,ds \to 0.$$

Writing $u_g(x) - u_h(x)$ in the form

$$\int_{-1}^{x} \Bigg\{ \psi^{-1}\Big(\psi\Big(\frac{b-a}{2}\Big) + \int_{\xi_g}^{t} g(s)\,ds\Big) - \psi^{-1}\Big(\psi\Big(\frac{b-a}{2}\Big) + \int_{\xi_g}^{t} g(s)\,ds + \int_{\xi_g}^{t} [h(s) - g(s)]\,ds + \int_{\xi_h}^{\xi_g} h(s)\,ds\Big)\Bigg\}\,dt,$$

we have at once that $||u_g - u_h||_0 \to 0$ as $h \to g$ uniformly.

Next we show that $||u'_g - u'_h||_0 \to 0$ as $h \to g$ uniformly. But since

$$\begin{aligned} u'_g(x) - u'_h(x) =& \psi^{-1}\Big(\psi\Big(\frac{b-a}{2}\Big) + \int_{\xi_g}^{x} g(s)\,ds\Big) \\ & - \psi^{-1}\Big(\psi\Big(\frac{b-a}{2}\Big) + \int_{\xi_h}^{x} h(s)\,ds\Big), \end{aligned}$$

this follows by the same argument.

Finally, we must show that $||u''_g - u''_h||_0 \to 0$ as $h \to g$. Since

$$\begin{aligned} u''_g(x) - u''_h(x) =& (\psi^{-1})'\Big(\psi\Big(\frac{b-a}{2}\Big) + \int_{\xi_g}^{x} g(s)\,ds\Big) g(x) \\ & - (\psi^{-1})'\Big(\psi\Big(\frac{b-a}{2}\Big) + \int_{\xi_h}^{x} h(s)\,ds\Big) h(x), \end{aligned}$$

this also follows. Thus $||u_g - u_h||_2 \to 0$ as $||g-h||_0 \to 0$, and the lemma is established. □

We return to the proof of the main theorem. Let M_1, M_2 be the constants of Lemma 1, set $M = b + M_1 + M_2 + 1$, and define

$$V = \{u \in C_B^2[-1,1] : ||u||_2 < M\};$$

V is an open subset of $C_B^2[-1,1]$. Define

$$H_\rho : \overline{V} \to C_B^2[-1,1]$$

by

$$H_\rho u = L^{-1} F_\rho j u;$$

then H_ρ is a compact homotopy whose fixed points are the solutions of $(1)_\rho$. For suppose $u = L^{-1}F_\rho ju$; then $Lu = \psi(u')' = F_\rho ju = \rho f(t,u,u')$. By Lemma 1 and the choice of V, this homotopy has no fixed points on the boundary of V. Moreover,

$$H_0 u = a + \frac{1}{2}(b-a)(x+1)$$

is a constant map to an interior point of $\overline{V}$. Thus H_0 is essential [**6**]. By the topological transversality theorem [**6**], H_1 is also essential, and accordingly has a fixed point. This fixed point is a solution of (1). □

In the physical problem of turbulent flow with reaction, the right-hand side of the differential equation does not depend on the gradient u'. If we require that f be independent of u', then we can easily establish existence under conditions weaker than $\mathrm{H}_1 - \mathrm{H}_4$; in particular, these conditions will allow $\psi(s) = |s|^{p-1}s$ for all $p > 0$. The essence of the following argument seems to be of general applicability wherever, as here, it is not possible to bound u'' *a priori*.

By a solution of

$$\psi(u')' = f(x,u), \qquad u(-1) = a, u(1) = b, \tag{7}$$

we now mean a function $u \in C^1[-1,1]$ such that $\psi(u')' \in C[-1,1]$ and (7) holds. Our weakened hypotheses are

(H_1') ψ is continuous and strictly increasing on $(-\infty,\infty)$, and the range of ψ is $(-\infty,\infty)$;

(H_3') f is continuous on $[-1,1] \times [0,\infty)$, $f(x,u) > 0$ for $u > 0$ and $x \in [-1,1]$, and $f(x,0) = 0$.

Theorem 2. *Let* H_1' *and* H_3' *hold. Then a solution to the problem* (7) *exists.*

Proof. Observe that H_4 holds with θ a suitable constant. Consider instead of (7) the family of problems

$$\psi(u')' = \rho f(x,u), \qquad u(-1) = a, u(1) = b, \tag*{$(7)_\rho$}$$

for $0 \le \rho \le 1$. The argument of Lemma 1 goes through to the point of showing that there exist constants M_1 and M_2 independent of ρ such that

$$0 \le u(x) \le b, \qquad |u'(x)| \le M_1, \qquad |\psi(u'(x))'| \le M_2$$

for any solution $u = u_\rho$ of $(7)_\rho$. However, since $\psi'(u'(x))$ could exist and vanish at a point of $[-1,1]$, no bound on u'' follows even if ψ is differentiable.

To get around this, we set $C^1_B[-1,1] = \{u \in C^1[-1,1] : u(-1) = a, u(1) = b\}$, a convex subset of the Banach space $C^1[-1,1]$, and consider the following triangle of maps

$$\begin{array}{ccc} & & C[-1,1] \\ & \overset{j}{\nearrow} & \Big\downarrow F_\rho \\ C^1_B[-1,1] & \xrightarrow[L]{} & C[-1,1] \end{array}$$

where j, F_ρ, and L are as before, except that L is defined only on $\operatorname{dom}(L) = \{u \in C^1_B[-1,1] : \psi(u') \in C^1[-1,1]\}$. Exactly as in the proof of Lemma 2, j is completely continuous and F_ρ is continuous. Also, as in the proof of Lemma 2, we have that L^{-1} exists, is well-defined, and is continuous as a map from $C[-1,1]$ into $C^1_B[-1,1]$. But we cannot conclude, as was done in Lemma 2, that u'' exists, because ψ^{-1} is not assumed differentiable.

Now set $M = b + M_1 + 1$ and define

$$V = \{u \in C^1_B[-1,1] : ||u||_1 < M\},$$

$$H_\rho u = L^{-1} F_\rho j u,$$

where

$$H_\rho : \overline{V} \to C^1_B[-1,1].$$

Then H_ρ is again a compact homotopy whose fixed points are the solutions of $(7)_\rho$, and H_ρ has no fixed points on the boundary of V. Again the constant map $H_0 u = a + (b-a)(x+1)/2$ is essential, so the topological transversality theorem asserts that H_1 has a fixed point that is a solution of (7). Since the fixed point u of H_1 lies in the range of $L^{-1}, \psi(u')'$ exists. □

For use in the next section, we shall briefly formulate the extension of this theorem to the differential equation $\psi(\phi(u)')' = f(t,u)$. By a solution we shall mean a function $u \in C[-1,1]$ such that $\phi(u) \in C^1[-1,1]$ and $\psi(\phi(u)') \in C^1[-1,1]$. In order to reduce this problem to the previously considered case by using the substitution $z = \phi(u)$ it is necessary to assume only that ϕ is a continuous increasing function that vanishes at zero and is positive for positive values of its argument. The hypotheses on ψ and f remain unaltered.

We turn now to uniqueness; the following simple result is adequate for our purposes.

Theorem 3. *Let, in addition to our other hypotheses, $f(x,u,p)$ be continuously differentiable with respect to u and p for $u>0$, and let $f_u(x,u,p)>0$ for $u>0$. Then there is only one positive solution of*

$$\psi(u')' = f(x,u,u'), \qquad u(-1)=a\geq 0, u(1)=b\geq a.$$

Proof. Suppose u and v are distinct solutions and that $u-v$ has a positive maximum at $\xi\in(-1,1)$; then $u'(\xi)-v'(\xi)=0$. Since

$$f(x,u,u')-f(x,v,v') = f_u(x,p_1,p_2)(u-v)+f_{u'}(x,p_3,p_4)(u'-v')$$

where $p_1,\ldots,p_4$ lie in a bounded set and $p_1(\xi)>0$, we have that

$$\psi(u'(\xi))'-\psi(v'(\xi))' = f_u(\xi,p_1,q_1)(u(\xi)-v(\xi))>0.$$

Therefore $\psi(u')-\psi(v')$ is increasing at ξ, and so, for $x>\xi$, x sufficiently close to ξ, we have $\psi(u'(x))-\psi(v'(x))>\psi(u'(\xi))-\psi(v'(\xi))=0$. Since ψ is increasing, $(u(x)-v(x))'>0$ for $x-\xi>0$ and small. But this contradicts the choice of ξ. □

Dead cores. In this section we are concerned with the general steady-state equation for reaction and turbulent flow in a one-dimensional, homogeneous porous medium,

$$\psi(\phi(u)')' = \lambda f(u), \qquad u(-1)=a\geq 0, u(1)=b\geq a, \tag{8}$$

where $\lambda>0$ is proportional to reaction rate divided by the diffusion coefficient. Since the change of variable $z=\phi(u)$ reduces this problem to the similar one

$$\psi(z')' = \lambda f(\phi^{-1}(z)), \qquad z(-1)=\phi(a), z(1)=\phi(b), \tag{9}$$

it suffices to study (8) with $\phi(u)\equiv u$ as long as ϕ is a continuous increasing function vanishing at 0. Existence and uniqueness have already been dealt with; here we are concerned with the possibility that, for large λ, u might vanish on some subinterval of $[-1,1]$, called the *dead core*.

Our hypotheses here are H_1', $\psi(0)=0$, and

(H_3'') f is continuous and nondecreasing on $[0,b]$, $f(0)=0$, and $f(u)>0$ for $u\in(0,b]$.

Note that for (9) we require $f(\phi^{-1}(\cdot))$ nondecreasing.

We consider first the case $a>0$.

In this case we can show that u' vanishes somewhere on $(-1,1)$ if λ is sufficiently large. This clearly holds for all $\lambda>0$ if $a=b$. If $b>a$ and $u'>0$ on $(-1,1)$, then $\psi(u'(x))'\geq\lambda f(a)$, so $\psi(u'(x))\geq\lambda f(a)(x+1)$. A further integration yields

(10)
$$b-a\geq\int_{-1}^{1}\psi^{-1}(\lambda f(a)(x+1))dx>\int_{0}^{1}\psi^{-1}(\lambda f(a))dx=\psi^{-1}(\lambda f(a)).$$

But this is impossible if λ is sufficiently large. We suppose throughout the following that either $a=b$ or λ is large enough that the first inequality of (10) is violated.

Suppose then that $u'(\overline{x})=0$ but $u'(x)>0$ on $(\overline{x},1]$. Integration of the differential equation produces, for $x>\overline{x}$,

$$\int_{\overline{x}}^{x}u'(s)\psi(u'(s))'\,ds=\lambda\int_{\overline{x}}^{x}f(u(s))u'(s)\,ds. \tag{11}$$

The integral on the left side can be written as a Riemann-Stieltjes integral and integrated by parts to yield

$$\int_{\overline{x}}^{x}u'(s)d\psi(u'(s))=u'(x)\psi(u'(x))-\int_{\overline{x}}^{x}\psi(u'(s))du'(s).$$

The substitutions $\xi=u'(s)$ in the final integral above and $\zeta=u(s)$ in the second integral of (11) yield

$$u'(x)\psi(u'(x))-\int_{u'(\overline{x})}^{u'(x)}\psi(\xi)\,d\xi=\lambda\int_{m}^{u(x)}f(\zeta)\,d\zeta, \tag{12}$$

where we put $m=\min_{[-1,1]}u(x)$.

Define a function G by

$$G(\mu) = \mu\psi(\mu) - \int_0^\mu \psi(\xi)\, d\xi.$$

If $\nu > \mu \geq 0$ then

$$\begin{aligned} G(\nu) - G(\mu) &= \mu[\psi(\nu) - \psi(\mu)] + (\nu - \mu)\psi(\nu) - \int_\mu^\nu \psi(\xi)\, d\xi \\ &= \mu[\psi(\nu) - \psi(\mu)] + (\nu - \mu)[\psi(\nu) - \psi(\theta)] \geq 0, \end{aligned}$$

where $\mu < \theta < \nu$. Here we have used the mean value theorem and the fact that ψ is increasing. Thus G is increasing and, hence, invertible. Therefore (12) can be written as

$$u'(x) = G^{-1}\Big[\lambda \int_m^{u(x)} f(\zeta)\, d\zeta\Big].$$

From this we get that

$$\int_{\overline{x}}^1 \frac{u'(x)\, dx}{G^{-1}[\lambda \int_m^{u(x)} f(\zeta)\, d\zeta]} = \int_m^b \frac{dz}{G^{-1}[\lambda \int_m^z f(\zeta) d\zeta]} = 1 - \overline{x}. \tag{13}$$

It is convenient to introduce the notation

$$I(m, b; \lambda) = \int_m^b \frac{dz}{G^{-1}[\lambda \int_m^z f(\zeta)\, d\zeta]}.$$

If $u'(\underline{x}) = 0$ but $u'(x) < 0$ for $x \in [-1, \underline{x})$, we get in an analogous way that

$$I(m, a; \lambda) = 1 + \underline{x}. \tag{14}$$

Change of variables shows that, for $\Delta m > 0$,

$$I(m, a; \lambda) = \int_{m+\Delta m}^{a+\Delta m} \frac{d\mu}{G^{-1}[\lambda \int_{m+\Delta m}^{\mu} f(\tau - \Delta m) d\tau]} \geq I(m + \Delta m, a; \lambda)$$

since f is increasing. Also, I is a decreasing function of λ as well.

If $m > 0$, then $\underline{x} = \overline{x}$ and addition of (13) and (14) yields

$$I(m, b; \lambda) + I(m, a; \lambda) = 2; \tag{15}$$

if $m = 0$, then $I(m, b; \lambda) + I(m, a; \lambda) < 2$. (Thus $I(m, b; \lambda) + I(m, a; \lambda) \leq 2$ always holds, since a solution exists.) If $I(0+, b; \lambda) + I(0+, a; \lambda) > 2$, then (15) admits a positive solution for m; that is, $\min_{[-1,1]} u > 0$, and there is no dead core. This includes, in particular, the case $I(0+, 1; \lambda) = \infty$. But if $I(0+, a; \lambda) + I(0+, b; \lambda) < 2$, there is no solution to (15), and hence it must be that $m = 0$ and a dead core exists. Note that if a dead core exists for λ_0, a dead core exists for $\lambda > \lambda_0$.

Let $u \equiv 0$ on $[\underline{x}, \overline{x}] \subset [-1, 1]$ and $u > 0$ on $[-1, \alpha) \cup (\beta, 1]$, so that $[\underline{x}, \overline{x}]$ is the dead core. Then (13) and (14) in the forms $I(0+, b; \lambda) = 1 - \overline{x}, I(0+, a; \lambda) = 1 + \overline{x}$ determine the location and extent of the dead core. It follows readily that $\underline{x}$ is a decreasing, and $\overline{x}$ an increasing, function of λ.

Suppose now that $a = 0$, i.e., we consider the problem

$$\overline{\psi}(u')' = \lambda f(u), \qquad u(-1) = 0, u(1) = b > 0, \tag{16}$$

where $\overline{\psi}$ enjoys the properties previously required of ψ. Let us change the dependent variable from z to $x = (z + 1)/2$ and set $u(z) = v(x)$; then (16) becomes

$$\psi(v')' = \lambda f(v), \qquad v(0) = 0, v(1) = b, \tag{17}$$

where $\psi(s) = \overline{\psi}(s/2)/2$. Rather than study (17) directly, we consider the problem

$$\psi(v')' = \lambda f(v), \qquad v(-1) = v(1) = b.$$

Our previous analysis applies, but now we know that if m denotes the minimum of v on $[-1, 1]$, then $m = v(0)$. If $m > 0$, then we must take $\overline{x} = 0$ in (13), and we conclude that $I(m, b; \lambda) = 1$. As before, if $I(0+, b; \lambda) > 1$ this equation has a unique positive solution for m, but there is no solution if $I(0+, b; \lambda) < 1$. Thus if $I(0+, b; \lambda) < 1$ it must be that $m = v(0) = 0$. Let then v vanish on $[0, \overline{x}]$ but $v > 0$ on $(\overline{x}, 1]$; then (13) yields $I(0+, b; \lambda) = 1 - \overline{x}$, an equation for $\overline{x} > 0$, and $u(z) = v(2x - 1)$ has a dead core on $[-1, 2\overline{x} - 1]$. The size of the dead core increases with λ.

As an example of the preceding results, we consider the specific equation for reaction and turbulent flow in a porous medium:

$$\frac{d}{dx}\Big(\Big|\frac{du^q}{dx}\Big|^{p-1}\frac{du^q}{dx}\Big) = u^r, \qquad u(-1) = u(1) = 1,$$

$p, q, r > 0$. For convenience we set $s = r/q$. Direct calculation shows that $I(0+, 1; \lambda) = \infty$, and hence no dead core exists if $s \geq p$; if $p > s$, then a dead core exists provided

$$\frac{(p(p+1)^p(s+1))^{\frac{1}{p+1}}}{p-s} < \lambda^{\frac{1}{p+1}}.$$

This condition reduces to that of [**3**] for $p = q = 1$. The condition $pq > r$ for the existence of a dead core for sufficiently large λ is similar in form to the condition $pq > 1$ given in [**5**] for finite velocity in the nonreacting, time-dependent problem.

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