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RIEMANN INTEGRATION IN BANACH SPACES

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In this paper we will consider the Riemann integration of functions mapping a closed interval into a Banach space. This problem was first studied by Graves [**4**]. As new results in the theory of Banach spaces appeared, various authors added to the theory of vector-valued Riemann integration. Most of the results in this paper are a compilation of the works of Graves [**4**], Alexiewicz and Orlicz [**1**], Rejouani [**9, 10**], Nemirovski, Ochan, and Rejouani [**7**], and da Rocha [**11**].

Many of the real-valued results concerning the Riemann integral remain valid in the vector case. However, in the vector case a Riemann integrable function need not be continuous almost everywhere. It is an interesting problem to determine which spaces have the property that every Riemann integrable function is continuous almost everywhere and an analysis of this problem will be one of the main focuses of this paper. We will also examine the relationship between the Riemann integral and other vector-valued integrals.

We begin with some terminology and notation. Throughout this paper X will denote a real Banach space and X^* its dual.

Definition 1. A partition of the interval $[a,b]$ is a finite set of points $\{t_i : 0 \le i \le N\}$ in $[a,b]$ that satisfy $a = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = b$. A tagged partition of $[a,b]$ is a partition $\{t_i : 0 \le i \le N\}$ of $[a,b]$ together with a set of points $\{s_i : 1 \le i \le N\}$ that satisfy $s_i \in [t_{i-1}, t_i]$ for each i. Let $\mathcal{P} = \{(s_i, [t_{i-1}, t_i]) : 1 \le i \le N\}$ be a tagged partition of $[a,b]$. The points $\{t_i : 0 \le i \le N\}$ are called the points of the partition, the intervals $\{[t_{i-1}, t_i] : 1 \le i \le N\}$ are called the intervals of the partition, the points $\{s_i : 1 \le i \le N\}$ are called the tags of the partition, and the norm $|\mathcal{P}|$ of the partition is defined by $|\mathcal{P}| = \max\{t_i - t_{i-1} : 1 \le i \le N\}$. If $f : [a,b] \to X$, then $f(\mathcal{P})$ will denote the Riemann sum $\sum_{i=1}^N f(s_i)(t_i - t_{i-1})$. Finally, the (tagged) partition $\mathcal{P}_1$ is a refinement of the (tagged) partition $\mathcal{P}_2$ if the points

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of $\mathcal{P}_2$ form a subset of the points of $\mathcal{P}_1$. In this case we say that $\mathcal{P}_1$ refines $\mathcal{P}_2$.

Definition 2. Let $f : [a,b] \to X$.

(a) The function f is R_δ integrable on $[a,b]$ if there exists a vector z in X with the following property: for each $\varepsilon > 0$ there exists $\delta > 0$ such that $||f(\mathcal{P}) - z|| < \varepsilon$ whenever $\mathcal{P}$ is a tagged partition of $[a,b]$ that satisfies $|\mathcal{P}| < \delta$.

(b) The function f is R_Δ integrable on $[a,b]$ if there exists a vector z in X with the following property: for each $\varepsilon > 0$ there exists a partition $\mathcal{P}_\varepsilon$ of $[a,b]$ such that $||f(\mathcal{P}) - z|| < \varepsilon$ whenever $\mathcal{P}$ is a tagged partition of $[a,b]$ that refines $\mathcal{P}_\varepsilon$.

It is clear that the vector z is unique in either case and a standard argument shows that a function integrable in either sense must be bounded. An R_δ integrable function is necessarily R_Δ integrable. The converse is true as well.

Theorem 3. *A function $f : [a,b] \to X$ is R_Δ integrable on $[a,b]$ if and only if it is R_δ integrable on $[a,b]$.*

Proof. Suppose that f is R_Δ integrable on $[a,b]$. Let z be the R_Δ integral of f on $[a,b]$ and let M be a bound for f on $[a,b]$. Let $\varepsilon > 0$ and choose a partition $\mathcal{P}_\varepsilon = \{t_k : 0 \le k \le N\}$ of $[a,b]$ such that $||f(\mathcal{P}) - z|| < \varepsilon/2$ whenever $\mathcal{P}$ is a tagged partition of $[a,b]$ that refines $\mathcal{P}_\varepsilon$. Let $\delta = \varepsilon/(4MN)$. We will show that $||f(\mathcal{P}) - z|| < \varepsilon$ whenever $\mathcal{P}$ is a tagged partition of $[a,b]$ that satisfies $|\mathcal{P}| < \delta$. It then follows that f is R_δ integrable on $[a,b]$.

Let $\mathcal{P}$ be such a tagged partition. Form a tagged partition $\mathcal{P}_1$ of $[a,b]$ as follows. The points of $\mathcal{P}_1$ are the points of both $\mathcal{P}$ and $\mathcal{P}_\varepsilon$. The tag of each interval of $\mathcal{P}_1$ that coincides with an interval of $\mathcal{P}$ is the same as the tag for $\mathcal{P}$. The tags of $\mathcal{P}_1$ for the remaining intervals are arbitrary. Let $\{[c_k, d_k] : 1 \le k \le K\}$ be the intervals of $\mathcal{P}$ that contain points of $\mathcal{P}_\varepsilon$ in their interiors and note that $K \le N - 1$. In the interval $[c_k, d_k]$ let $c_k = u_0^k < u_1^k < \cdots < u_{n_k-1}^k < u_{n_k}^k = d_k$ where the points $\{u_i^k : 1 \le i \le n_k - 1\}$ are the points of $\mathcal{P}_\varepsilon$ in (c_k, d_k). Let s_k be the tag of $\mathcal{P}$ for $[c_k, d_k]$ and let v_i^k be the tag of $\mathcal{P}_1$ for $[u_{i-1}^k, u_i^k]$. Then

$$\begin{aligned}||f(\mathcal{P}) - f(\mathcal{P}_1)|| &= \left\| \sum_{k=1}^{K} \{f(s_k)(d_k - c_k) - \sum_{i=1}^{n_k} f(v_i^k)(u_i^k - u_{i-1}^k)\} \right\| \\ &\le \sum_{k=1}^{K} \sum_{i=1}^{n_k} ||f(s_k) - f(v_i^k)||(u_i^k - u_{i-1}^k) \\ &\le 2M \sum_{k=1}^{K} (d_k - c_k) \\ &\le 2M(N-1)\delta \\ &< \frac{\varepsilon}{2}.\end{aligned}$$

Since $\mathcal{P}_1$ is a refinement of $\mathcal{P}_\varepsilon$, we obtain

$$||f(\mathcal{P}) - z|| \le ||f(\mathcal{P}) - f(\mathcal{P}_1)|| + ||f(\mathcal{P}_1) - z|| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This completes the proof. □

Definition 4. The function $f : [a, b] \to X$ is Riemann integrable on $[a, b]$ if f is either R_δ or R_Δ integrable on $[a, b]$.

The next theorem presents several Cauchy criteria for the existence of the Riemann integral. As in the scalar case, these conditions are quite useful in proving other properties of the Riemann integral.

Theorem 5. *Let $f : [a, b] \to X$. The following are equivalent:*

(1) *The function f is Riemann integrable on $[a, b]$.*

(2) *For each $\varepsilon > 0$ there exists $\delta > 0$ such that $||f(\mathcal{P}_1) - f(\mathcal{P}_2)|| < \varepsilon$ for all tagged partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $[a, b]$ with norms less than δ.*

(3) *For each $\varepsilon > 0$ there exists a partition $\mathcal{P}_\varepsilon$ of $[a, b]$ such that $||f(\mathcal{P}_1) - f(\mathcal{P}_2)|| < \varepsilon$ for all tagged partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $[a, b]$ that refine $\mathcal{P}_\varepsilon$.*

(4) *For each $\varepsilon > 0$ there exists a partition $\mathcal{P}_\varepsilon$ of $[a, b]$ such that $||f(\mathcal{P}_1) - f(\mathcal{P}_2)|| < \varepsilon$ for all tagged partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $[a, b]$ that have the same points as $\mathcal{P}_\varepsilon$.*

Proof. The proofs that (1) and (2) are equivalent and that (1) and (3) are equivalent follow standard advanced calculus arguments using the R_δ and R_Δ definitions, respectively, of the Riemann integral and will be omitted. To complete the proof we will show that (3) and (4) are equivalent. Since (3) clearly implies (4) all that remains to be proved is that (4) implies (3).

Let $\varepsilon > 0$ and choose a partition $\mathcal{P}_\varepsilon = \{t_i : 0 \le i \le N\}$ of $[a,b]$ such that $||f(\mathcal{P}_1) - f(\mathcal{P}_2)|| < \varepsilon/2$ for all tagged partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $[a,b]$ that have the same points as $\mathcal{P}_\varepsilon$. Let $\mathcal{P}_0$ be the tagged partition $\{(t_i, [t_{i-1}, t_i]) : 1 \le i \le N\}$. For each i, let W_i be the set $\{(t_i - t_{i-1})f(t) : t \in [t_{i-1}, t_i]\}$ and let $W = \sum_{i=1}^N W_i$. Note that $||x|| < \varepsilon/2$ for all x in $\mathrm{co}\,(W - W)$ where $\mathrm{co}\,A$ denotes the convex hull of A.

Let $\mathcal{P} = \{(v_k, [u_{k-1}, u_k]) : 1 \le k \le M\}$ be a tagged partition of $[a,b]$ that refines $\mathcal{P}_\varepsilon$. For each i, let k_i be the index k for which $u_k = t_i$. Then

$$\begin{aligned} f(\mathcal{P}_0) - f(\mathcal{P}) &= \sum_{i=1}^N \{f(t_i)(t_i - t_{i-1}) - \sum_{k=k_{i-1}+1}^{k_i} f(v_k)(u_k - u_{k-1})\} \\ &= \sum_{i=1}^N \sum_{k=k_{i-1}+1}^{k_i} \frac{u_k - u_{k-1}}{t_i - t_{i-1}} \{(t_i - t_{i-1})f(t_i) \\ &\qquad - (t_i - t_{i-1})f(v_k)\} \\ &\in \sum_{i=1}^N \mathrm{co}\,(W_i - W_i) = \mathrm{co}\,(W - W) \end{aligned}$$

and it follows that $||f(\mathcal{P}_0) - f(\mathcal{P})|| < \varepsilon/2$. Now let $\mathcal{P}_1$ and $\mathcal{P}_2$ be tagged partitions of $[a,b]$ that refine $\mathcal{P}_\varepsilon$ and compute

$$||f(\mathcal{P}_1) - f(\mathcal{P}_2)|| \le ||f(\mathcal{P}_1) - f(\mathcal{P}_0)|| + ||f(\mathcal{P}_0) - f(\mathcal{P}_2)|| < \varepsilon.$$

This completes the proof. □

A collection of standard definitions appears next. Measure and measurable refer to standard Lebesque measure.

Definition 6. Let $f : [a,b] \to X$.

(a) The function f is scalarly measurable if x^*f is measurable for each x^* in X^*.

(b) The function f is of weak bounded variation on $[a,b]$ if x^*f is of bounded variation on $[a,b]$ for each x^* in X^*.

(c) The function f is of outside bounded variation on $[a,b]$ if $\sup\{\|\sum_i (f(d_i) - f(c_i))\|\}$ is finite where the supremum is taken over all finite collections $\{[c_i, d_i]\}$ of nonoverlapping intervals in $[a,b]$.

(d) The function f is a scalar derivative of $F : [a,b] \to X$ on $[a,b]$ if for each x^* in X^* the function x^*F is differentiable almost everywhere on $[a,b]$ and $(x^*F)' = x^*f$ almost everywhere on $[a,b]$.

The adjective outside is used in (c) to emphasize that the norm is outside of the sum. It is well known that the notions of outside bounded variation and weak bounded variation are equivalent.

The next two theorems summarize the basic properties of the Riemann integral. All of the proofs are straightforward and will be omitted.

Theorem 7. *Let $f : [a,b] \to X$ be Riemann integrable on $[a,b]$.*

(a) *The function f is Riemann integrable on every subinterval of $[a,b]$.*

(b) *If M is a bound for f, then $\|\int_a^b f\| \le M(b-a)$.*

(c) *If $T : X \to Y$ is a continuous linear operator, then Tf is Riemann integrable on $[a,b]$ and $\int_a^b Tf = T(\int_a^b f)$.*

(d) *For each x^* in X^*, the function x^*f is Riemann integrable on $[a,b]$ and $\int_a^b x^*f = x^*\int_a^b f$. Hence, the function f is scalarly measurable, and for each x^* in X^* the function x^*f is continuous almost everywhere on $[a,b]$.*

Theorem 8. *Let $f : [a,b] \to X$ be Riemann integrable on $[a,b]$ and let $F(t) = \int_a^t f$. Then F is absolutely continuous on $[a,b]$ and f is a scalar derivative of F on $[a,b]$. Furthermore, at each point t of continuity of f the function F is differentiable and $F'(t) = f(t)$.*

A Riemann integrable function need be neither measurable nor continuous almost everywhere. Before presenting examples to illustrate this, we prove a theorem that gives a useful criterion for determining whether or not a function is Riemann integrable. It is not difficult to prove that a real-valued function of bounded variation is Riemann integrable. The vector analogue of this was first noticed by Alexiewicz and Orlicz [**1**].

Theorem 9. *If $f : [a,b] \to X$ is of outside bounded variation on $[a,b]$, then f is Riemann integrable on $[a,b]$. Consequently, a function of weak bounded variation is Riemann integrable.*

Proof. We will show that f satisfies (4) of Theorem 5. Let $\varepsilon > 0$ be given. Let M be the outside variation of f on $[a,b]$ and choose a positive integer N such that $(b-a)/N < \varepsilon/M$. Let $\mathcal{P}_\varepsilon = \{t_i : 0 \leq i \leq N\}$ be the partition of $[a,b]$ for which $t_i = a + (i/N)(b-a)$. Let $\mathcal{P}_1 = \{(u_i, [t_{i-1}, t_i]) : 1 \leq i \leq N\}$ and $\mathcal{P}_2 = \{(v_i, [t_{i-1}, t_i]) : 1 \leq i \leq N\}$ be tagged partitions of $[a,b]$. These partitions have the same points as $\mathcal{P}_\varepsilon$ and we have

$$\begin{aligned} ||f(\mathcal{P}_1) - f(\mathcal{P}_2)|| &= \left\| \sum_{i=1}^{N} (f(u_i) - f(v_i))(t_i - t_{i-1}) \right\| \\ &= \frac{b-a}{N} \left\| \sum_{i=1}^{N} (f(u_i) - f(v_i)) \right\| \\ &\leq \frac{b-a}{N} M \\ &< \varepsilon. \end{aligned}$$

Hence (4) is satisfied and it follows that f is Riemann integrable on $[a,b]$. □

The next few examples illustrate pathological characteristics of the vector-valued Riemann integral. Examples 10 and 11 are due to Rejouani [**9**]. Example 12 is due to Graves [**4**]. Example 13 is a corrected version of an example of Alexiewicz and Orlicz [**1**]. Example 14 is due to Pettis [**8**]. We will use θ to denote the zero in a Banach space.

Example 10. A measurable, Riemann integrable function that is not continuous almost everywhere.

Let $\{r_n\}$ be a listing of the rational numbers in $[0,1]$ and define $f : [0,1] \to c_0$ by $f(t) = \theta$ if t is irrational and $f(t) = e_n$ if $t = r_n$. This function is Riemann integrable on $[0,1]$ since it is of outside bounded variation on $[0,1]$, and it is clear that f is not continuous almost everywhere on $[0,1]$.

Example 11. A measurable, Riemann integrable function that is not of outside bounded variation:

Let $\{r_n\}$ be a listing of the rational numbers in $[0,1]$ and define $f : [0,1] \to \ell_2$ by $f(t) = \theta$ if t is irrational and $f(t) = e_n$ if $t = r_n$. We show first that f is Riemann integrable on $[0,1]$. Let $\varepsilon > 0$ and let $\delta = \varepsilon^2$. Let $\mathcal{P} = \{(s_i, [t_{i-1}, t_i]) : 1 \le i \le N\}$ be a tagged partition of $[0,1]$ with $|\mathcal{P}| < \delta$ and compute

$$\begin{aligned} ||f(\mathcal{P})|| &= \left\| \sum_{i=1}^{N} f(s_i)(t_i - t_{i-1}) \right\| \\ &\le \left\{ \sum_{i=1}^{N} (t_i - t_{i-1})^2 \right\}^{\frac{1}{2}} \\ &\le |\mathcal{P}|^{\frac{1}{2}} \left\{ \sum_{i=1}^{N} (t_i - t_{i-1}) \right\}^{\frac{1}{2}} \\ &< \varepsilon. \end{aligned}$$

Hence, the function f is Riemann integrable on $[0,1]$ with integral θ.

To show that f is not of outside bounded variation on $[0,1]$, let N be a positive integer and for each positive integer i, let c_i be an irrational number in the interval $(1/(i+1), 1/i)$. Then

$$\left\| \sum_{i=1}^{N} \left(f\left(\frac{1}{i}\right) - f(c_i) \right) \right\| = \left(\sum_{i=1}^{N} 1 \right)^{\frac{1}{2}} = \sqrt{N}$$

and this shows that f is not of outside bounded variation on $[0,1]$.

Note that f is weakly continuous almost everywhere on $[0,1]$. Furthermore, all of the conclusions about f remain valid if the range of f is any ℓ_p space for $1 < p < \infty$.

Example 12. A Riemann integrable function that is not measurable and not weakly continuous almost everywhere:

Define $f : [0,1] \to \ell_\infty[0,1]$ by $f(t) = \chi_{[0,t]}$. It is easy to verify that f is not measurable and that f is not weakly continuous almost everywhere on $[0,1]$. Since f is of outside bounded variation on $[0,1]$ it is Riemann integrable on $[0,1]$.

Example 13. A measurable, Riemann integrable function that is not weakly continuous almost everywhere:

Define $f : [0,1] \to \mathcal{C}[0,1]$ as follows. If t is a dyadic rational number of the form $(2m-1)2^{-k}$ with $2 \le m \le 2^{k-1}$, then $f(t)$ is the function that equals 1 on the set $\{0, 1, t-2^{-k+1}, t+2^{-k}\}$, equals 0 on the set $\{t-2^{-k}, t\}$, and is linear on the intervals between these points. If t is any other number, then $f(t)$ is the constant function 1.

To show that f is Riemann integrable on $[0,1]$, we will show that f satisfies condition (4) of Theorem 5. Let $\varepsilon > 0$ and choose a positive integer $K \ge 2$ such that $2^{-K} < \varepsilon/5$. For each $n \in \{1, 2, \ldots, 2^K - 1\}$ let $I_n = [n2^{-K} - 2^{-2K}, n2^{-K} + 2^{-2K}]$ and let $\{J_n : 1 \le n \le 2^K\}$ be the remaining intervals of $[0,1]$ listed in increasing order. Let u_n and v_n be arbitrary points of I_n and compute

$$\begin{aligned}\left\| \sum_{n=1}^{2^K-1} (f(v_n) - f(u_n))\mu(I_n) \right\| &\le \sum_{n=1}^{2^K-1} \|f(v_n) - f(u_n)\| \mu(I_n) \\ &\le (2^K - 1)(2 \cdot 2^{-2K}) \\ &\le 2 \cdot 2^{-K}.\end{aligned}$$

Now let u_n and v_n be arbitrary points of J_n. It is easy to verify that for each point $s \in [0,1]$ there are at most three integers n for which $(f(v_n) - f(u_n))(s) \ne 0$. It follows that for each point s in $[0,1]$ there are at most three integers n for which $(f(v_n) - f(u_n))(s) \ne 0$. It follows that

$$\left\| \sum_{n=1}^{2^K} (f(v_n) - f(u_n))\mu(J_n) \right\| \le 3 \max_n \{\mu(J_n)\} < 3 \cdot 2^{-K}.$$

Let $\mathcal{P}_\varepsilon$ be the partition of $[0,1]$ formed from the intervals $\{I_n\}$ and $\{J_n\}$. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two tagged partitions of $[0,1]$ that have the same points as $\mathcal{P}_\varepsilon$. The inequalities of the last paragraph show that $||f(\mathcal{P}_1) - f(\mathcal{P}_2)|| \le 5 \cdot 2^{-K} < \varepsilon$. Therefore, the function f is Riemann integrable on $[0,1]$.

To show that f is not weakly continuous almost everywhere on $[0,1]$, we will show that f is not weakly continuous at any irrational point of $(0,1)$. Let s be an irrational number in $(0,1)$ and let $z^* \in \mathcal{C}^*[0,1]$ be defined by $z^*\phi = \phi(s)$. For infinitely many integers k, there exists an integer $m \ge 2$ such that $(2m-2)2^{-k} < s < (2m-1)2^{-k}$. For such k let $t_k = (2m-1)2^{-k}$. The seqence $\{t_k\}$ converges to s and

$$\lim_{k\to\infty} z^* f(t_k) = \lim_{k\to\infty} f(t_k)(s) = 0 \ne 1 = z^* f(s).$$

Thus, the function z^*f is not continuous at s and this shows that f is not weakly continuous at s.

Example 14. A Riemann integrable function f such that $||f||$ is not measurable and, hence, neither Riemann nor Lebesgue integrable:

Let E be a nonmeasurable set in $[0,1]$ and define $f : [0,1] \to \ell_\infty[0,1]$ by $f(t) = \theta$ if $t \notin E$ and $f(t) = \chi_{\{t\}}$ if $t \in E$. Then it is easy to see that f is of outside bounded variation and hence Riemann integrable on $[0,1]$. However, the function $||f|| = \chi_E$ is not measurable.

A real-valued Riemann integrable function is Lebesgue integrable, but a Riemann integrable function in the vector case is not always Bochner integrable. Nevertheless, it is always Pettis integrable. This fact is not obvious and Rejouani [**10**] appears to have the Dunford integral in mind when he claims that it is obvious. (The reader should see Diestel and Uhl [**3**] for the definitions of the Bochner, Pettis, and Dunford integrals.)

Theorem 15. *If $f : [a,b] \to X$ is Riemann integrable on $[a,b]$, then f is Pettis integrable on $[a,b]$. If, in addition, f is measurable, then f is Bochner integrable on $[a,b]$.*

Proof. Since f is bounded and scalarly measurable, it is Dunford integrable on $[a,b]$. The fact that f is bounded implies that the family $\{x^*f : ||x^*|| \le 1\}$ is uniformly integrable on $[a,b]$ and the Riemann integrability of f implies that $(D)\int_I f \in X$ for every interval $I \subset [a,b]$.

Using these two facts, it is a standard measure theory argument to show that $(D)\int_E f \in X$ for every measurable set $E \subset [a,b]$. Hence, the function f is Pettis integrable on $[a,b]$.

If f is measurable, then the bounded, measurable function $||f||$ is Lebesgue integrable on $[a,b]$. It follows that f is Bochner integrable on $[a,b]$. $\square$

The next result is due to Graves [**4**].

Theorem 16. *Let* $F : [a,b] \to X$ *be differentiable at each point of* $[a,b]$. *If* F' *is Riemann integrable on* $[a,b]$, *then* $F(t) = F(a) + \int_a^t F'$.

Proof. Since F is differentiable at each point of $[a,b]$, the function F is continuous on $[a,b]$ and this in turn implies that F' is measurable. By Theorem 15, the function F' is Bochner integrable on $[a,b]$. Since F is absolutely continuous on $[a,b]$ (F' is bounded), we have $F(t) = F(a) + (B)\int_a^t F'$. Since the integrals $(B)\int_a^t F'$ and $(R)\int_a^t F'$ are equal, the proof is complete. $\square$

It should be remarked that Graves did not prove the above theorem using the Bochner integral because he did not have access to it. He proves the theorem directly from the definition of the Riemann integral using an argument that requires no properties of Banach spaces other than the triangle inequality. His argument is a bit tedious and we have chosen to omit it in favor of the above proof.

For real-valued functions the Riemann integral can be defined using upper and lower sums. This Darboux approach leads to an integral that is equivalent to the Riemann integral. For arbitrary vector-valued functions the collection of Darboux integrable functions is contained in but not equal to the collection of Riemann integrable functions. The containment is proper since a function is Darboux integrable if and only if it is bounded and continuous almost everywhere. We will prove this fact after the necessary definitions. If f is defined on $[a,b]$ and

$\mathcal{P} = \{t_i : 0 \le i \le N\}$ is a partition of $[a, b]$, then

$$\omega(f, \mathcal{P}) = \sum_{i=1}^{N} \omega(f, [t_{i-1}, t_i])(t_i - t_{i-1})$$

where $\omega(f, [t_{i-1}, t_i]) = \sup\{||f(v) - f(u)|| : u, v \in [t_{i-1}, t_i]\}$ is the oscillation of the function f on the interval $[t_{i-1}, t_i]$.

Definition 17. Let $f : [a, b] \to X$.

(a) The function f is D_δ integrable on $[a, b]$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\omega(f, \mathcal{P}) < \varepsilon$ whenever $\mathcal{P}$ is a partition of $[a, b]$ that satisfies $|\mathcal{P}| < \delta$.

(b) The function f is D_Δ integrable on $[a, b]$ if for each $\varepsilon > 0$ there exists a partition $\mathcal{P}_\varepsilon$ of $[a, b]$ such that $\omega(f, \mathcal{P}) < \varepsilon$ whenever $\mathcal{P}$ is a partition of $[a, b]$ that refines $\mathcal{P}_\varepsilon$.

(c) The function $f : [a, b] \to X$ is Darboux integrable on $[a, b]$ if f is either D_δ or D_Δ integrable on $[a, b]$.

The proof that a function is D_δ integrable on $[a, b]$ if and only if it is D_Δ integrable on $[a, b]$ is similar to the proof that a function is R_δ integrable on $[a, b]$ if and only if it is R_Δ integrable on $[a, b]$ and will be omitted. It is not difficult to show that a function is Darboux integrable on every subinterval of $[a, b]$ if it is Darboux integrable on $[a, b]$. The standard advanced calculus proof is valid here and shows that every Darboux integrable function is Riemann integrable. The value of the Darboux integral is the value of the Riemann integral.

In order to prove that a function is Darboux integrable if and only if it is bounded and continuous almost everywhere, we introduce some standard notation. Let $f : [a, b] \to X$. For each t in (a, b), let $\omega(f, t) = \lim_{\delta \to 0^+} \omega(f, [t - \delta, t + \delta])$ be the oscillation of f at t. Let $\omega(f, a) = \lim_{\delta \to 0^+} \omega(f, [a, a + \delta])$ and $\omega(f, b) = \lim_{\delta \to 0^+} \omega(f, [b - \delta, b])$. Note that f is continuous at t if and only if $\omega(f, t) = 0$. In addition, the set $\{t \in [a, b] : \omega(f, t) \ge \alpha\}$ is closed for each real number α.

Theorem 18. *A function $f : [a, b] \to X$ is Darboux integrable on $[a, b]$ if and only if it is bounded and continuous almost everywhere on $[a, b]$.*

Proof. The proof is standard, but we present it in detail.

Suppose that f is Darboux integrable on $[a, b]$. Then it is clear that f is bounded. To show that f is continuous almost everywhere on $[a, b]$, let $E_n = \{t \in [a, b] : \omega(f, t) \geq 1/n\}$ for each positive integer n and let $E = \cup_n E_n$. Since each E_n is closed, the set E is measurable and we must show that $\mu(E) = 0$. If $\mu(E) \neq 0$, then there exist $\eta > 0$ and a positive integer N such that $\mu(E_N) = \eta$. Let $\mathcal{P}$ be any partition of $[a, b]$ and let $\mathcal{P}_1$ be the collection of intervals of $\mathcal{P}$ that contain points of E_N in their interior. Then

$$\omega(f, \mathcal{P}) \geq \sum_{I \in \mathcal{P}_1} \omega(f, I)\mu(I) \geq \frac{1}{N}\mu(E_N) = \frac{\eta}{N},$$

a contradiction to the Darboux integrability of f. Thus, the function f is continuous almost everywhere on $[a, b]$.

Now suppose that f is bounded and continuous almost everywhere on $[a, b]$, and let M be a bound for f. We will show that f is D_Δ integrable on $[a, b]$.

Let $\varepsilon > 0$ and choose a positive integer N such that $(b-a)/N < \varepsilon/2$. Let $E_N = \{t \in [a, b] : \omega(f, t) \geq 1/N\}$. We will construct a partition $\mathcal{P}_\varepsilon$ of $[a, b]$ such that the sum of the lengths of the intervals of $\mathcal{P}_\varepsilon$ that intersect E_N is less than $\varepsilon/(4M)$ and the oscillation of f on each interval of $\mathcal{P}_\varepsilon$ that does not intersect E_N is less than $1/N$. Denote the intervals of $\mathcal{P}_\varepsilon$ that intersect E_N by $\mathcal{P}'_\varepsilon$ and the remaining intervals $\mathcal{P}''_\varepsilon$. Since $\mu(E_N) = 0$, there exists a sequence $\{(c_i, d_i)\}$ of disjoint open intervals such that $E_N \subset \cup_i(c_i, d_i)$ and $\sum_i(d_i - c_i) < \varepsilon/(4M)$. Since the set E_N is closed and bounded, it is compact and therefore a finite number of the intervals $\{(c_i, d_i)\}$ cover E_N. The closure of each interval in the finite subcover intersected with $[a, b]$ is an element of $\mathcal{P}'_\varepsilon$. Let $[\alpha, \beta]$ be an interval in $[a, b]$ that is contiguous to the intervals of $\mathcal{P}'_\varepsilon$. Since $[\alpha, \beta] \cap E_N = \phi$, for each t in $[\alpha, \beta]$ there exists $\delta_t > 0$ such that $\omega(f, [t - \delta_t, t + \delta_t]) < 1/N$. The collection $\{(t - \delta_t, t + \delta_t) : t \in [\alpha, \beta]\}$ is an open cover of $[\alpha, \beta]$ and thus there exists a finite subcover. The endpoints of the intervals comprising the finite subcover that belong to (α, β) together with α and β form a partition of $[\alpha, \beta]$. Put the intervals of this partition into $\mathcal{P}''_\varepsilon$ and do this for all of the intervals in $[a, b]$ that are contiguous to $\mathcal{P}'_\varepsilon$. It is easily checked that the intervals of $\mathcal{P}'_\varepsilon$ and $\mathcal{P}''_\varepsilon$ combine to form a partition $\mathcal{P}_\varepsilon$ of $[a, b]$ with the desired properties.

Let $\mathcal{P}$ be a partition of $[a,b]$ that refines $\mathcal{P}_\varepsilon$. Let $\mathcal{P}'$ and $\mathcal{P}''$ be the intervals of $\mathcal{P}$ that are entirely contained within intervals of $\mathcal{P}'_\varepsilon$ and $\mathcal{P}''_\varepsilon$, respectively, and compute

$$\begin{aligned}\omega(f,\mathcal{P}) &= \sum_{I\in\mathcal{P}'}\omega(f,I)\mu(I) + \sum_{I\in\mathcal{P}''}\omega(f,I)\mu(I)\\ &\le 2M\cdot\frac{\varepsilon}{4M} + \frac{1}{N}(b-a)\\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Thus the function f is D_Δ integrable on $[a,b]$. □

Corollary 19. *If $f:[a,b]\to X$ is Darboux integrable on $[a,b]$, then f is measurable and $||f||$ is Riemann integrable on $[a,b]$. Consequently, the function f is Bochner integrable on $[a,b]$.*

Proof. By Theorem 18, the function f is continuous almost everywhere on $[a,b]$ and hence measurable. The function $||f||$ is bounded and continuous almost everywhere on $[a,b]$ and hence Riemann integrable on $[a,b]$. □

It is an interesting problem to determine spaces in which every Riemann integrable function is continuous almost everywhere. This is equivalent to determining those spaces in which a function is Riemann integrable if and only if it is Darboux integrable. Since Lebesgue proved that **R** has this property, we make the following definition.

Definition 20. A Banach space X has the property of Lebesgue if every Riemann integrable function $f:[a,b]\to X$ is continuous almost everywhere on $[a,b]$.

The next theorem is useful in determining whether or not a given space has the property of Lebesgue. Its easy proof will be omitted.

Theorem 21. *Let Y be a subspace of X.*

(a) *If X has the property of Lebesgue, then Y has the property of Lebesgue.*

(b) *If Y does not have the property of Lebesgue, then X does not have the property of Lebesgue.*

We begin by identifying those spaces that do not have the property of Lebesgue.

Theorem 22. *The following spaces do not have the property of Lebesgue*:

(a) *the spaces $c_0, c, \ell_\infty, \mathcal{C}[a,b], \ell_\infty[a,b]$, and $L_\infty[a,b]$,*

(b) *the spaces ℓ_p for $1 < p < \infty$,*

(c) *the space $L_1[a,b]$, and*

(d) *the dual X^* if X contains a copy of ℓ_1.*

Proof. By Example 10, the space c_0 does not have the property Lebesgue. Since c_0 imbeds in the remaining spaces of (a), we can apply Theorem 21(b). Example 11 shows that ℓ_p for $1 < p < \infty$ does not have the property of Lebesgue. The theorem now follows since ℓ_2 imbeds in $L_1[a,b]$ and in X^* if X contains a copy of ℓ_1. □

The next theorem is due to da Rocha [**11**]. In the proof we will use the following special case of a result due to James [**5**]. If $\{x_n\}$ is a normalized basis of the uniformly convex space X, then there exist $M > 0$ and $r > 1$ such that $||\sum_n \alpha_n x_n|| \le M(\sum_n |\alpha_n|^r)^{1/r}$ for all finitely nonzero sequences $\{\alpha_n\}$ of real numbers.

Theorem 23. *An infinite dimensional, uniformly convex Banach space does not have the property of Lebesgue.*

Proof. Let X be an infinite dimensional, uniformly convex Banach space. Since X is infinite dimensional, it contains a basic sequence $\{x_n\}$

and we may assume that $||x_n|| = 1$ for every n. Let Y be the closed linear subspace generated by $\{x_n\}$. Then Y is uniformly convex and $\{x_n\}$ is a normalized basis of Y. To complete the proof it is sufficient to prove that Y does not have the property of Lebesgue and apply Theorem 21(b).

Let $\{r_n\}$ be a listing of the rational numbers in $[0,1]$ and define $f : [0,1] \to Y$ by $f(t) = \theta$ if t is irrational and $f(t) = x_n$ if $t = r_n$. We will show that f is Riemann integrable on $[0,1]$ with integral θ. Choose M and r as in the remark preceding the theorem and let $\varepsilon > 0$ be given. Let $\delta = (\varepsilon/M)^{r/(r-1)}$, and let $\mathcal{P} = \{(s_k, [t_{k-1}, t_k]) : 1 \le k \le N\}$ be a tagged partition of $[0,1]$ that satisfies $|\mathcal{P}| < \delta$. Then

$$\begin{aligned} ||f(\mathcal{P})|| &= \left\| \sum_{k=1}^{N} f(s_k)(t_k - t_{k-1}) \right\| \\ &\le M \left(\sum_{k=1}^{N} (t_k - t_{k-1})^r \right)^{\frac{1}{r}} \\ &\le M \delta^{\frac{r-1}{r}} \left(\sum_{k=1}^{N} (t_k - t_{k-1}) \right)^{\frac{1}{r}} \\ &\le M \cdot \frac{\varepsilon}{M} = \varepsilon. \end{aligned}$$

Therefore, the function f is Riemann integrable on $[0,1]$. Since it is clear that f is not continuous almost everywhere on $[0,1]$, the space Y does not have the property of Lebesgue. □

Corollary 24. *The following spaces do not have the property of Lebesgue.*

(a) *Infinite dimensional Hilbert spaces.*

(b) *The spaces $L_p[a,b]$ for $1 < p < \infty$.*

Proof. These spaces are infinite dimensional and uniformly convex. □

We next prove that a Lorentz sequence space does not have the property of Lebesgue. Let $1 \le p < \infty$ and let $w = \{w_n\}$ be a nonincreasing sequence of positive numbers such that $w_1 = 1$, $\lim_{n\to\infty} w_n = 0$, and $\sum_n w_n = \infty$. The Lorentz sequence space $d(w,p)$ is the Banach space of all sequences $\{a_n\}$ of real numbers for which $||\{a_n\}|| = \sup\{(\sum_{n=1}^{\infty} |a_{\sigma(n)}|^p w_n)^{1/p}\}$ is finite where the supremum is taken over all permutations σ of the positive integers. No Lorentz sequence space is isomorphic to an ℓ_p space and for $1 < p < \infty$ the space $d(w,p)$ is reflexive. For the proofs of these results see Lindenstrauss and Tzafriri [**6**].

Da Rocha [**11**] states that $d(w,p)$ does not have the property of Lebesgue if $p > 1$ and offers a proof of this result, but the details of his proof appear to be lacking. We offer another proof of this result and extend the result to the case $p = 1$.

Theorem 25. *A Lorentz sequence space does not have the property of Lebesgue.*

Proof. Let $d(w,p)$ be a Lorentz sequence space. Let $\{r_n\}$ be a listing of the rational numbers in $[0,1]$ and define $f : [0,1] \to d(w,p)$ by $f(t) = \theta$ if t is irrational and $f(t) = e_n$ if $t = r_n$. It is clear that f is not continuous almost everywhere on $[0,1]$. To complete the proof it is sufficient to prove that f is Riemann integrable on $[0,1]$. To this end, we will show that f satisfies (4) of Theorem 5.

Suppose first that $p > 1$ and let $\varepsilon > 0$ be given. Choose a positive integer N such that $N^{1/p-1} < \varepsilon/2$ and let $\mathcal{P}_\varepsilon = \{i/N : 0 \le i \le N\}$. Let $\mathcal{P}$ be a tagged partition of $[0,1]$ that has the same points as $\mathcal{P}_\varepsilon$ and compute

$$||f(\mathcal{P})|| \le \left(\sum_{i=1}^{N}\left(\frac{1}{N}\right)^p w_i\right)^{\frac{1}{p}} \le \frac{1}{N}\left(\sum_{i=1}^{N} w_i\right)^{\frac{1}{p}} \le \frac{1}{N}\cdot N^{\frac{1}{p}} < \frac{\varepsilon}{2}.$$

If $\mathcal{P}_1$ and $\mathcal{P}_2$ are tagged partitions of $[0,1]$ that have the same points as $\mathcal{P}_\varepsilon$, then $||f(\mathcal{P}_1) - f(\mathcal{P}_2)|| \le ||f(\mathcal{P}_1)|| + ||f(\mathcal{P}_2)|| < \varepsilon$.

Now suppose that $p = 1$ and let $\varepsilon > 0$ be given. Since the Cesaro sums $(1/n)\sum_{k=1}^{n} w_k$ converge to 0, there exists a positive integer N such that $(1/n)\sum_{k=1}^{n} w_k < \varepsilon/2$ for $n \ge N$. Set $\mathcal{P}_\varepsilon = \{i/N : 0 \le i \le N\}$. Let $\mathcal{P}$ be a tagged partition of $[0,1]$ that has the same points as $\mathcal{P}_\varepsilon$ and note that $||f(\mathcal{P})|| \le \sum_{i=1}^{N}(1/N)w_i < \varepsilon/2$. If $\mathcal{P}_1$ and $\mathcal{P}_2$ are tagged partitions of $[0,1]$ that have the same points as $\mathcal{P}_\varepsilon$, then $||f(\mathcal{P}_1) - f(\mathcal{P}_2)|| < \varepsilon$. This completes the proof. $\square$

As mentioned earlier, the space $\mathbf{R}$ has the property of Lebesgue. By arguing in each coordinate, it is not difficult to prove that all finite dimensional spaces have the property of Lebesgue. Nemirovski, Ochan, and Rejouani [**7**] proved that ℓ_1 has the property of Lebesgue. This fact was discovered independently by da Rocha [**11**]. The two proofs use the same technique and we reproduce this argument below.

Theorem 26. *The space ℓ_1 has the property of Lebesgue.*

Proof. It is sufficient to prove that a bounded function $f:[0,1]\to\ell_1$ that is not continuous almost everywhere on $[0,1]$ is not Riemann integrable on $[0,1]$. Let $f:[0,1]\to\ell_1$ be bounded but not continuous almost everywhere on $[0,1]$. There exist positive numbers α and β such that $\mu(H)=\alpha$ where $H=\{t\in[0,1]:\omega(f,t)\geq\beta\}$. We will prove that for each $\delta>0$ there exist tagged partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $[0,1]$ with $|\mathcal{P}_1|<\delta$ and $|\mathcal{P}_2|<\delta$ such that $||f(\mathcal{P}_1)-f(\mathcal{P}_2)||\geq\alpha\beta/4$. By Theorem 5, the proof will be complete.

Let $\delta>0$ be given. Choose a positive integer N such that $1/N<\delta$ and let $\mathcal{P}_N=\{k/N:0\leq k\leq N\}$. Let $\{[c_i,d_i]:1\leq i\leq p\}$ be all of the intervals of $\mathcal{P}_N$ for which $\mu(H\cap(c_i,d_i))>0$ and note that $p/N\geq\alpha$. For each positive integer j, let G_j be the set of discontinuities of e_jf on $[0,1]$. If $u(G_j)\neq 0$, then e_jf and consequently f is not Riemann integrable on $[0,1]$. Otherwise the set $G=\cup_j G_j$ has measure zero and every e_jf is continuous on $[0,1]-G$. Let $\varepsilon=\alpha\beta/16$. We will construct sets $\{u_i:1\leq i\leq p\}$ where $u_i\in(H-G)\cap(c_i,d_i)$ for each i, $\{v_i:1\leq i\leq p\}$ where $v_i\in(c_i,d_i)$ for each i, and $\{n_i:0\leq i\leq p\}$ where each n_i is an integer and $0=n_0<n_1<\cdots<n_p$ that have the following properties. Let $z_i=f(u_i)-f(v_i)=\{a_j^i\}$. Then $||z_i||\geq\beta/2$ for all $i\geq 1$, $\sum_{j=n_i}^{\infty}|a_j^i|<\varepsilon 2^{-i}$ for all $i\geq 1$, and $\sum_{j=1}^{n_{i-1}}|a_j^i|<\varepsilon 2^{-i}$ for all $i\geq 2$.

We proceed as follows. Let $n_0=0$ and choose $u_1\in(H-G)\cap(c_1,d_1)$. Since $\omega(f,u_1)\geq\beta$ there exists a point $v_1\in(c_1,d_1)$ such that $||f(u_1)-f(v_1)||\geq\beta/2$. Let $z_1=f(u_1)-f(v_1)=\{a_j^1\}$ and choose an integer $n_1>n_0$ such that $\sum_{j=n_1}^{\infty}|a_j^1|<\varepsilon/2$. Now choose

$u_2 \in (H-G)\cap(c_2,d_2)$. Since $\omega(f,u_2)\geq\beta$ and since e_jf is continuous at u_2 for each $1\leq j\leq n_1$, there exists $v_2\in(c_2,d_2)$ such that $||f(u_2)-f(v_2)||\geq\beta/2$ and $\sum_{j=1}^{n_1}|e_jf(u_2)-e_jf(v_2)|<\varepsilon/4$. Let $z_2=f(u_2)-f(v_2)=\{a_j^2\}$; then $\sum_{j=1}^{n_1}|a_j^2|<\varepsilon/4$. Choose an integer $n_2>n_1$ such that $\sum_{j=n_2}^{\infty}|a_j^2|<\varepsilon/4$. We continue this process for p steps and arrive at the desired sets.

Let $y_i=\sum\limits_{j=n_{i-1}+1}^{n_i-1}a_j^ie_j$ for each $1\leq i\leq p$. Then

$$||z_i-y_i||=\sum_{j=1}^{n_{i-1}}|a_j^i|+\sum_{j=n_i}^{\infty}|a_j^i|<2\varepsilon 2^{-i}$$

and

$$||y_i||=||z_i||-||z_i-y_i||\geq\frac{1}{2}\beta-2\varepsilon 2^{-i}$$

for all $1\leq i\leq p$. Therefore,

$$\begin{aligned}\left\|\sum_{i=1}^{p}z_i\right\|&\geq\left\|\sum_{i=1}^{p}y_i\right\|-\left\|\sum_{i=1}^{p}(y_i-z_i)\right\|\\&\geq\sum_{i=1}^{p}||y_i||-\sum_{i=1}^{p}||y_i-z_i||\\&\geq\sum_{i=1}^{p}\left(\frac{1}{2}\beta-2\varepsilon 2^{-i}\right)-\sum_{i=1}^{p}2\varepsilon 2^{-i}\\&\geq\frac{1}{2}p\beta-4\varepsilon.\end{aligned}$$

Now let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two tagged partitions of $[0,1]$ that have the same points as $\mathcal{P}_N$. The tags of $\mathcal{P}_1$ and $\mathcal{P}_2$ are u_i and v_i respectively in the intervals $[c_i,d_i]$ for $1\leq i\leq p$ and the tags of $\mathcal{P}_1$ and $\mathcal{P}_2$ are the same in the remaining intervals. Then $|\mathcal{P}_1|<\delta$ and $|\mathcal{P}_2|<\delta$ and we have

$$||f(\mathcal{P}_1)-f(\mathcal{P}_2)||=\left\|\sum_{i=1}^{p}\frac{1}{N}z_i\right\|\geq\frac{1}{2}\frac{p}{N}\beta-\frac{4}{N}\varepsilon\geq\frac{1}{2}\alpha\beta-\frac{1}{4}\alpha\beta=\frac{\alpha\beta}{4}.$$

This completes the proof. □

The list of spaces that do not have the property of Lebesgue contains all of the usual examples of infinite dimensional reflexive spaces. It is natural to ask whether or not there are any infinite dimensional reflexive spaces that have the property of Lebesgue. Such a space can contain no copy of c_0 and can contain no infinite dimensional, uniformly convex subspaces. The Tsirelson space has these properties and da Rocha [**11**] proved that the Tsirelson space has the property of Lebesgue. We will prove this result after presenting the definition of the Tsirelson space as given by Casazza and Shura [**2**].

Let A and B be nonempty finite sets of positive integers. We will write $A \leq B$ if $\max\{n : n \in A\} \leq \min\{n : n \in B\}$ and write $A < B$ if the inequality is strict. Let c_{00} be the vector space of all sequences of real numbers that are finitely nonzero, and let $\{e_n\}$ be the standard unit vector basis of c_{00}. Given $x = \{a_n\} \in c_{00}$ and a subset A of positive integers let $Ax = \sum_{n \in A} a_n$. We define a sequence of norms $\{||\cdot||_m : 0 \leq m < \infty\}$ inductively on c_{00} by letting $||x||_0 = \max\{||a_n|| : 1 \leq n < \infty\}$ and $||x||_{m+1} = \max\{||x||_m, (1/2)\sup\{\sum_{i=1}^k ||A_i x||_m\}\}$ for each $m \geq 0$ where the supremum is taken over all collections of finite subsets $\{A_i\}$ of positive integers such that $\{k\} \leq A_1 < A_2 < \cdots < A_k$ and all positive integers k. It is easy to verify that these norms increase with m on c_{00} and that $||x||_{\ell_\infty} \leq ||x||_m \leq ||x||_{\ell_1}$ for all $x \in c_{00}$ and for all m. Define $||x|| = \lim_{m\to\infty} ||x||_m$. The Tsirelson space T is the $||\cdot||$ completion of c_{00}.

We make note of the following properties of T. All of the proofs can be found in Casazza and Shura. The sequence $\{e_n\}$ is a normalized, unconditional, Schauder basis of T and for each $x \in T$ we have $||x|| = \max\{||x||_{\ell_\infty}, (1/2)\sup\{\sum_{i=1}^k ||A_i x||\}\}$ where the supremum is taken as above. The space T is reflexive, contains no copy of c_0 or ℓ_p for $1 \leq p < \infty$, and contains no infinite dimensional, uniformly convex subspaces.

The proof that Tsirelson space has the property of Lebesgue is similar to the proof that ℓ_1 has the property of Lebesgue. The idea for the proof given below is due to da Rocha [**11**].

Theorem 27. *The Tsirelson space T has the property of Lebesgue.*

Proof. It is sufficient to prove that a bounded function $f : [0,1] \to T$ that is not continuous almost everywhere on $[0,1]$ is not Riemann integrable on $[0,1]$. Let $f : [0,1] \to T$ be bounded but not continuous almost everywhere on $[0,1]$. There exist positive numbers α and β such that $\mu(H) = \alpha$ where $H = \{t \in [0,1] : \omega(f,t) \geq \beta\}$. We will prove that for each $\delta > 0$ there exist tagged partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $[0,1]$ with $|\mathcal{P}_1| < \delta$ and $|\mathcal{P}_2| < \delta$ such that $||f(\mathcal{P}_1) - f(\mathcal{P}_2)|| \geq \alpha\beta/8$. By Theorem 5, the proof will be complete.

Let $\delta > 0$ be given. Choose a positive integer N such that $1/N < \delta$ and let $\mathcal{P}_N = \{k/N : 0 \leq k \leq N\}$. Let $\{[c_i, d_i] : 1 \leq i \leq p\}$ be all of the intervals of $\mathcal{P}_N$ for which $\mu(H \cap (c_i, d_i)) > 0$ and note that $p/N \geq \alpha$. For each positive integer j, let G_j be the set of discontinuities of $e_j f$ on $[0,1]$. If $u(G_j) \neq 0$, then $e_j f$ and consequently f is not Riemann integrable on $[0,1]$. Otherwise the set $G = \cup_j G_j$ has measure zero and every $e_j f$ is continuous on $[0,1] - G$. Let $\varepsilon = \alpha\beta/24$. We will construct sets $\{u_i : 1 \leq i \leq p\}$, where $u_i \in (H - G) \cap (c_i, d_i)$ for each i, $\{v_i : 1 \leq i \leq p\}$, where $v_i \in (c_i, d_i)$ for each i, and $\{n_i : 0 \leq i \leq p\}$, where each n_i is an integer and $p + 1 = n_0 < n_1 < \cdots < n_p$, that have the following properties. Let $z_i = f(u_i) - f(v_i) = \{a_j^i\} = w_i + x_i + y_i$ where $w_i = \sum_{j=1}^{n_{i-1}} a_j^i e_j$, $x_i = \sum_{j=n_{i-1}+1}^{n_i - 1} a_j^i e_j$, and $y_i = \sum_{j=n_i}^{\infty} a_j^i e_j$ for $1 \leq i \leq p$. Then $||z_i|| \geq \beta/2$, $||w_i|| \leq \varepsilon 2^{-i}$, and $||y_i|| \leq \varepsilon 2^{-i}$.

We proceed as follows. Let $n_0 = p + 1$ and choose $u_1 \in (H - G) \cap (c_1, d_1)$. Since $\omega(f, u_1) \geq \beta$ and since $e_j f$ is continuous at u_1 for each $1 \leq j \leq n_0$, there exists a point $v_1 \in (c_1, d_1)$ such that $||f(u_1) - f(v_1)|| \geq \beta/2$ and $\sum_{j=1}^{n_0} |e_j f(u_1) - e_j f(v_1)| < \varepsilon/2$. Let $z_1 = f(u_1) - f(v_1) = \{a_j^1\}$; then $|| \sum_{j=1}^{n_0} a_j^1 e_j || \leq \sum_{j=1}^{n_0} |a_j^1| < \varepsilon/2$. Since $\{e_j\}$ is an unconditional Schauder basis of T, there exists an integer $n_1 > n_0$ such that $|| \sum_{j=n_1}^{\infty} a_j^1 e_j || < \varepsilon/2$. Now choose $u_2 \in (H - G) \cap (c_2, d_2)$. Since $\omega(f, u_2) \geq \beta$, and since $e_j f$ is continuous at u_2 for each $1 \leq j \leq n_1$, there exists a point $v_2 \in (c_2, d_2)$ such that $||f(u_2) - f(v_2)|| \geq \beta/2$ and $\sum_{j=1}^{n_1} |e_j f(u_2) - e_j f(v_2)| < \varepsilon/4$. Let $z_2 = f(u_2) - f(v_2) = \{a_j^2\}$; then $|| \sum_{j=1}^{n_1} a_j^2 e_j || \leq \sum_{j=1}^{n_1} |a_j^2| \leq \varepsilon/4$. Choose an integer $n_2 > n_1$ such that $|| \sum_{j=n_2}^{\infty} a_j^2 e_j || < \varepsilon/4$. We continue this process for p steps and arrive at the desired sets.

Let $A_i = \{n_{i-1}+1, \dots, n_i - 1\}$ for each $1 \le i \le p$. By the property of the norm on T we find that

$$\left\| \sum_{i=1}^{p} x_i \right\| \ge \frac{1}{2} \sum_{j=1}^{p} \left\| A_j \left(\sum_{i=1}^{p} x_i \right) \right\| = \frac{1}{2} \sum_{j=1}^{p} ||x_i||.$$

We then have

$$\begin{aligned}
\left\| \sum_{i=1}^{p} z_i \right\| &\ge \left\| \sum_{i=1}^{p} x_i \right\| - \left\| \sum_{i=1}^{p} (w_i + y_i) \right\| \\
&\ge \frac{1}{2} \sum_{j=1}^{p} ||x_i|| - \sum_{i=1}^{p} (||w_i|| + ||y_i||) \\
&\ge \frac{1}{2} \sum_{j=1}^{p} (||z_i|| - (||w_i|| + ||y_i||)) - \sum_{i=1}^{p} (||w_i|| + ||y_i||) \\
&\ge \frac{1}{2} \sum_{j=1}^{p} ||z_i|| - \frac{3}{2} \sum_{i=1}^{p} (||w_i|| + ||y_i||) \\
&\ge \frac{1}{4} p\beta - 3\varepsilon.
\end{aligned}$$

Now let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two tagged partitions of $[0,1]$ that have the same points as $\mathcal{P}_N$. The tags of $\mathcal{P}_1$ and $\mathcal{P}_2$ are u_i and v_i, respectively, in the intervals $[c_i, d_i]$ for $1 \le i \le p$, and the tags of $\mathcal{P}_1$ and $\mathcal{P}_2$ are the same in the remaining intervals. Then $|\mathcal{P}_1| < \delta$ and $|\mathcal{P}_2| < \delta$ and we have

$$||f(\mathcal{P}_1) - f(\mathcal{P}_2)|| = \left\| \sum_{i=1}^{p} \frac{1}{N} z_i \right\| \ge \frac{1}{4} \frac{p}{N} \beta - \frac{3}{N} \varepsilon \ge \frac{1}{4} \alpha\beta - \frac{1}{8} \alpha\beta = \frac{\alpha\beta}{8}.$$

This completes the proof. $\square$

We now take a brief look at weak versions of the Riemann integral. Most of the results in this section are due to Alexiewicz and Orlicz [**1**].

Definition 28. Let $f : [a,b] \to X$. The function f is scalarly Riemann integrable on $[a,b]$ if $x^* f$ is Riemann integrable on $[a,b]$ for each x^* in X^*. If, in addition, for each interval $I \subset [a,b]$ there is a vector x_I in X such that $x^*(x_I) = \int_I x^*f$ for all x^* in X^*, then f is Riemann–Pettis integrable on $[a,b]$.

It is clear that every Riemann integrable function is Riemann–Pettis integrable and that every scalarly Riemann integrable function is Dunford integrable. A simple application of the Uniform Boundedness Principle shows that every scalarly Riemann integrable function is bounded. Consequently, a measurable, scalarly Riemann integrable function is Bochner integrable and a Riemann–Pettis integrable function is Pettis integrable (see the proof of Theorem 15).

Theorem 29. *A measurable, scalarly Riemann integrable function is Riemann–Pettis integrable. Consequently, in a separable space every scalarly Riemann integrable function is Riemann–Pettis integrable.*

Proof. Let $f : [a,b] \to X$ be measurable and scalarly Riemann integrable on $[a,b]$. Then f is Bochner integrable and hence Pettis integrable on $[a,b]$. It follows that f is Riemann-Pettis integrable on $[a,b]$. □

Corollary 30. *A bounded function that is weakly continuous almost everywhere is Riemann–Pettis integrable.*

Proof. Let $f : [a,b] \to X$ be bounded and weakly continuous almost everywhere on $[a,b]$. Then f is measurable and scalarly Riemann integrable on $[a,b]$. By Theorem 29, the function f is Riemann–Pettis integrable on $[a,b]$. □

Theorem 31. *In a weakly sequentially complete space every scalarly Riemann integrable function is Riemann–Pettis integrable.*

Proof. Let X be weakly sequentially complete and let $f : [a,b] \to X$ be scalarly Riemann integrable on $[a,b]$. Let $[c,d] \subset [a,b]$. We must show that there exists a vector z in X such that $x^*(z) = \int_c^d x^*f$ for all x^* in X^*.

For each positive integer n, let $\mathcal{P}_n$ be a tagged partition of $[c,d]$ with points $\{c + (k/n)(d - c) : 0 \leq k \leq n\}$. Since each x^*f is Riemann integrable on $[c,d]$, the sequence $\{f(\mathcal{P}_n)\}$ is a weak Cauchy sequence and since X is weakly sequentially complete this sequence converges weakly to a vector $z \in X$. For each x^* in X^* we have $x^*(z) = \lim_{n\to\infty} x^* f(\mathcal{P}_n) = \int_c^d x^* f$. This completes the proof. □

The distinction between the Riemann integral and the Riemann–Pettis integral is simply whether or not the Riemann sums converge in the norm topology or in the weak topology. This observation leads to the next two results. The first is due to Alexiewicz and Orlicz [**1**].

Theorem 32. *A scalarly Riemann integrable function that has a relatively compact range is Riemann integrable and in fact Darboux integrable.*

Proof. Let $f : [a,b] \to X$ be scalarly Riemann integrable on $[a,b]$ and suppose that the range of f is relatively compact. It is easily seen that f is measurable and hence Riemann–Pettis integrable on $[a,b]$ by Theorem 29. Let z be the vector in X such that $x^*(z) = \int_a^b x^* f$ for all x^* in X^*. We first show that f is Riemann integrable on $[a,b]$.

Let $V = \{f(t) : t \in [a,b]\}$, let V_1 be the closed convex hull of the closure of V, and let $W = (b - a)V_1$. Then W is a compact set and W contains all of the Riemann sums of f. Suppose that f is not Riemann integrable on $[a,b]$. Then there exists $\eta > 0$ such that for each $\delta > 0$ there exists a tagged partition $\mathcal{P}_\delta$ of $[a,b]$ such that $|P_\delta| < \delta$ and $||f(\mathcal{P}_\delta) - z|| \geq \eta$. For each positive integer n, choose a tagged partition $\mathcal{P}_n$ of $[a,b]$ such that $|\mathcal{P}_n| < 1/n$ and $||f(\mathcal{P}_n) - z|| \geq \eta$. Since z is the Riemann–Pettis integral of f on $[a,b]$, the sequence $\{f(\mathcal{P}_n)\}$ converges weakly to z, and since W is compact the sequence $\{f(\mathcal{P}_n)\}$ must converge in norm to z. This contradiction establishes the Riemann integrability of f on $[a,b]$.

Since f is bounded, to prove that f is Darboux integrable on $[a,b]$, it is sufficient to prove that f is continuous almost everywhere on $[a,b]$. Since V_1 is separable, there exists a sequence $\{x_n^*\}$ in X such that $||v|| = \sup_n |x_n^*(v)|$ for all v in V_1. For each n, let D_n be the set of discontinuities of $x_n^* f$ on $[a, b]$ and let $D = \cup_n D_n$. Then $\mu(D) = 0$ and we will show that f is continuous on $[a, b] - D$.

Let $t \in [a, b] - D$ and let $\{t_k\}$ be a sequence in $[a, b]$ that converges to t. For each n, the sequence $\{x_n^* f(t_k)\}$ converges to $x_n^* f(t)$. Since $\{x_n^*\}$ separates the points of V_1 and since V_1 is compact, the sequence $\{f(t_k)\}$ converges in norm to $f(t)$. This shows that f is continuous at t. □

Theorem 33. *If X is a Schur space, then every function $f : [a, b] \to X$ that is scalarly Riemann integrable on $[a, b]$ is Riemann integrable on $[a, b]$.*

Proof. Since a Schur space is weakly sequentially complete, the function f is Riemann–Pettis integrable on $[a, b]$ by Theorem 31. Since every weakly convergent sequence in X converges in norm, the Riemann integrability of f on $[a, b]$ follows as in the first part of the proof of Theorem 32. □

The next theorem is due to da Rocha [**11**]. It shows in particular that in ℓ_1 scalar Riemann integrability and Darboux integrability are equivalent notions.

Theorem 34. *A Banach space X is a Schur space and has the property of Lebesgue if and only if every scalarly Riemann integrable function $f : [a, b] \to X$ is Darboux integrable.*

Proof. Suppose that X is a Schur space and has the property of Lebesgue, and let $f : [a, b] \to X$ be scalarly Riemann integrable on $[a, b]$. By the previous theorem, the function f is Riemann integrable on $[a, b]$ and hence Darboux integrable on $[a, b]$ since X has the property of Lebesgue.

Now suppose that X is not a Schur space. There exists a sequence $\{x_n\}$ in X such that $||x_n|| \geq 1$ for all n and $\{x_n\}$ converges weakly to θ. Let $\{r_n\}$ be a listing of the rational numbers in $[0, 1]$ and define $f : [0, 1] \to X$ by $f(t) = \theta$ if t is irrational and $f(t) = x_n$ if $t = r_n$. Since f is not continuous almost everywhere on $[0, 1]$, it is not Darboux integrable on $[0,1]$. However, the function f is bounded and weakly continuous almost everywhere on $[0,1]$ and, therefore, scalarly Riemann integrable on $[0,1]$. This completes the proof since the case in which X does not have the property of Lebesgue is trivial. □

We conclude this paper with an example due to Alexiewicz and Orlicz [**1**] to illustrate that weak continuity need not imply Riemann integrability. Since by Corollary 30 a weakly continuous function is Riemann–Pettis integrable, the example also shows that a Riemann–Pettis integrable function may fail to be Riemann integrable.

Example 35. A weakly continuous function that is not Riemann integrable:

Let H be a perfect, nowhere dense subset of $[0,1]$ with $\mu(H) \geq 3/4$, and let $(0,1) - H = \cup_k (a_k, b_k)$. For each pair of positive integers k and $n \geq 2$, let

$$E_k^n = \{a_k, a_k + (b_k - a_k)/(2n), a_k + (b_k - a_k)/n, \\ b_k - (b_k - a_k)/n, b_k - (b_k - a_k)/(2n), b_k\},$$

and let ϕ_k^n be the function that equals 1 at $a_k + (b_k - a_k)/(2n)$ and $b_k - (b_k - a_k)/(2n)$, equals 0 at the other points of E_k^n, and is linear on the intervals contiguous to E_k^n. For each n, let $f_n(t) = \sum_{k=1}^n \phi_k^n(t)$. Then the sequence $\{f_n\}$ converges pointwise to the zero function on $[0,1]$ and $\int_0^1 f_n = \sum_{k=1}^n \int_0^1 \phi_k^n(t) = (1/n)\sum_{k=1}^n (b_k - a_k)$.

Define $f : [0,1] \to c_0$ by $f(t) = \{f_n(t)\}$. We will first prove that f is weakly continuous. Let $x^* = \{\alpha_n\} \in \ell_1$, then $x^* f = \sum_n \alpha_n f_n$. Since each $\alpha_n f_n$ is continuous on $[0,1]$ and $|\alpha_n f_n| \leq |\alpha_n|$ on $[0,1]$, the function $x^* f$ is continuous on $[0,1]$ being the uniform limit of continuous functions. This shows that f is weakly continuous.

To prove that f is not Riemann integrable on $[0,1]$, it is sufficient to prove that for each $\delta > 0$ there exist a tagged partition $\mathcal{P}$ of $[0,1]$ and an integer j_0 such that $|\mathcal{P}| < \delta$ and $||f_{j_0}(\mathcal{P}) - \int_0^1 f_{j_0}|| \geq 1/2$. Let $\delta > 0$ be given. Since H is nowhere dense, there exists a partition $\{t_m : 0 \leq m \leq M\}$ of $[0,1]$ such that $t_m \notin H$ for $1 \leq m \leq M-1$ and $t_m - t_{m-1} < \delta$ for $1 \leq m \leq M$. Let $\{I_k : 1 \leq k \leq N\}$ be the intervals of this partition that contain points of H in their interiors and let $\{K_i : 1 \leq i \leq L\}$ be the remaining intervals. For each k, there exists n_k such that $(a_{n_k}, b_{n_k}) \subset I_k$. Let $j_0 = \max\{n_k : 1 \leq k \leq N\}$, and for each k choose $t_k \in (a_{n_k}, b_{n_k})$ such that $\phi_{n_k}^{j_0}(t_k) = 1$. Let $s_i \in K_i$ be arbitrary and let $\mathcal{P} = \{(t_k, I_k) : 1 \leq k \leq N\} \cup \{(s_i, K_i) : 1 \leq i \leq L\}$. Then $\mathcal{P}$ is a tagged partition of $[0,1]$ with $|\mathcal{P}| < \delta$, and we have

$$\begin{aligned} f_{j_0}(\mathcal{P}) - \int_0^1 f_{j_0} &= \sum_{k=1}^{N} f_{j_0}(t_k)\mu(I_k) + \sum_{i=1}^{L} f_{j_0}(s_i)\mu(K_i) - \int_0^1 f_{j_0} \\ &= \sum_{k=1}^{N} \mu(I_k) - \frac{1}{j_0}\sum_{k=1}^{j_0}(b_k - a_k) \\ &\geq \mu(H) - (1 - \mu(H)) \\ &\geq \frac{1}{2}. \end{aligned}$$

This shows that f is not Riemann integrable on $[0,1]$.

REFERENCES

1. A. Alexiewicz and W. Orlicz, *Remarks on Riemann-integration of vector-valued functions*, Studia Math. **12** (1951), 125–132.

2. P.G. Casazza and T. Shura, *Tsireĺson's space*, Springer-Verlag, Berlin-New York, 1989.

3. J. Diestel and J.J. Uhl, *Vector measures*, American Mathematical Society, Providence, RI, 1977.

4. L.M. Graves, *Riemann integration and Taylor's theorem in general analysis*, Trans. Amer. Math. Soc. **29** (1927), 163–177.

5. R.C. James, *Super-reflexive spaces with bases*, Pacific J. Math. (2) **41** (1972), 409–419.

6. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*, I, Springer-Verlag, New York, 1977.

7. A.S. Nemirovskii, M.Yu. Ochan, and R. Rejouani, *Conditions for Riemann integrability of functions with values in a Banach space*, Vestnik Moskov. Univ. Ser. I. Mat. Meh. no. 4, **27** (1972), 62–65.

8. B.J. Pettis, *On integration in vector spaces*, Trans. Amer. Math. Soc. **44** (1938), 277–304.

9. R. Rejouani, *On the question of the Riemann integrability of functions with values in a Banach space*, Vestnik Moskov. Univ. Ser. I Mat. Meh. no. 4 **26** (1971), 75–79.

10. ———, *On the question of strong and weak (R)-integrability of abstract functions*, Vestnik Moskov. Univ. Ser. I Mat. Meh. no. 6, **26** (1971), 26–31.

11. G.C. da Rocha, *Integral de Riemann vetoriel e geometri de espaços de Banach*, Ph.D. thesis, Universidade de Sao Paulo, 1979.

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