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ON EXTENDED WALLMAN TYPE SPACES

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ABSTRACT. In this paper the usual construction of the Wallman topology on the set of all $(0-1)$-valued, lattice regular measures is extended to the set of all nontrivial, nonnegative, bounded, lattice regular measures. Furthermore, the notion of repleteness is extended to this more general situation.

0. Introduction. In the usual Wallman construction of a compact T_1 space associated with an arbitrary set X and an arbitrary disjunctive lattice of subsets of X, $\mathcal{L}$, one considers the pair $\langle IR(\mathcal{L}), W(\mathcal{L})\rangle$, where $IR(\mathcal{L})$ is the set of $(0-1)$-valued, $\mathcal{L}$-regular measures on $\mathcal{A}(\mathcal{L})$, the algebra of subsets of X generated by $\mathcal{L}$, and $W(\mathcal{L})$ is a certain lattice of subsets of $IR(\mathcal{L})$ (see below for definition). $W(\mathcal{L})$ is then taken as a base for the collection of closed sets of a topology on $IR(\mathcal{L})$, and it turns out that $IR(\mathcal{L})$ with respect to this topology is compact and T_1. (See [**6**].) It is T_2 if and only if $\mathcal{L}$ is normal. If, moreover, $\mathcal{L}$ is separating and X is given the topology with $\mathcal{L}$ as the base for the closed sets, then $IR(\mathcal{L})$ is a compactification of X. Specific cases, where X is a given topological space, give rise to such well-known compactifications of X as ωX, the Wallman compactification of X, βX, the Stone-Čech compactification of X, $\beta_0 X$, the Banaschewski compactification of X, etc.

Considering the set of σ-smooth elements of $IR(\mathcal{L})$, $IR(\sigma,\mathcal{L})$, and the restriction of $W(\mathcal{L})$ to $IR(\sigma,\mathcal{L})$, $W_\sigma(\mathcal{L})$, in [**2**] it was shown that $W_\sigma(\mathcal{L})$ is replete, i.e., for every element of $IR(\sigma,\mathcal{L})$, ν, the support of ν is nonempty. (See also the remark after Theorem 2.4.) If, moreover, $\mathcal{L}$ is separating and X is given the topology mentioned above, then $IR(\sigma,\mathcal{L})$ with the relative topology "contains" X densely, under a suitable

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identification. Specific cases, where X is a given topological space, give rise to such well-known spaces as υX, the realcompactification of X, $\upsilon_0 X$, the N-compactification of X, etc.

In this paper we attempt to construct analogous spaces associated with $M^+R(\mathcal{L}) - \{0\}$, the set of nontrivial, nonnegative, bounded, $\mathcal{L}$-regular measures on $\mathcal{A}(\mathcal{L})$ and with $M^+R(\sigma, \mathcal{L}) - \{0\}$, the set of σ-smooth elements of $M^+R(\mathcal{L}) - \{0\}$.

In particular, we consider certain lattices of subsets of $M^+R(\mathcal{L}) - \{0\}$ and $M^+R(\sigma, \mathcal{L}) - \{0\}$, $H(\mathcal{L})$ and $H_\sigma(\mathcal{L})$, respectively. However, unlike $W(\mathcal{L})$ and $W_\sigma(\mathcal{L})$, these are not lattices with respect to the usual set-theoretic operations $\cap, \cup$, but with respect to $\cap, \vee$, where $H(A) \vee H(B) = H(A \cup B)$. (See details below.) Nevertheless, when restricted to $IR(\mathcal{L})$ and $IR(\sigma, \mathcal{L})$, they yield, respectively, the lattices $W(\mathcal{L})$ and $W_\sigma(\mathcal{L})$ with respect to the usual set-theoretic operations.

We prove (Theorem 2.4) that if $\mathcal{L}$ is disjunctive, then $H_\sigma(\mathcal{L})$ is support-measure replete, i.e., for every element of $M^+R(\sigma, H_\sigma(\mathcal{L})) - \{0\}$, ν, the support of ν is nonempty. This gives a large category of abstract, support-measure replete lattices. We then give conditions under which the usual set-theoretic lattice generated by $H_\sigma(\mathcal{L})$ is support-measure replete. This in turn gives many topological examples of support-measure replete lattices in the usual set-theoretic sense.

Below, we give the terminology and notation which will be used throughout the paper and some basic facts.

1. Terminology and notation and some basic facts.

(A) Consider any set X and any lattice of subsets of $X, \mathcal{L}$. Assume $\varnothing$, $X \in \mathcal{L}$. The definitions of the following terms are found in [**2**]: $\mathcal{L}$ is separating, disjunctive, regular, normal, Lindelöf, compact, countably compact. (See also [**4**].)

(B) For an arbitrary function f, the domain of f is denoted by D_f.

The set whose general element is the intersection of an arbitrary subset of $\mathcal{L}$ is denoted by $t\mathcal{L}$. The algebra of subsets of X generated by $\mathcal{L}$ is denoted by $\mathcal{A}(\mathcal{L})$.

(C) Consider any algebra of subsets of $X, \mathcal{A}$. A measure on $\mathcal{A}$ is defined to be a function μ, from $\mathcal{A}$ to R, such that μ is finitely additive and bounded. (See [**1**, p. 567].) The set whose general element is a measure on $\mathcal{A}(\mathcal{L})$ is denoted by $M(\mathcal{L})$.

For an arbitrary element of $M(\mathcal{L})$, μ, the support of μ is defined to be $\cap\{L \in \mathcal{L}/|\mu|(L) = |\mu|(X)\}$ and is denoted by $S(\mu)$.

An element of $M(\mathcal{L})$, μ, is said to be $\mathcal{L}$-regular if and only if, for every element of $\mathcal{A}(\mathcal{L})$, E, for every positive number, ε, there exists an element of $\mathcal{L}$, L, such that $L \subset E$ and $|\mu(E) - \mu(L)| < \varepsilon$. The set whose general element is an element of $M(\mathcal{L})$ which is $\mathcal{L}$-regular is denoted by $MR(\mathcal{L})$. An element of $M(\mathcal{L})$, μ, is said to be $\mathcal{L}$-(σ-smooth) if and only if for every sequence in $\mathcal{A}(\mathcal{L})$, $\langle A_n\rangle$, if $\langle A_n\rangle$ is decreasing and $\lim_n A_n = \varnothing$, then $\lim_n \mu(A_n) = 0$. The set whose general element is an element of $M(\mathcal{L})$ which is $\mathcal{L}$-(σ-smooth) is denoted by $M(\sigma, \mathcal{L})$. An element of $M(\mathcal{L})$, μ, is said to be $\mathcal{L}$-(τ-smooth) if and only if, for every net in $\mathcal{L}$, $\langle \mathcal{L}_\alpha\rangle$, if $\langle L_\alpha\rangle$ is decreasing and $\lim_\alpha L_\alpha = \varnothing$, then $\lim_\alpha \mu(L_\alpha) = 0$. The set whose general element is an element of $M(\mathcal{L})$ which is $\mathcal{L}$-(τ-smooth) is denoted by $M(\tau, \mathcal{L})$.

The set whose general element is an element of $M(\mathcal{L})$, μ, such that $\mu(\mathcal{A}(\mathcal{L})) = \{0, 1\}$, is denoted by $I(\mathcal{L})$. For an arbitrary element of $\mathcal{A}(\mathcal{L})$, A, $\{\mu \in IR(\mathcal{L})/\mu(A) = 1\}$ is denoted by $W(A)$ and $\{\mu \in IR(\sigma, \mathcal{L})/\mu(A) = 1\}$ by $W_\sigma(A)$. At this point, consider the topological space $\langle IR(\mathcal{L}), tW(\mathcal{L})\rangle$. $tW(\mathcal{L})$ is called the Wallman topology.

(D) $\mathcal{L}$ is said to be replete if and only if whenever $\mu \in IR(\sigma, \mathcal{L})$, then $S(\mu) \neq \varnothing$. $\mathcal{L}$ is said to be support-measure replete if and only if whenever $\mu \in MR(\sigma, \mathcal{L}) - \{0\}$, then $S(\mu) \neq \varnothing$. $\mathcal{L}$ is said to be measure replete if and only if $MR(\sigma, \mathcal{L}) = MR(\tau, \mathcal{L})$. The following statement is true:

If $\mathcal{L}$ is separating and disjunctive, then $\mathcal{L}$ is measure replete if and only if $\mathcal{L}$ is support-measure replete. (For a related result, see [**5**].)

Remarks. (1) Consider any topological space X such that X is $T_{3\frac{1}{2}}$ and denote its collection of zero sets by $\mathcal{Z}$. Then the statement "$\mathcal{Z}$ is support-measure replete" is equivalent to "X is measure compact."

(2) Consider any topological space X such that X is T_1 and denote its collection of closed sets by $\mathcal{F}$. Then the statement "$\mathcal{F}$ is support-measure replete" is equivalent to "X is Borel measure compact."

(E) A premeasure on $\mathcal{L}$ is defined to be a function π, from $\mathcal{L}$ to R, such that (i) $\pi(\mathcal{L}) = \{0,1\}$ and $\pi(\varnothing) = 0$; (ii) For every two elements of $\mathcal{L}, L_1, L_2$, if $L_1 \subset L_2$, then $\pi(L_1) \leq \pi(L_2)$; and (iii) For every two elements of $\mathcal{L}, L_1, L_2$, if $\pi(L_1) = 1$ and $\pi(L_2) = 1$, then $\pi(L_1 \cap L_2) = 1$. The set whose general element is a premeasure on $\mathcal{L}$ is denoted by $\Pi(\mathcal{L})$. An element of $\Pi(\mathcal{L})$, π, is said to be $\mathcal{L}$-(σ-smooth) if and only if for every sequence in $\mathcal{L}$, $\langle L_n \rangle$, if $\langle L_n \rangle$ is decreasing and $\lim_n L_n = \varnothing$, then $\lim_n \pi(L_n) = 0$. The set whose general element is an element of $\Pi(\mathcal{L})$ which is $\mathcal{L}$-(σ-smooth) is denoted by $\Pi(\sigma, \mathcal{L})$. $\mathcal{L}$ is said to be an I-lattice if and only if for every element of $\Pi(\sigma, \mathcal{L})$, π, there exists an element of $IR(\sigma, \mathcal{L})$, μ, such that $\pi \leq \mu$.

We note that there exists a one-to-one correspondence between $\Pi(\mathcal{L})$ and the set of all $\mathcal{L}$-filters, and there exists a one-to-one correspondence between $\Pi(\sigma, \mathcal{L})$ and the set of all $\mathcal{L}$-filters with the Countable Intersection Property (C.I.P.). (Details can be found in [**3**].)

2. Topologize $M^+R(\mathcal{L})$ as follows:

> For every element of $\mathcal{A}(\mathcal{L})$, A, consider $\{\mu \in M^+R(\mathcal{L}) - \{0\} / \mu(A) = \mu(X)\}$ and denote it by $H(A)$.

Proposition 2.1. (a) $H(\varnothing) = \varnothing$.

(b) *For every element of* $\mathcal{A}(\mathcal{L})$, A, $H(A') \subset H(A)'$.

(c) *For every two elements of* $\mathcal{A}(\mathcal{L})$, A, B, *if* $A \subset B$, *then* $H(A) \subset H(B)$.

(d) *If* $\mathcal{L}$ *is disjunctive, then for every two elements of* $\mathcal{A}(\mathcal{L})$, A, B, *if* $H(A) \subset H(B)$, *then* $A \subset B$.

(e) *For every two elements of* $\mathcal{A}(\mathcal{L})$, A, B, $H(A) \cap H(B) = H(A \cap B)$.

(f) *For every two elements of* $\mathcal{A}(\mathcal{L})$, A, B, $H(A) \cup H(B) \subset H(A \cup B)$.

(Proof omitted.)

Next, consider $\{H(L); L \in \mathcal{L}\}$ and denote it by $H(\mathcal{L})$.

Note that the algebraic system $\langle H(\mathcal{L}), \subset, \cap \rangle$ is a semi-lattice; however, the algebraic system $\langle H(\mathcal{L}), \subset, \cap, \cup \rangle$ is not a lattice in the usual set-theoretic sense, because the following statement is false: For every two elements of $\mathcal{L}$, L_1, L_2, $H(L_1) \cup H(L_2) = \sup\{H(L_1), H(L_2)\}$ relative to $\subset$. (See Proposition 2.1 (f).)

Consider the lattice of subsets of $M^+R(\mathcal{L}) - \{0\}$ generated by $H(\mathcal{L})$ and denote it by $\tilde{H}(\mathcal{L})$. (Note that the general element of $\tilde{H}(\mathcal{L})$ is of the form $\cup_{i=1}^n H(L_i)$, where $L_i \in \mathcal{L}$ for $i = 1, \dots, n$.)

Now, consider $t\tilde{H}(\mathcal{L})$ and regard it as a topology on $M^+R(\mathcal{L})$.

Proposition 2.2. *The relativization of $t\tilde{H}(\mathcal{L})$ to $IR(\mathcal{L})$ is $tW(\mathcal{L})$ (the Wallman topology).*

Proof. Show $IR(\mathcal{L}) \cap t\tilde{H}(\mathcal{L}) = tW(\mathcal{L})$.

(α) Show $IR(\mathcal{L}) \cap t\tilde{H}(\mathcal{L}) \subset tW(\mathcal{L})$. Consider any element of $\tilde{H}(\mathcal{L})$, $\cup_{i=1}^n H(L_i)$. Note that $IR(\mathcal{L}) \cap \cup_{i=1}^n H(L_i) = \cup_{i=1}^n IR(\mathcal{L}) \cap H(L_i) = \cup_{i=1}^n W(L_i) = W(\cup_{i=1}^n L_i) \in tW(\mathcal{L})$. Consequently, $IR(\mathcal{L}) \cap t\tilde{H}(\mathcal{L}) \subset tW(\mathcal{L})$.

(β) Show $tW(\mathcal{L}) \subset IR(\mathcal{L}) \cap t\tilde{H}(\mathcal{L})$. (Proof omitted.)

(γ) Consequently, $IR(\mathcal{L}) \cap t\tilde{H}(\mathcal{L}) = tW(\mathcal{L})$.

Now, topologize $M^+R(\mathcal{L})$ as follows:

(α) Consider any net in $M^+R(\mathcal{L})$, $\langle \mu_m \rangle$, and any element of $M^+R(\mathcal{L})$, ν. $\langle \mu_m \rangle$ is said to converge to ν if and only if

(i) For every element of $\mathcal{L}$, L, $\overline{\lim}_m \mu_m(L) \le \nu(L)$ and

(ii) $\lim_m \mu_m(X) = \nu(X)$.

The statement "$\langle \mu_m \rangle$ converges to ν" is also expressed as $\lim_m \mu_m = \nu$.

(β) Define an operator on $\mathcal{P}(M^+R(\mathcal{L}))$ as follows: Consider any element of $\mathcal{P}(M^+R(\mathcal{L}))$, A. Now, consider the element of $\mathcal{P}(M^+R(\mathcal{L}))$, $\overline{A}$, described by $\overline{A} = \{\nu \in M^+R(\mathcal{L}) /$ there exists a net in A, $\langle \mu_m \rangle$, such that $\lim_m \mu_m = \nu\}$. The following statement is true: The operator "$-$" is a closure operator. (Proof omitted.)

(γ) Consider the topology on $M^+R(\mathcal{L})$ associated with this closure operator and denote it by $\mathcal{T}$. □

Proposition 2.3. *The relativization of* $t\tilde{H}(\mathcal{L})$ *to* $M^+R(\mathcal{L}) - \{0\}$ *is contained in the relativization of* $\mathcal{T}$ *to* $M^+R(\mathcal{L}) - \{0\}$. (*Proof omitted.*)

The following discussion leads to a "lattice-theoretic" result on measure repleteness:

Note that the algebraic system $\langle H(\mathcal{A}(\mathcal{L})), \subset, \cap, \cup \rangle$ is not a lattice for the same reason for which $\langle H(\mathcal{L}), \subset, \cap, \cup \rangle$ is not a lattice.

Assume $\mathcal{L}$ is disjunctive and define a lattice on $H(\mathcal{A}(\mathcal{L}))$ as follows:

(α) $H(\mathcal{A}(\mathcal{L}))$ is partially ordered by $\subset$. Denote $\subset$ by $\leq$.

(β) For every two elements of $\mathcal{A}(\mathcal{L})$, A, B, $H(A) \cap H(B) = \inf\{H(A), H(B)\}$ relative to $\leq$. Denote $\cap$ by $\wedge$.

(γ) Show, for every two elements of $\mathcal{A}(\mathcal{L}), A, B$, that $H(A \cup B) = \sup\{H(A), H(B)\}$ relative to $\leq$. Consider any two elements of $\mathcal{A}(\mathcal{L})$, A, B. Note that $H(A) \subset H(A \cup B)$ and $H(B) \subset H(A \cup B)$. Now, consider any element of $\mathcal{A}(\mathcal{L}), C$, such that $H(A) \subset H(C)$ and $H(B) \subset H(C)$. Then, since $\mathcal{L}$ is disjunctive, by Proposition 2.1 (d), $A \subset C$ and $B \subset C$. Hence, $A \cup B \subset C$. Hence, by Proposition 2.1 (c), $H(A \cup B) \subset H(C)$. Consequently, $H(A \cup B) = \sup\{H(A), H(B)\}$ relative to $\leq$. Set $H(A \cup B) = H(A) \vee H(B)$.

(Note, in connection with the pair $\langle IR(\mathcal{L}), W(\mathcal{L}) \rangle$, for every two elements of $\mathcal{A}(\mathcal{L})$, A, B, that $W(A) \cup W(B) = W(A \cup B)$. Hence, in this case $\vee = \cup$.)

Consequently, the algebraic system $\langle H(\mathcal{A}(\mathcal{L})), \leq, \wedge, \vee \rangle$ is a lattice.

Further, this lattice is distributive and complemented.

Consequently, this lattice is a Boolean algebra.

Next, for every element of $\mathcal{A}(\mathcal{L})$, A, consider $\{\mu \in M^+R(\sigma, L) - \{0\} / \mu(A) = \mu(X)\}$ and denote it by $H_\sigma(A)$.

Observation. If, in each statement of Proposition 2.1, the letter H is replaced by H_σ, the resulting statement is true.

Next, consider $H_\sigma(\mathcal{A}(\mathcal{L}))$ and define a lattice on it in the same way as for $H(\mathcal{A}(\mathcal{L}))$, thus obtaining the lattice $\langle H_\sigma(\mathcal{A}(\mathcal{L})), \leq, \wedge, \vee \rangle$. This lattice is a Boolean algebra. Also, consider the lattice $\langle H_\sigma(\mathcal{L}), \leq, \wedge, \vee \rangle$.

Now, assume $\mathcal{L}$ is disjunctive and consider any element of $M(\mathcal{L}), \mu$. For any two elements of $\mathcal{A}(\mathcal{L})$, A, B, if $H_\sigma(A) = H_\sigma(B)$, then, since $\mathcal{L}$ is disjunctive, $A = B$. Consider the function μ', which is such that $D_{\mu'} = H_\sigma(\mathcal{A}(\mathcal{L}))$ and, for every element of $H_\sigma(\mathcal{A}(\mathcal{L}))$, $H_\sigma(A)$, $\mu'(H_\sigma(A)) = \mu(A)$. Note $\mu' \in M(H_\sigma(\mathcal{L}))$. Conversely, consider any element of $M(H_\sigma(\mathcal{L})), \rho$, and the function μ which is such that $D_\mu = \mathcal{A}(\mathcal{L})$ and, for every element of $\mathcal{A}(\mathcal{L})$, A, $\mu(A) = \rho(H_\sigma(A))$. Note $\mu \in M(\mathcal{L})$ and $\rho = \mu'$. Further, note $\mu \in MR(\mathcal{L})$ if and only if $\mu' \in MR(H_\sigma(\mathcal{L}))$. Finally, note $\mu \in MR(\sigma, \mathcal{L})$ if and only if $\mu' \in MR(\sigma, H_\sigma(\mathcal{L}))$.

Theorem 2.4. *If $\mathcal{L}$ is disjunctive, then the lattice $\langle H_\sigma(\mathcal{L}), \leq, \wedge, \vee\rangle$ is support-measure replete.*

Proof. Assume $\mathcal{L}$ is disjunctive. Consider any element of $M^+R(\sigma, H_\sigma(\mathcal{L})) - \{0\}$, ρ, and show $S(\rho) \neq \varnothing$. By the definition of support, $S(\rho) = \cap\{H_\sigma(L)/L \in L \text{ and } \rho(H_\sigma(L)) = \rho(M^+R(\sigma, \mathcal{L}) - \{0\})\}$. Since $\rho \in MR(\sigma, H_\sigma(\mathcal{L}))$, there exists an element of $MR(\sigma, \mathcal{L})$, μ, such that $\rho = \mu'$ and μ is unique. Then $S(\rho) = \cap\{H_\sigma(L)/L \in \mathcal{L} \text{ and } \mu'(H_\sigma(L)) = \mu'(M^+R(\sigma, \mathcal{L}) - \{0\})\}$. Now, consider any element of $H_\sigma(\mathcal{L})$, $H_\sigma(L)$, such that $\mu'(H_\sigma(L)) = \mu'(M^+R(\sigma, \mathcal{L}) - \{0\})$. Then, since $\mu'(H_\sigma(L)) = \mu(L)$ and $\mu'(M^+R(\sigma, \mathcal{L}) - \{0\}) = \mu'(H_\sigma(X)) = \mu(X)$, $\mu(L) = \mu(X)$. Moreover, since $\mu' = \rho$ and $\rho \neq 0$, $\mu \neq 0$. Consequently, $\mu \in H_\sigma(L)$. Consequently, $S(\rho) \neq \varnothing$. Hence, $\langle H_\sigma(\mathcal{L}), \leq, \wedge, \vee\rangle$ is support-measure replete. □

Remark . When $\langle H_\sigma(\mathcal{L}), \leq, \wedge, \vee\rangle$ is restricted to the case of $(0-1)$-valued measures, the following well-known result is obtained: If $\mathcal{L}$ is disjunctive, then $W_\sigma(\mathcal{L})$ is replete.

Proposition 2.5. *$H(\mathcal{L})$ is compact.*

Proof. Consider any subset of $H(\mathcal{L})$, $\{H(L_\alpha); \alpha \in A\}$ such that $\cap\{H(L_\alpha); \alpha \in A\} = \varnothing$ and show there exists a subset of A, A^*, such that $\cap\{H(L_\alpha); \alpha \in A^*\} = \varnothing$ and A^* is finite. Assume the contrary. Then $\{H(L_\alpha); \alpha \in A\}$ has the F.I.P. Now, consider $\{L_\alpha; \alpha \in A\}$

and show it has the F.I.P. (Proof omitted.) Hence, there exists an element of $I_R(\mathcal{L})$, μ, such that, for every $\alpha, \mu(L_\alpha) = 1$ and μ is unique. Consequently, for every $\alpha, \mu \in H(L_\alpha)$. Hence, $\cap\{H(L_\alpha); \alpha \in A\} \neq \varnothing$. Thus, a contradiction has been reached. Consequently, there exists a subset of A, A^*, such that $\cap\{H(L_\alpha); \alpha \in A^*\} = \varnothing$ and A^* is finite. Hence, $H(\mathcal{L})$ is compact. □

Corollary 2.6. *$\tilde{H}(\mathcal{L})$ is compact.*

Proof. Since $\tilde{H}(\mathcal{L})$ is the lattice of subsets of $M^+R(\mathcal{L}) - \{0\}$ generated by $H(\mathcal{L})$ and $H(\mathcal{L})$ is compact (Proposition 2.5), it follows readily that $\tilde{H}(\mathcal{L})$ is compact. □

OBSERVATIONS. (1) $t\tilde{H}(\mathcal{L})$ is compact.

(2) $\tilde{H}(\mathcal{L})$ is measure replete.

Another result on measure repleteness is given by the following theorem.

Theorem 2.7. *If $\mathcal{L}$ is disjunctive and $W_\sigma(\mathcal{L})$ is an I-lattice, then $\tilde{H}_\sigma(\mathcal{L})$ is support-measure replete.*

Proof. Assume $\mathcal{L}$ is disjunctive and $W_\sigma(\mathcal{L})$ is an I-lattice. To show $\tilde{H}_\sigma(\mathcal{L})$ is support-measure replete, consider any element of $MR(\sigma, \tilde{H}_\sigma(\mathcal{L})) - \{0\}, \rho$, and show $S(\rho) \neq \varnothing$. Assume $S(\rho) = \varnothing$. By the definition of support, $S(\rho) = \cap\{E \in \tilde{H}_\sigma(\mathcal{L})/\rho(E) = \rho(M^+R(\sigma, \mathcal{L}) - \{0\})\}$. Set $\{E \in \tilde{H}_\sigma(\mathcal{L})/\rho(E) = \rho(M^+R(\sigma, \mathcal{L}) - \{0\})\} = \{S_\alpha; \alpha \in \Lambda\}$. Note, for every α, $\rho(S_\alpha) = \rho(M^+R(\sigma, \mathcal{L}) - \{0\})$ and $\cap\{S_\alpha; \alpha \in \Lambda\} = \varnothing$. For every α, since $S_\alpha \in \tilde{H}_\sigma(\mathcal{L})$, $S_\alpha = \cup\{H_\sigma(A_{\alpha i});\ A_{\alpha i} \in \mathcal{L}$ for $i = 1, \dots, n_\alpha\}$; then $S_\alpha \subset H_\sigma(\cup\{A_{\alpha i}; i = 1, \dots, n_\alpha\})$ with $\cup\{A_{\alpha i};$ $i = 1, \dots, n_\alpha\} \in L$; set $\cup\{A_{\alpha i}; i = 1, \dots, n_\alpha\} = L_\alpha$; then $S_\alpha = H_\sigma(L_\alpha)$. Consequently, for every α, $\rho(H_\sigma(L_\alpha)) = \rho(M^+R(\sigma, \mathcal{L}) - \{0\})$ and $\phi = IR(\sigma, \mathcal{L}) \cap \cap\{S_\alpha; \alpha \in \Lambda\} = \cap\{IR(\sigma, \mathcal{L}) \cap S_\alpha; \alpha \in \Lambda\} = \cap\{W_\sigma(L_\alpha); \alpha \in \Lambda\}$.

Now consider $\{W_\sigma(L_\alpha); \alpha \in \Lambda\}$. Show $\{W_\sigma(L_\alpha); \alpha \in \Lambda\}$ has the C.I.P. Assume the contrary. Then there exists a sequence in

$\{W_\sigma(L_\alpha); \alpha \in \Lambda\}$, $\langle W_\sigma(\hat{L}_i)\rangle$, such that $\cap\{W_\sigma(\hat{L}_i); i \in \mathbf{N}\} = \varnothing$ and $\langle W_\sigma(\hat{L}_i)\rangle$ is decreasing. Consider any such $\langle W_\sigma(\hat{L}_i)\rangle$. Then, since $\cap\{W_\sigma(\hat{L}_i); i \in \mathbf{N}\} = W_\sigma(\cap\{\hat{L}_i; i \in \mathbf{N}\})$, $W_\sigma(\cap\{\hat{L}_i; i \in \mathbf{N}\}) = \varnothing$. Hence, $\cap\{\hat{L}_i; i \in \mathbf{N}\} = \varnothing$. Then $H_\sigma(\cap\{\hat{L}_i; i \in \mathbf{N}\}) = \varnothing$. Hence, since $H_\sigma(\cap\{\hat{L}_i; i \in \mathbf{N}\}) = \cap\{H_\sigma(\hat{L}_i); i \in \mathbf{N}\}$, $\cap\{H_\sigma(\hat{L}_i); i \in \mathbf{N}\} = \varnothing$. Consequently, $\langle H_\sigma(\hat{L}_i)\rangle$ is decreasing and $\lim_i H_\sigma(\hat{L}_i) = \varnothing$. Hence, since $\rho \in MR(\sigma, \tilde{H}_\sigma(\mathcal{L}))$, $\lim_i \rho(H_\sigma(\hat{L}_i)) = 0$. Further, note, since $\langle H_\sigma(\hat{L}_i)\rangle$ is in $\{H_\sigma(L_\alpha); \alpha \in \Lambda\}$, for every i, $\rho(H_\sigma(\hat{L}_i)) = \rho(M^+R(\sigma, \mathcal{L}) - \{0\}) \neq 0$. Hence, $\lim_i \rho(H_\sigma(\hat{L}_i)) \neq 0$. Thus, a contradiction has been reached. Consequently, $\{W_\sigma(L_\alpha); \alpha \in \Lambda\}$ has the C.I.P.

Hence, there exists an element of $\Pi_\sigma(W_\sigma(\mathcal{L}))$, π, such that, for every $\alpha, \pi(W_\sigma(L_\alpha)) = 1$ and π is unique. Hence, since $W_\sigma(\mathcal{L})$ is an I-lattice, there exists an element of $IR(\sigma, W_\sigma(\mathcal{L}))$, τ, such that $\pi \leq \tau$ on $W_\sigma(\mathcal{L})$. Consider any such τ. Then, since $\pi \leq \tau$ on $W_\sigma(\mathcal{L})$, $S(\tau) \subset S(\pi)$. Since $\mathcal{L}$ is disjunctive, $W_\sigma(\mathcal{L})$ is replete. (See [**2**].)

Hence, since $\tau \in IR(\sigma, W_\sigma(\mathcal{L}))$, $S(\tau) \neq \varnothing$. Consequently, $S(\pi) \neq \varnothing$. Since for every α, $\pi(W_\sigma(L_\alpha)) = 1$, $S(\pi) \subset \cap\{W_\sigma(L_\alpha); \alpha \in \Lambda\}$. Consequently, $\cap\{W_\sigma(L_\alpha); \alpha \in \Lambda\} \neq \varnothing$. Thus, a contradiction has been reached. Consequently, $S(\rho) \neq \varnothing$. Hence, $\tilde{H}_\sigma(\mathcal{L})$ is support-measure replete. $\square$

Proposition 2.8. *If $\mathcal{L}$ is disjunctive and Lindelöf, then $\mathcal{L}$ is an I-lattice.* (*Known.*)

Proposition 2.9. *If $\mathcal{L}$ is an I-lattice, then $W_\sigma(\mathcal{L})$ is Lindelöf.* (*Known.*)

Corollary 2.10. *If $\mathcal{L}$ is an I-lattice, then $W_\sigma(\mathcal{L})$ is an I-lattice.*

Proof. Assume $\mathcal{L}$ is an I-lattice. Then, by Proposition 2.9, $W_\sigma(\mathcal{L})$ is Lindelöf. Hence, since $W_\sigma(\mathcal{L})$ is disjunctive, by Proposition 2.8, $W_\sigma(\mathcal{L})$ is an I-lattice. $\square$

Corollary 2.11. *If $\mathcal{L}$ is disjunctive and Lindelöf, then $\tilde{H}_\sigma(\mathcal{L})$ is support-measure replete.*

Proof. Assume $\mathcal{L}$ is disjunctive and Lindelöf. Then, by Proposition 2.8 and Corollary 2.10, $W_\sigma(\mathcal{L})$ is an I-lattice. Hence, by Theorem 2.7, $\tilde{H}_\sigma(\mathcal{L})$ is support-measure replete.

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