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STABILITY OF A FAKE TOPOLOGICAL HILBERT SPACE

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ABSTRACT. The space under consideration is the basic fake Hilbert space Y of Anderson, Curtis and van Mill. It is shown that the product of an arbitrary space A with Y is homeomorphic to Y if and only if A is a compact absolute retract. Furthermore, we prove that the complement of $Y \times Y$ is a capset in $Q \times Q$, which implies the known result that $Y \times Y$ is homeomorphic to Hilbert space.

1. Introduction. We are interested in the basic fake Hilbert space Y that was constructed by Anderson, Curtis and van Mill [**1**]. The space Y is the complement of a σZ-set in the Hilbert cube Q and, hence, a complete AR. The following properties can be found in [**1**] and illustrate the closeness of Y to the Hilbert space ℓ^2: (a) Y is homogeneous, (b) $Y \times Y$ is homeomorphic to ℓ^2, and (c) Y has the weak discrete approximation property. The space has proved to be a very useful basis for the construction of other peculiar spaces and counterexamples as is witnessed by the papers of Anderson et al. [**1**], Dijkstra and van Mill [**8**], Dijkstra [**7**], and Bowers [**3**]. More information on Y can be found in Dijkstra [**6**, Chapters 4 and 5]. The most important results here are the Unknotting Theorem (homeomorphisms between compacta in Y can be extended with control) and the Negligibility Theorem (the negligible compacta in Y are precisely the compacta with the shape of a finite set).

In this article we investigate the stability of Y under multiplication. The result $Y \times Y \approx \ell^2$ can be improved by showing that the complement of $Y \times Y$ in $Q \times Q$ is a capset. We are mainly interested, however, in determining for which spaces A the product $Y \times A$ is homeomorphic to Y. We show that this is the case precisely if A is a compact absolute retract.

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All spaces in this paper are Tychonoff. Absolute retracts are assumed to be separable metric. For undefined terms from infinite-dimensional topology, see, e.g., Bessaga and Pełczynski [**2**].

2. Preliminaries. In this section we introduce the fake Hilbert space Y and we state several facts that will be used in the next sections.

If $\varepsilon > 0$, then $Q(\varepsilon)$ denotes the space $\prod_{i=1}^{\infty}[-\varepsilon,\varepsilon]_i$ equipped with the product topology. The standard representation of the Hilbert cube Q is $Q(1) = J^{\mathbf{N}}$, where $J = [-1,1]$. Let $\mathcal{R}$ be the set of all sequences $p_1, p_2, p_3, \dots$ in the interval $(0,1)$ such that $\lim_{i\to\infty} p_i = 1$ and let $\mathcal{R}^{\dagger}$ be the subset consisting of all strictly increasing sequences. If $(p_i)_{i=1}^{\infty}$ and $(q_i)_{i=1}^{\infty}$ are elements of $\mathcal{R}$, then $(p_i)_{i=1}^{\infty} < (q_i)_{i=1}^{\infty}$ means that $p_i < q_i$ for every $i \in \mathbf{N}$. Select a $p = (p_i)_{i=1}^{\infty}$ in $\mathcal{R}$. For every natural number n we define the *shrunken endface in the n-coordinate direction* by

$$W_n = [-p_n,p_n]_1 \times \cdots \times [-p_n,p_n]_{n-1} \times \{1\}_n \times [-p_n,p_n]_{n+1} \times [-p_n,p_n]_{n+2} \times \cdots \subset Q.$$

Note that W_n is itself a Hilbert cube and that it is a Z-set in Q. Observe, furthermore, that the W_n's are pairwise disjoint. Let A be an infinite subset of $\mathbf{N}$. We define

$$Y(A) = Q \Big\backslash \bigcup_{n\in A} W_n.$$

Since $\lim_{n\to\infty} p_n = 1$ there exists a sequence of maps $\alpha_n : Q \to W_n$ such that $\lim_{n\to\infty} \alpha_n = 1_Q$, where 1_Q denotes the identity mapping on Q. This implies that the complement of $Y(A)$ in Q is both dense and connected. Moreover, it follows that every compact subset of $Y(A)$ is a Z-set in Q. The fake Hilbert space Y is represented by $Y(\mathbf{N})$.

Definition. If $X' \subset X$ and $Z' \subset Z$ then we say that the pair (X',X) is homeomorphic to the pair (Z',Z), notation $(X',X) \approx (Z',Z)$, if there is a homeomorphism $h : X \to Z$ such that $h(X') = Z'$.

It is shown in Dijkstra [**6**, 4.4.3] that every $p \in \mathcal{R}$ leads to the same topological type (Y,Q). If X is a space, then $\mathcal{H}(X)$ denotes the group of autohomeomorphisms of X. The following theorem was taken from Dijkstra [**6**, 4.3.6].

The Unknotting Theorem. *Let $\mathcal{U}$ be an open covering of Q, let C be a compact metric space and let $F : C \times [0,1] \to Q$ be a homotopy that is limited by $\mathcal{U}$ (i.e., the paths $F(\{c\} \times [0,1])$ are contained in elements of $\mathcal{U}$). If F_0 and F_1 are imbeddings of C in Y, then there is an $h \in \mathcal{H}(Q)$ such that $h \circ F_0 = F_1$, h is $\mathcal{U}$-close to 1_Q and $h(W_n) = W_n$ for every n.*

The following theorem will be used several times in Section 3.

The Sierpiński Theorem [**12**]. *No continuum can be partitioned into countably many pairwise disjoint nonempty closed sets.*

A space is called *continuum-connected* if for every two points of the space there is a continuum that contains them both. The Sierpiński Theorem is also valid for continuum-connected spaces. The *continuum-components* of a space are maximally continuum-connected subspaces.

3. Stability of Y. In this section we show that Y is stable under multiplication with compact absolute retracts only. We also consider a few other fake Hilbert spaces.

Lemma 1. *If A is an infinite subset of* $\mathbf{N}$, *then* $(Y(A), Q) \approx (Y, Q)$.

Proof. It is shown in Dijkstra [**6**, 4.4.4] that individual shrunken endfaces can be deleted. This proves that $(Y(A), Q) \approx (Y, Q)$ if the complement of A is finite.

Consider now the case that A has an infinite complement. Precisely as for (Y, Q), we have that every choice of $p \in \mathcal{R}$ leads to the same topological type $(Y(A), Q)$. Moreover, if B is another set with infinite complement, then a simple permutation of coordinates shows that $(Y(B), Q) \approx (Y(A), Q)$. So it suffices to show that (Y, Q) is homeomorphic to, for instance, $(Y(\mathbf{N}_{\text{even}}), Q)$, or, separating even and odd coordinates, that (Y, Q) is homeomorphic to the pair $(X, Q \times Q)$ where

$$X = (Q \times Q) \Big\backslash \bigcup_{i=1}^{\infty} (W_i \times Q(p_i)).$$

Consider the pair (Y,Q). According to Dijkstra [**6**, 4.4.4] there exists a $\chi \in \mathcal{H}(Q)$ such that $\chi(W_{2i-1}) = W_{2i-1}$ and

$$\begin{aligned}\chi(W_{2i}) = W^{-}_{2i-1} = &[-p_{2i},p_{2i}]_1 \times \cdots \times [-p_{2i},p_{2i}]_{2i-2} \\ &\times \{-1\}_{2i-1} \times [-p_{2i},p_{2i}]_{2i} \times [-p_{2i},p_{2i}]_{2i+1} \times \cdots\end{aligned}$$

for every $i \in \mathbf{N}$. This means that in the pair $(\chi(Y),Q)$ the even coordinates are interchangeable. Doubling the even coordinates we see that $(\chi(Y),Q)$ is homeomorphic to $(Z,Q\times Q)$ where

$$Z = (Q\times Q)\Big\backslash \bigcup_{i=1}^{\infty}((W^{-1}_{2i-1} \times Q(p_{2i})) \cup (W_{2i-1}\times Q(p_{2i-1}))).$$

Observe now that $\chi^{-1}\times 1_Q$ is a homeomorphism from $(Z,Q\times Q)$ onto $(X,Q\times Q)$. This proves the Lemma. □

Lemma 2. $(Y,Q) \approx (Y\times Q, Q\times Q)$.

Proof. This proof uses Anderson's Convergence Criterion which states that if every element of a sequence $(h_i)_{i=1}^{\infty}$ in $\mathcal{H}(Q)$ can be chosen arbitrarily close to 1_Q, then there is a selection possible such that $\lim_{n\to\infty} h_n \circ \cdots \circ h_1 \in \mathcal{H}(Q)$ (see Dijkstra [**6**, 1.1.2] for a precise formulation).

Consider the Unknotting Theorem. Observe that the shrunken endfaces are the continuum-components of $Q\backslash Y$ (Sierpiński) and, hence, the Theorem is a topological statement about the pair (Y,Q). This means that the Theorem is valid for any pair (C,D) that is homeomorphic to (Y,Q). Specifically, consider a pair $(Z,Q\times Q)$ that is homeomorphic to (Y,Q) and let ρ be a standard convex metric on $Q\times Q$. Using a straight line homotopy the Unknotting Theorem leads to: if $\varepsilon > 0$, C and D are compact subsets of Z and h is a homeomorphism from C onto D with $\rho(h, 1_{Q\times Q}) < \varepsilon$, then there is a $\bar{h} \in \mathcal{H}(Q\times Q)$ which extends h and which has the properties $\rho(\bar{h}, 1_{Q\times Q}) < \varepsilon$ and $\bar{h}(W) = W$ for every continuum-component W of $(Q\times Q)\backslash Z$.

Consider now $(X, Q\times Q)$ where

$$X = Q\times Q\Big\backslash \bigcup_{i=1}^{\infty} W_i \times Q(p_i).$$

Let n_1 be a natural number. Lemma 1 implies that $(Q\times Q\backslash\cup_{i>n_1} W_i\times Q(p_i), Q\times Q)\approx(Y,Q)$, and, hence, there exists an $h_1\in\mathcal{H}(Q\times Q)$ with $h_1(W_i\times Q(p_i))=W_i\times Q(p_i)$ if $i>n_1$ and $h_1(W_{n_1}\times Q(p_{n_1}))=W_{n_1}\times Q$. By choosing n_1 large we can get p_{n_1} as close to 1 as we want, which means that we can make the distance of h_1 towards $1_{Q\times Q}$ arbitrarily small. As step 2 of the induction select an $n_2>n_1$ and an $h_2\in\mathcal{H}(Q\times Q)$ such that h_2 fixes $W_{n_1}\times Q$, $h_2(W_{n_2}\times Q(p_{n_2}))=W_{n_2}\times Q$ and $h_2(W_i\times Q(p_i))=W_i\times Q(p_i)$ if $i>n_2$. Continue this process. Since every h_n can be chosen arbitrarily close to $1_{Q\times Q}$ we may assume that $h=\lim_{n\to\infty} h_n\circ\cdots\circ h_1\in\mathcal{H}(Q\times Q)$. Note that h has the property $h(W_{n_i}\times Q(p_{n_i}))=W_{n_i}\times Q$ for $i\in\mathbf{N}$. Consequently, we have $(Q\times Q\backslash\cup_{i=1}^{\infty} W_{n_i}\times Q(p_{n_i}), Q\times Q)\approx(Y(A)\times Q, Q\times Q)$ where $A=\{n_i\mid i\in\mathbf{N}\}$. According to Lemma 1, the first pair is homeomorphic to (Y,Q) and the second to $(Y\times Q, Q\times Q)$.

Lemma 3. *If a connected space is a product of two noncompact spaces, then it has a continuum-connected remainder in any compactification.*

Proof. Consider two connected spaces X_1 and X_2, neither of them compact. Let C be a compactification of $X_1\times X_2$ and put $R=C\backslash(X_1\times X_2)$. Select for $i=1,2$ a filter $\mathcal{F}_i$ in X_i without cluster points. Let p be a cluster point in C of the filter $\mathcal{F}$ that is generated by $\{F_1\times F_2\mid F_i\in\mathcal{F}_i\}$ and note that $p\in R$. We shall show that for any $q\in R$ there is a continuum in R that contains both p and q.

Let q be an arbitrary point of R and select an ultrafilter $\mathcal{G}$ in C that converges to q and contains the set $X_1\times X_2$. Let for $i=1,2$ the ultrafilter $\mathcal{G}_i$ be given by

$$\mathcal{G}_i=\{\pi_i(G)\mid G\in\mathcal{G}\text{ and }G\subset X_1\times X_2\},$$

where $\pi_i: X_1\times X_2\to X_i$ is the projection. Observe that since $\mathcal{G}_1$ and $\mathcal{G}_2$ are ultrafilters and $\mathcal{G}$ has no cluster points in $X_1\times X_2$ we find that $\mathcal{G}_1$ or $\mathcal{G}_2$ has no cluster points in X_1, respectively, X_2. We may assume without loss of generality that $\mathcal{G}_2$ has no cluster points. Consider the collection

$$\mathcal{K}=\{\mathrm{cl}_C((X_1\times G)\cup(F\times X_2))\mid F\in\mathcal{F}_1\text{ and }G\in\mathcal{G}_2\}.$$

Note that since X_1 and X_2 are connected and F and G are nonempty, the set $(X_1 \times G) \cup (F \times X_2)$ is connected and, hence, $\mathcal{K}$ is a collection of continua. Observe that the inclusion relation makes an inverse system of $\mathcal{K}$ and, hence, the limit $\cap\mathcal{K}$ is a continuum, see Engelking [**9**, 6.1.18]. Since $\mathcal{K} \subset \mathcal{F} \cap \mathcal{G}$, we have $\{p, q\} \subset \cap\mathcal{K}$. It remains to verify that $\cap\mathcal{K}$ is contained in R. Let x be an arbitrary point of $X_1 \times X_2$. Since $\mathcal{F}_1$ and $\mathcal{G}_2$ have no cluster points there exists a neighborhood $U \times V$ of x in $X_1 \times X_2$ such that $X_1 \backslash U \in \mathcal{F}_1$ and $X_2 \backslash V \in \mathcal{G}_2$. Consequently, the set

$$\mathrm{cl}_C((X_1 \times (X_2 \backslash V)) \cup ((X_1 \backslash U) \times X_2))$$

is an element of $\mathcal{K}$ that does not contain x. So we may conclude that $\cap\mathcal{K} \cap (X_1 \times X_2) = \varnothing$.

Stability Theorem. *The product $Y \times A$ is homeomorphic to Y if and only if A is a compact absolute retract.*

Proof. Let A be a compact AR. According to Edwards (see Chapman [**4**, §44]) we have $A \times Q \approx Q$. This leads to

$$Y \approx Y \times Q \approx Y \times Q \times A \approx Y \times A.$$

Now assume that $Y \times A \approx Y$. The space A is obviously a complete AR. Since $Q \backslash Y$ is a countable disjoint union of Hilbert cubes, we have according to Sierpiński that it is not continuum-connected. With Lemma 3 we find that A is compact. □

A more ambitious task is to find all factors of Y. We make the following

Conjecture. *If $Y \cong A \times B$, then $Y \cong A$ and B is a compact* AR *or vice versa.*

Note that if $Y \approx A \times B$ then according to Lemma 3 one of the factors, say B, is compact and, hence, $A \times Q \approx A \times B \times Q \approx Y \times Q \approx Y$. So the conjecture is equivalent with the statement: if $A \times Q \approx Y$, then $A \approx Y$. Note that this statement is true for the Hilbert space ℓ^2 instead of Y, Mogilski [**11**]. Unfortunately, Mogilski's proof leans heavily on the unknotting theorem for noncompact Z-sets in ℓ^2. A similar theorem does not exist for Y (see Dijkstra [**6**, 4.3.10]).

We shall now consider two derived fake Hilbert spaces. The first one appears in Anderson et al. [**1**] and is obtained by deleting a countable dense subset D from Y. The second one can be found in Dijkstra and van Mill [**9**] and is obtained by deleting a so-called 0-dimensional capset A_0 from Y (A_0 is a dense copy of the product of the Cantor set and the set of rational numbers). Both spaces are complete AR's.

Proposition 1. *If X is either $Y\backslash D$ or $Y\backslash A_0$, then $X \times A \approx X$ only if A is a singleton.*

Proof. Assume that $X \times A \approx X$ and note that $Q\backslash X$ is not continuum-connected. In fact, the Sierpiński Theorem implies that the continuum-components of $Q\backslash X$ are the shrunken endfaces plus the singletons of D, respectively A_0. So, according to Lemma 3 the factor A is compact.

Let f be the homeomorphism from X onto $X \times A$. Since $Q \times A$ is a Hilbert cube (Edwards) we may apply Lemma 3.6 of Anderson et al. [**1**] to find a compact space M and monotone maps $\alpha : M \to Q$ and $\beta : M \to Q \times A$ such that $\alpha^{-1}(X) = \beta^{-1}(X \times A)$ and $f \circ \alpha \mid \alpha^{-1}(X) = \beta \mid \alpha^{-1}(X)$. Recall that a monotone map is a continuous, closed surjection with the property that the preimage of each connected set is also connected. This means that the preimage under α (or β) of a continuum-component of $Q\backslash X$ (or $(Q\backslash X) \times A$) is a continuum-component of $M\backslash\alpha^{-1}(X)$. Since the singletons of D, respectively A_0, are continuum-components in $Q\backslash X$, it is possible to find a sequence $(C_i)_{i=1}^{\infty}$ of continuum-components of $Q\backslash X$ that has only one cluster point in Q, namely, some point x in X. So we have that every $\alpha^{-1}(C_i)$ is a continuum-component of $M\backslash\alpha^{-1}(X)$ and that every $\beta(\alpha^{-1}(C_i))$ is a continuum-component of $(Q\backslash X) \times A$. Since A is a continuum this implies that each $\beta(\alpha^{-1}(C_i))$ has the form $F_i \times A$. This means that the set of cluster points of $\beta(\alpha^{-1}(C_i))_{i=1}^{\infty}$ in $Q \times A$ also has the form $F \times A$. On the other hand, $F \times A$ is contained in $\beta(\alpha^{-1}(\{x\})) = \{f(x)\}$. Since $F \times A$ is nonempty we may conclude that A is a singleton.

4. The pair $(\mathbf{Y} \times \mathbf{Y}, \mathbf{Q} \times \mathbf{Q})$. In Anderson et al. [**1**] it is shown that $Y \times Y$ is homeomorphic to ℓ^2. The aim of this section is to prove a slightly stronger result, namely that $(Y \times Y, Q \times Q)$ is homeomorphic to (s, Q), where s is the pseudo-interior of Q. This means that we have to show that $Q \times Q \backslash Y \times Y$ is a capset in $Q \times Q$. We shall use the following theorem.

The Capset Characterization Theorem (Curtis [**5**, Corollary 4.9]). *Let $(B_i)_{i=1}^{\infty}$ be a sequence of subsets of Q. If $(B_i)_{i=1}^{\infty}$ satisfies the properties*

(1) *each B_i is a* Z-*set in Q,*

(2) *each B_i is homeomorphic to Q,*

(3) *each B_i is a* Z-*set in B_{i+1}, and*

(4) *there is a homotopy $H : Q \times [0,1] \to Q$ such that $H_0 = 1_Q$ and, for every $t \in (0,1]$, there is an $n \in \mathbf{N}$ such that $H(Q \times [t,1]) \subset B_n$,*

then $(B_i)_{i=1}^{\infty}$ is a capset.

Proposition 2. $(Y \times Y, Q \times Q) \approx (s, Q)$.

Proof. In order to show that $Q \times Q \backslash Y \times Y$ is a capset in $Q \times Q$ it suffices to prove that it contains a capset, see Bessaga and Pełczyński [**2**, Theorem IV.4.2]. We introduce some notations. If $p \in \mathcal{R}^{\dagger}$, then the shrunken endfaces it determines are denoted by $W_n(p)$. Furthermore, let $C_n(p)$ stand for the set

$$[-p_{n+1}, p_{n+1}]_1 \times \cdots \times [-p_{n+1}, p_{n+1}]_{n-1} \times J_n \times J_{n+1} \\ \times [-p_{n+1}, p_{n+1}]_{n+2} \times [-p_{n+1}, p_{n+1}]_{n+3} \times \cdots .$$

Note that $C_n(p)$ is just as $W_n(p)$ a convex Z-set in Q that is homeomorphic to Q. Moreover, $C_n(p)$ contains $W_n(p)$ and $W_{n+1}(p)$ as Z-sets but does not meet any of the other shrunken endfaces. Let $B_n(p)$ be given by

$$B_n(p) = (W_1(p) \times C_1(p)) \cup (C_1(p) \times W_2(p)) \cup (W_2(p) \times C_2(p)) \\ \cup (C_2(p) \times W_3(p)) \cup \cdots \cup (W_n(p) \times C_n(p)) \cup (C_n(p) \times W_{n+1}(p)).$$

Note that in this union only adjacent terms have a nonempty intersection. We need the following fact: if X is a space that is the union of two Hilbert cubes X_1 and X_2 such that $X_1 \cap X_2 = X_0$ is also a Hilbert cube and, moreover, a Z-set in both X_1 and X_2, then X is a Hilbert cube. This result can easily be obtained by observing that each pair (X_0, X_1) and (X_0, X_2) is homeomorphic to (F, Q) where F is an end-face of Q, or it can be seen as a special case of a much stronger theorem by Handel [**10**]. Moreover, if Z is a subset of X such that $Z \cap X_i$ is a Z-set in X_i for $i = 0, 1, 2$, then one easily verifies that Z is a Z-set in X. Note that $(W_i(p) \times C_i(p)) \cap (C_i(p) \times W_{i+1}(p)) = W_i(p) \times W_{i+1}(p)$ and that $(C_i(p) \times W_{i+1}(p)) \cap (W_{i+1}(p) \times C_i(p)) = W_{i+1}(p) \times W_{i+1}(p)$ and, hence, every set $B_n(p)$ is a Hilbert cube. If $p < q \in \mathcal{R}^\dagger$, then $W_n(p)$ is a Z-set in $W_n(q)$ and $C_n(p)$ is a Z-set in $C_n(q)$. A tedious but straightforward argument involving the second part of the aforementioned fact now yields that $B_n(p)$ is a Z-set in $B_n(q)$.

Select a sequence $p^1 < p^2 < p^3 < \cdots$ in $\mathcal{R}^\dagger$ that has an upper bound $q \in \mathcal{R}^\dagger$. The sequence that satisfies the Capset Characterization Theorem is $(B_n(p^n))_{n=1}^\infty$. It is obvious that every $B_n(p^n)$ is contained in

$$\bigcup_{i=1}^\infty (W_i(q) \times Q) \cup (Q \times W_i(q)) = Q \times Q \backslash Y \times Y$$

and, hence, every $B_n(p^n)$ is a Z-set in $Q \times Q$. Every $B_n(p^n)$ is a Hilbert cube and $B_n(p^n)$ is contained in $B_{n+1}(p^n)$ which is a Z-set in $B_{n+1}(p^{n+1})$. It remains to show that $(B_n(p^n))_{n=1}^\infty$ satisfies property (4). Obviously, it suffices to show that $(B_n(p^1))_{n=1}^\infty$ has this property. Select maps $\alpha_i : Q \to W_i(p^1)$ such that $\lim_{i\to\infty} \alpha_i = 1_Q$. Consider the following sequence of maps from $Q \times Q$ into $Q \times Q$

$$\alpha_1 \times \alpha_1, \alpha_1 \times \alpha_2, \alpha_2 \times \alpha_2, \alpha_2 \times \alpha_3, \alpha_3 \times \alpha_3, \ldots .$$

Connect adjacent maps in this sequence by straight line homotopies. Since $C_n(p^1)$ is a convex set which contains $W_n(p^1)$ and $W_{n+1}(p^1)$ we have that the image of the homotopy connecting $\alpha_n \times \alpha_n$ with $\alpha_n \times \alpha_{n+1}$ is contained in $W_n(p^1) \times C_n(p^1) \subset B_n(p^1)$ and that the image of the homotopy that connects $\alpha_n \times \alpha_{n+1}$ with $\alpha_{n+1} \times \alpha_{n+1}$ is contained in $C_n(p^1) \times W_{n+1}(p^1) \subset B^n(p^1)$. "Glued" together these homotopies form an H as in property (4) of the Capset Characterization Theorem. This proves the proposition.

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