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ON MAPS WITH DENSE ORBITS AND THE DEFINITION OF CHAOS

STEPHEN SILVERMAN

1. Introduction and definitions. Though the concept of chaos in dynamics goes back to at least Poincaré, it has only been during the flood of activity of the past decade that precise definitions of chaotic behavior have come forth [**2,3,8,18**]. The object of this article is to examine the relationship between the axioms for the most popular definition of chaos in discrete systems. The focus will be on a definitive analysis in the case of one-dimensional manifolds.

Let M be a metric space and let the distance between two points $x, y \in M$ be denoted $|x-y|$. A discrete dynamical system at its simplest is the set of iterates of a map $Q : M \to M$, i.e., $\{Q^0, Q, Q^2, Q^3, \dots\}$, where Q^0 is the identity function and Q^n denotes Q composed with itself n times.

The *orbit* of a point $x \in M$ is the set $\{x, Q(x), Q^2(x), Q^3(x), \dots\}$ and will often be written $\{x_n\}_{n=0}^{\infty}$ or $\{x_n\}$ when there is no ambiguity about which map Q is being used. A point x is said to be *periodic* if $Q^n(x) = x$ for some positive n. The minimum such n is called the *period* of x.

Frequently, the map Q depends on further parameters σ in an index set Σ, and one studies the behavior of the orbits obtained from Q_σ as σ is varied. The definition of chaos that we will use can be motivated by the following standard example. Let M be the closed interval $[0, 1]$. For every number $\sigma \in [0, 4]$, define

$$Q_\sigma(x) = \sigma x(1-x) \qquad x \in [0, 1].$$

When $\sigma \in [0, 1]$ the orbit of any point converges to the fixed point 0; hence, the set $\{0\}$ is called an attractor. For $\sigma \in (1, 3]$ the orbit of $x \neq 0, 1$ converges to $(\sigma - 1)/\sigma$. As σ increases to 4, Q_σ undergoes bifurcations and many periodic orbits emerge and higher periods occur. For $\sigma = 4$, there are no longer any attractors. In fact, for any point x there are points arbitrarily close to x whose orbits drift far away from

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the orbit of x. There are even points whose orbits are dense in M and the set of periodic points is dense as well (see Section 4).

With this in mind, we make the following definition.

A continuous mapping f taking a metric space M to itself is *chaotic* if

SIC. f is sensitive to initial conditions, i.e., there is a $\delta > 0$ so that for any $x \in M$ and any neighborhood N of x there is a point $y \in N$ and a positive integer n satisfying

$$|f^n(x) - f^n(y)| \geq \delta.$$

DO. f has a dense orbit, i.e., there is a point $x_0 \in M$ whose orbit $\{f_n(x_0)\}_{n=0}^{\infty} = \{x_n\}$ is dense in M.

DPP. f has a dense set of periodic points, i.e., $\{x | f^n(x) = x$ for some $n > 0\}$ is dense in M.

This is essentially the definition given by Devaney [**8**] and Barnsley [**2**] with two minor differences. First off, for **DO** they say

TT. f is topologically transitive, i.e., if U and V are nonempty open subsets of M there exists $n > 0$ such that $f^n(U) \cap V \neq \varnothing$.

The following proposition binds the two.

Proposition (1.1). *Let M be a perfect (has no isolated points). Then* **DO** *implies* **TT**. *Furthermore, if M is separable and second category, then* **TT** *implies* **DO**.

Proof. If M is perfect and $\{x_n\}$ is a dense orbit, then there is some $x_k \in U$ and some $x_m \in V \backslash \{x_0, x_1, \dots, x_k\}$ which is open and nonempty. But $m > k$ and $f^{m-k}(U) \cap V \neq \varnothing$.

For the second half, suppose that f has no dense orbit and $\{V_n\}_{n=1}^{\infty}$ is a countable base. For each $x \in M$ there is some $V_{n(x)}$ such that $f^k(x) \notin V_{n(x)}$ for all $k \geq 0$. But

$$\bigcup_{k=0}^{\infty} f^{-k}(V_{n(x)})$$

is open and meets every open set since f is topologically transitive. If we let $A_{n(x)}$ be the complement of this union, then $A_{n(x)}$ contains

x and is closed and nowhere dense. However, $M = \cup_{x \in M} A_{n(x)}$ is a countable union, contradicting the fact that M is of second category. $\square$

The main idea here is known, e.g., see [**9**]. To see that separability and second category are necessary, check the end of Section 4.

This also shows that if M is perfect, and U and V are open, and n is positive, then there is $x_0 \in U$ with dense orbit and $x_m \in V$ for some $m > n$.

The second difference between the two definitions is that we assume continuity. Essentially all maps in the realm of dynamics meet that condition. However, if f is not continuous, so that f is divorced from the topology of M, then the axioms are independent. *Henceforth, all maps are continuous.*

Also note that **SIC** and **DPP** can never imply **DO**. Indeed, if f_1 is chaotic on M_1, f_2 is chaotic on M_2, and $M_1 \cap M_2 = \varnothing$ then $f_1 \cup f_2$ satisfies **SIC** and **DPP** but not **DO**.

2. Sensitivity to initial conditions is redundant.

Theorem (2.1). *If M is infinite, then* **DO** *and* **DPP** *imply* **SIC**.

Proof. It is not hard to check that if M is infinite, then **DO** and **DPP** imply that M is perfect so f is topologically transitive. Let p be a periodic point. Choose a point q that is not in the orbit of p (this choice is possible since M is infinite). Let p' be the point in the orbit of p that is closest to q and set $d = |p' - q|$.

Assume that **SIC** does not hold. Then there exists an x and a neighborhood $N(x)$ of x such that for all $n \geq 0$, diam $f^n(N(x)) < d/4$. Let $y \in N(x)$ be periodic with period Y. Because f is continuous, there is $N(p)$ such that if $z \in N(p)$, then

$$|f^k(z) - f^k(p)| < d/4, \quad \text{for } 0 \leq k \leq Y - 1.$$

Finally, let diam $(N(q)) < d/4$.

Now there exists m such that $f^m(N(x)) \cap N(p)$ is nonempty and there exists j such that $f^j(N(x)) \cap N(q)$ is nonempty. By the triangle inequality,

$$|f^{m+k}(y) - f^k(p)| < d/2 \quad \text{for } 0 \le k \le Y-1$$

and of course $|f^j(y) - q| < d/2$.

But since

$$f^m(y), f^{m+1}(y), \dots, f^{m+Y-1}(y)$$

exhausts the orbit of y there must be some k, $0 \le k < Y$, such that $f^{m+k}(y) = f^j(y)$. Another application of the triangle inequality shows that $|f^k(p) - q| < d$ which contradicts the definition of d. □

The proof of (2.1) implies that if f is chaotic then the δ in **SIC** can be chosen at least as large as

$$(1/4) \sup_{q,p_0 \in M} \min n \in \mathbf{Z}^+ \{|p_n - q| | p_0 \text{ periodic } \}.$$

For example, in the case $f = Q_4 = 4x(1-x)$, δ can be $3/16$ by letting $p_0 = 3/4$ (a fixed point so $\{p_n\} = \{p_0\}$) and $q = 0$. A more general conjecture at the end of Section 4 will improve on this.

Two maps $f : A \to A$ and $g : B \to B$ are said to be *conjugate* if there exists a homeomorphism $h : A \to B$ such that $h \circ f \circ h^{-1} = g$. When this holds, we write $f \sim g$. For example, if $T(x) = 1 - 2|x - 1/2|$, $x \in [0,1]$, then $T \sim Q_4$ as can be verified by a trigonometric calculation using the conjugating homeomorphism $h(x) = (1 - \cos(\pi x))/2$.

Since **DO** and **DPP** are preserved under conjugacy, the preceding theorem immediately implies that $\sim$ is a chaos-preserving equivalence relation:

Corollary (2.2). *If $f \sim g$ and f is chaotic, then g is chaotic.*

The metric on the metric space M plays a role in definition **SIC**, but definitions **DO** and **DPP** depend only on the underlying topology and not on the metric that induces that topology. Thus, the following corollary also follows immediately from the Theorem.

Corollary (2.3). *The determination of whether f is chaotic or not depends only on the topology of M and not on the metric.*

3. On intervals a dense orbit implies chaos. The theorem of this section relies heavily on the following feature of the unit interval: given any two points there is a unique minimal connected set containing them. If this fails, for instance in the case of the circle (Section 7), the Cantor set (Section 5), or higher dimensions (Section 9), then the class of maps with dense orbits widens considerably.

Lemma (3.1). *Let M be an arbitrary subinterval of* $\mathbf{R}$ *and let $f : M \to M$ have a dense orbit. If $(a,b) \subset M$ is free of periodic points, then so is $f^j(a,b)$ for all $j \geq 0$.*

Proof. Suppose the conclusion is false, i.e., there exists $r \in (a,b)$ with $f^n(r)$ a periodic point and consequently $f^i(r) \notin (a,b)$ for all $i \geq n$.

There exists x with dense orbit, in (a,r) and $m > n$ such that $x < f^m(x) < r$ and there exists y, with dense orbit, in (r,b) and $k > n$ such that $y > f^k(y) > r$.

Now $u < f^m(u)$ for all $u \in (a,b)$ for otherwise there would exist a fixed point $c \in (a,b)$ for f^m, contradicting the lack of periodic points in (a,b). Similarly, $f^k(u) < u$, $u \in (a,b)$. Because of continuity and $f^m(r) \geq b$, $f^k(r) \leq a$, we have

$$f^k([r,y]) \supset [x,r] \quad \text{so} \quad f^{k+m}([r,y]) \supset f^m([x,r]) \subset [r,y].$$

This guarantees a fixed point for f^{k+m} in $[r,y] \subset (a,b)$, again a contradiction. □

Convention. $\langle \alpha, \beta \rangle$ is one of the four intervals of $\mathbf{R}$ with endpoints α and β.

Theorem (3.2). *If M is a subinterval of* $\mathbf{R}$*, then* **DO** *implies* **DPP** *and* **SIC**.

Proof. By (2.1) all we need to show is that f has a dense set of periodic points. Assume otherwise, i.e., there is an interval $(a,b) \subset M$ free of periodic points. Then there is a maximum such interval $\langle \alpha, \beta \rangle \supset (a,b)$.

Case 1. α or $\beta \in M$. Say $\alpha \in M$ for definiteness. There is $x_0 \in (\alpha, \beta)$

with dense orbit and $m > 0$ such that

$$\alpha < x_m = f^m(x_0) < x_0.$$

Then $f^m(\alpha,\beta) \cap (\alpha,\beta) \neq \varnothing$, but due to Lemma (3.1) $f^m(\alpha,\beta)$ is free of periodic points, and, hence, by the maximality of $\langle\alpha,\beta\rangle$,

$$(*) \qquad f^m(\alpha,\beta) \subset \langle\alpha,\beta\rangle.$$

Since $f^m(x_0) < x_0$, we must have $f^m(x) < x$ for all $x \in (\alpha,\beta)$ to avoid a periodic point in (α,β). This implies $f^m(\alpha) \leq \alpha$ and, along with $(*)$, we get $f^m([\alpha,\beta)) \subset [\alpha,\beta\rangle$ which means $f^m(\alpha) = \alpha$ so $\langle\alpha,\beta\rangle$ and $f^m(\alpha,\beta) \subset (\alpha,\beta\rangle$.

Therefore, $x_0 > x_m > f^m(x_m) = x_{2m} > \alpha$. Continuing this in vain we get

$$x_0 > x_m > x_{2m} > x_{3m} > \cdots > \alpha$$

so there is a γ such that $x_{km} \searrow \gamma \geq \alpha$ as $k \to \infty$. Continuity implies

$$\gamma \leftarrow x_{(k+1)m} = f^m(x_{km}) \to f^m(\gamma) \quad \text{or} \quad \gamma = f^m(\gamma).$$

Again (α,β) periodic point free implies $\gamma = \alpha$.

The continuity of f yields

$$f(x_{km}) = x_{km+1} \to f(\alpha) \quad \text{and} \quad x_{km+i} \to f^i(\alpha).$$

But $f^i(\alpha)$, $i = 0, 1, \dots, m-1$ exhausts the orbit of α and thus a tail of $\{x_n\}$ stays arbitrarily close to the finite "attractor" $\{\alpha, f(\alpha), \dots, f^{m-1}(\alpha)\}$ and therefore cannot be dense. This contradiction establishes Case 1.

Case 2. $\alpha \notin M$ and $\beta \notin M$. Then $M = (\alpha,\beta)$ so M has no periodic points and consequently $f(x) > x$ for all x or $f(x) < x$ for all x. In either case, there can be no dense orbit since all orbits are monotone. □

This generalizes results in [**1, 14, 16**]; the compact case was handled by [**1**].

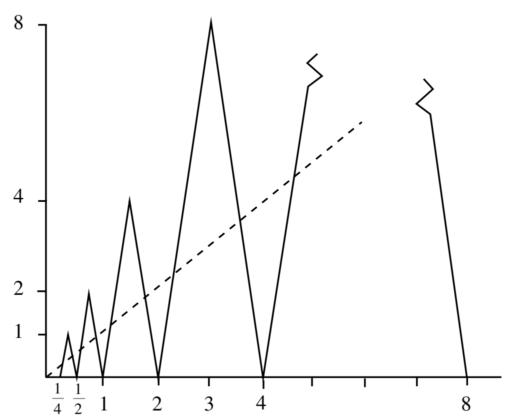
4. A chaotic map on R. The last theorem (3.2) can provide a fast track to showing maps to be chaotic. In particular, the introduction of symbolic dynamics is unnecessary in what follows.

For example, to see that $Q_4(x) = 4x(1-x)$ is chaotic, it is sufficient to show that
$$T(x) = 1 - 2|x - 1/2|$$
is chaotic (since they are conjugate by the map given in Section 2). If $x = .a_1a_2a_3\dots$ is the binary expansion of $x \in [0,1]$ and $\bar{a}_i = 1 - a_i$, then
$$T(x) = \begin{cases} .a_2a_3a_4\dots & \text{if } a_1 = 0, \\ .\bar{a}_2\bar{a}_3\bar{a}_4\dots & \text{if } a_1 = 1. \end{cases}$$
So if $x = a_1a_2\dots a_m000\dots$ then $f^n(x) = 0$ for $n \geq m+1$ and if $x' = a_1a_2\dots a_m1000$ then $f^{m+1}(x) = 1$.

Now any open $U \subset [0,1]$ contains an interval of the form $[x, x']$ and since T^{m+1} is continuous, $T^{m+1}([x, x^p]) = [0,1]$. Thus T is topologically transitive and, by (1.1) and (3.4), T is chaotic.

It is harder to exhibit chaotic maps on the whole real line. The function $(\sqrt{2}e^{5\pi}/4)\sin(\log|x|)$ is chaotic on $\mathbf{R}$, but the proof is quite messy. The following function, due to Ray Meyer, is chaotic, as can be verified as follows.

Let $W : [0,\infty) \to [0,\infty)$ with
$$W(2^n x) = 2^{n+2}T(x-1), \quad x \in [1,2], n \in \mathbf{Z}, W(0) = 0.$$



Then

$$W(2^n x) = 2^n \begin{cases} 2^3 x - 2^3 & \text{if } x \in [1, 3/2) \\ 2^4 - 2^3 x & \text{if } x \in [3/2, 2) \end{cases}$$

or, in binary,

$$\begin{aligned} W(2^n x) &= 2^n \begin{cases} 1a_2a_3.a_4a_5\ldots - 1000 & \text{if } a_1 = 0 \\ 10000 - 1a_1a_2a_3.a_4a_5\ldots & \text{if } a_1 = 1 \end{cases} \\ &= 2^n \begin{cases} a_2a_3.a_4a_5\ldots & \text{if } a_1 = 0 \\ \bar{a}_2\bar{a}_3.\bar{a}_4\bar{a}_5\ldots & \text{if } a_1 = 1. \end{cases} \end{aligned}$$

It is easy to check that if $\alpha = 2^n \cdot 1.a_1a_2\ldots a_m$, then $W^{2r}(\alpha) = 0$ if $r > m/2$, and if $\beta = 2^n \cdot 1.a_1a_2\ldots a_m 010101\ldots$, then

$$W^{2r}(\beta) = 2^{k+2r} \cdot 1.010101\ldots$$

for $r > m/2$ and some $k \in \mathbf{Z}$, since

$$W^2(2^{k+2r} \cdot 1.01010101\ldots) = 2^{k+2(r+1)} \cdot 1.01010101\ldots.$$

Any open subset U of $[0, \infty)$ contains an interval of the form $[\alpha, \beta]$ and for r large enough

$$W^{2r}([\alpha, \beta]) \supset [0, 2^{k+2r}], \qquad k \in \mathbf{Z}.$$

Hence, given any open $V \subset \mathbf{R}^+$ there is an r large enough so that $W^{2r}(U) \cap V \neq \varnothing$. Thus, W^2 is topologically transitive on $[0, \infty)$.

Let

$$f(x) = \begin{cases} -W(x) & x \geq 0 \\ W(x) & x \leq 0. \end{cases}$$

So $f^2|_{[0,\infty)} = W^2$ and thus f is topologically transitive on $\mathbf{R}$ and is thus chaotic.

We know of no other examples in the literature of maps on $\mathbf{R}$ with dense orbits.

The above "uniform expanding" property of T, namely, given $\varepsilon > 0$ there is n_ε satisfying

$$T^{n_\varepsilon}((a - \varepsilon, a + \varepsilon) \cap [0, 1]) = [0, 1],$$

has another nice application. It will provide us with an example of a nonseparable complete metric space M supporting a continuous topologically transitive function. Clearly, such a function cannot have a dense orbit since M is not separable (see Proposition (1.1)).

Indeed, suppose $M = [0,1]^{[0,1]} =$ all functions from $[0,1]$ to $[0,1]$ with the uniform metric, i.e.,

$$|f - g| = \sup_{x\in[0,1]} |f(x) - g(x)|.$$

Let $\tau : M \to M$ be defined by

$$\tau(f)(x) = T(f(x)).$$

Finally, let U be open in M; then there exists $f \in M$ and $\varepsilon > 0$ such that

$$\{g : |g(x) - f(x)| < \varepsilon \quad \forall x \in [0,1]\} \subset U.$$

Hence, $\tau^{n_\varepsilon}(U) = M$ and thus τ is an expanding topologically transitive map as well.

Also note that if $M' = \{x|x \text{ is a periodic point of } T\}$, then M' is not second category. Furthermore, T is topologically transitive on M' but there is no dense orbit.

It would be interesting to know what conditions to place on a chaotic map f on an interval J to insure that: If U is open in J and C is compact then there is an m such that $f^m(U) \supset C$.

This would generalize the phenomena we witnessed in the above examples and aid in finding the largest δ in the definition of sensitivity to initial conditions; e.e., $\delta = 3/4$ for Q_4. Demanding that f have a periodic point with odd period does insure that $\cup_{m\in\mathbf{Z}^+} f^m(U) \supset J^0$ by the dazzling theorem of Sarkovskii [**8**]. Is this part of the condition we are looking for? Note Preston's notion of "exact" [**16**].

5. On the Cantor set a dense orbit does not imply chaos. The Cantor set can be represented as all sequences $(a_1, a_2, \dots)$ of zeros and ones with the metric

$$|(a_1, a_2, \dots) - (b_1, b_2, \dots)| = \sum_{n=1}^{\infty} \frac{|a_n - b_n|}{2^n}.$$

We now introduce the "adding machine" of Misiurewicz (see [**14**, p. 14]).

Motivated by addition in the two-adic field we define (again following an idea of Mayer)

$$f(a_1, a_2, \dots) = 1 + (a_1 a_2 a_3 \dots).$$

This "Hebrew binary addition" map is best understood by seeing examples:

$$1 + (000\dots) = (1000\dots) \text{ and } 1 + (110001000\dots) = (001001000\dots).$$

The orbit of $\alpha = (000\dots)$ is dense, but if

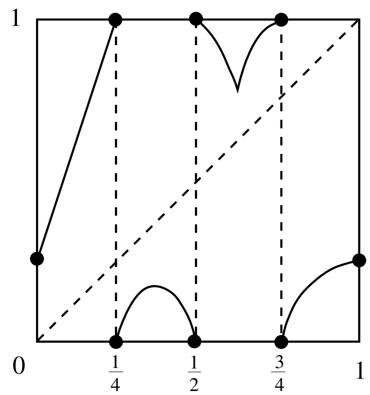
$$|\beta - \alpha| \le 2^{-k} \quad \text{then } |f^n(\beta) - f^n(\alpha)| \le 2^{-k+1}.$$

Thus f is not sensitive to initial conditions and hence is not chaotic. Thus by (2.1) f cannot have a dense set of periodic points. This illustrates the essential nature of connectedness in (3.2).

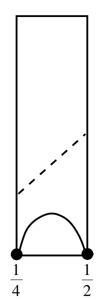
6. Observations about the circle. The theorems of Section 3 cannot be totally carried over to the circle. Rotation through an irrational angle (setting the total angle of the circle S^1 to 1) is an isometry and thus certainly not sensitive to initial conditions. There are no periodic points, but every point has a dense orbit [**8**].

The main obstruction is that there is more than one minimal connected set containing two points. However, if we restrict ourselves to a subarc or interval of S^1 then a unique minimal connected set does exist. In fact, if we think of S^1 as [0,1] with 0 and 1 identified, then $(0, 1)$ has the desired property.

When thinking of S^1 as $[0, 1]$ and we wish to draw the graph of a continuous function g, it may appear to be discontinuous as a function on **R**. For example:



If, on the other hand, we restrict our attention to an interval $\langle a, b\rangle \subset [0,1]$ for which the range of $g|_{\langle a,b\rangle}$ is contained in $(0,1)$, then $g|_{[a,b]}$ can be drawn to appear continuous (in the real sense) by omitting superfluous points. For example, let $\langle a, b\rangle = (1/4, 1/2)$; then $g|_{[1/4,1/2]}$ looks like the figure below, a pleasant enough continuous function on an interval. However, g itself, though continuous on S^1, does not have a fixed point in $[0,1]$ and this deficiency can be overcome if we make the above restriction. The key is that the order topology on a proper subinterval (a,b) of $[0,1]$ agrees with the circle topology on (a,b) whereas the order topology on $[0,1]$ (we will drop the $\bmod 1$) is not the circle topology.

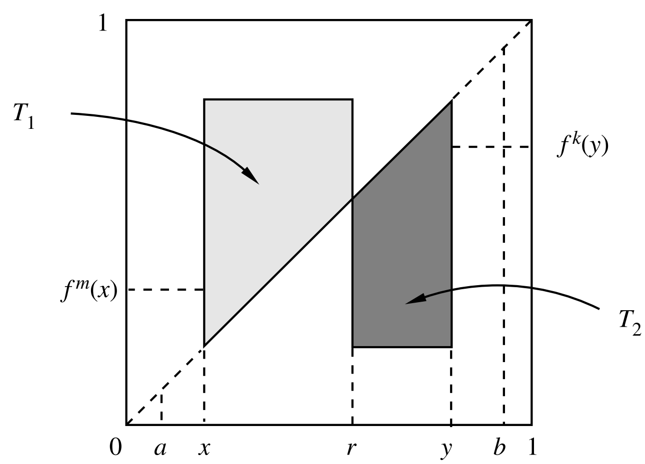


We now do (3.1) for the circle.

Lemma (6.1). *Let $f : S^1 \to S^1$ have a dense orbit. If $(a,b) \subset S^1$ is free of periodic points so is $f^j(a,b)$ for all $j \geq 0$.*

Proof. If $(a, b) = S^1$, then all is trivial so from now on we will assume that $(a, b) \subset S^1$ means $0 \le a < b \le 1$ so (a, b) is a proper subset of $[0, 1] = S^1$. Now suppose that the conclusion is false, i.e., there exists $r \in (a, b)$ with $f^n(r)$ a periodic point. Hence, $f^i(r) \notin (a, b)$ for all $i \ge n$.

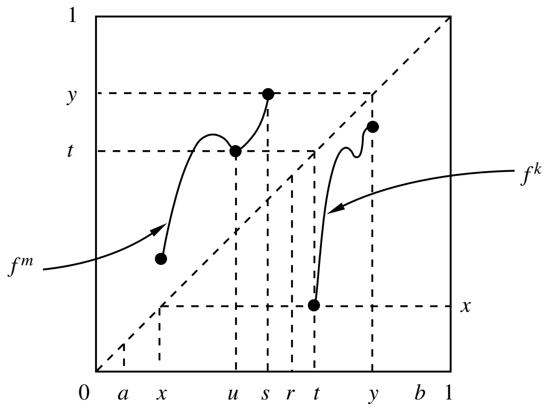
There exists x, with dense orbit, in (a, r) and $m > n$ such that $x < f^m(x) < r$ and there exists y, with dense orbit, in (r, b) and $k > n$ such that $y > f^k(y) > r$. Note that neither $f^m(r)$ nor $f^k(r)$ is in (a, b).



Let T_1 be the trapezoid determined by $(x, x), (r, r), (r, y)$, and (x, y), and let T_2 be the trapezoid determined by $(y, y), (r, r), (r, x)$, and (y, x).

Since $(r, f^m(r)) \notin T_1$ the graph of f^m must be on the boundary of T_1 for some $\alpha \in (x, r)$. It cannot be on the identity line for in that case $f^m(\alpha) = \alpha$ and α would be a periodic point in (a, b). Hence, $f^m(\alpha) = y$ and so there is a least $s \in (x, r)$ with $f^m(s) = y$.

By the same argument, there is a greatest $t \in (r, y)$ with $f^k(t) = x$ and consequently the graph of $f^k|_{[t,y]} \subset T_2$. Also there exists a greatest $v \in (x, s)$ with $f^m(v) = t$.



Now $f^m([v,s]) = [t,y]$ and $f^k([t,y]) \subset [x,y]$. Hence,

$$f^{m+k}([v,s]) \subset [x,y] \quad \text{and} \quad f^{m+k}(v) = x < v, \qquad f^{m+k}(s) = f^k(y) > s.$$

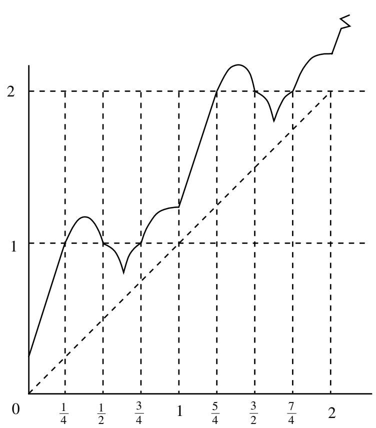
Thus, there exists $z \in (v,s)$ such that $f^{m+k}(z) = z$. This contradicts the fact that there are no periodic points in (a,b). □

Another idea that will prove useful is that of a lift. G is a lift of g if

(1) $G : \mathbf{R} \to \mathbf{R}$ is continuous and the function $G(x) - x$ is periodic with period 1.

(2) $G(x) \bmod 1 = g(x \bmod 1)$.

Note that if G is a lift then $G + k$, $k \in \mathbf{Z}$ is also a lift. Following is a picture of a lift for the g at the beginning of this section.



Lemma (6.2). *If $f : S^1 \to S^1$ has a dense orbit and F is a lift of f with $F(c) = F(d)$, $0 \le c < d < 1$, then f has a periodic point in $[c, d]$.*

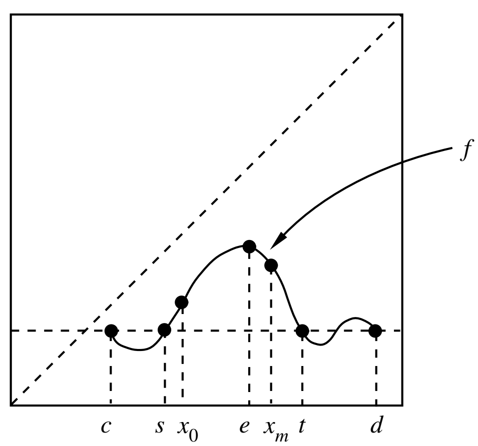
Proof. F cannot be constant on $[c, d]$, for otherwise f could not have a dense orbit, and without loss of generality we will suppose F has a maximum on $[c, d]$ at e with $F(e) > F(c)$. Choose the lift F so that $F(e) \in (0, 1]$ and let

$$w = \max\{0, F(c)\}.$$

The continuity of F allows us to find

$$c' \in [c, e) \quad \text{and} \quad d' \in (e, d]$$

with $F(c') = F(d') = w$. Now let s be the maximum of all such c' and let t be the minimum of all such d' so the graph of F on $[s, t]$ is identical to the graph of f and $f(x) > f(t)$ for $x \in (s, t)$.



Choose $x_0 \in (s, e)$ with dense orbit under f and close to s so that $f(x_0)$ is close to $f(s)$. There is an $m > 0$ such that $x_m = f^m(x_0)$ is close to e so

$$x_0 < x_m < t \quad \text{and} \quad f(x_0) < f(x_m).$$

Let $\alpha = \sup\{x | x \in [x_0, t] \text{ and } x < f^m(x) < t\}$. Now either $\alpha < t$ or $\alpha = t$ and in either case $\alpha \le f^m(\alpha) \le t$ with at least one of the inequalities being equality. If $\alpha = t$ then $\alpha = f^m(\alpha)$ is a periodic point in $[c,d]$. The same holds if $\alpha < t$ and $\alpha = f^m(\alpha)$ so the only case remaining is $\alpha < t$, and $\alpha < f^m(\alpha) = t$, $\alpha \in (s,t)$.

Consider the function

$$g(x) = f(f^m(x)) - f(x), \qquad x \in [x_0, \alpha].$$

Note that $g(x_0) > 0$ and $g(\alpha) < 0$ so that there exists $\beta \in (x_0, \alpha)$ with $g(\beta) = 0$. This implies $f^{m+1}(\beta) = f^m(f(\beta)) = f(\beta)$ is a periodic point. Hence, we have a periodic point in $f((c,d))$ and thus by (6.1), (c,d) must contain a periodic point as well. □

One further notion is that if h is a homeomorphism on S^1 then either h is orientation-preserving, i.e., a lift of h is monotone increasing, or h is orientation-reversing, i.e., a lift of h is monotone decreasing.

We now turn to the main theorem of the paper.

7. A dense orbit almost implies chaos on the circle.

Theorem (7.1). *If $f : S^1 \to S^1$ has a dense orbit, then any of the following are equivalent to f being chaotic*:

(a) *f has a periodic point*

(b) *f is not one-to-one*

(c) *f is sensitive to initial conditions*

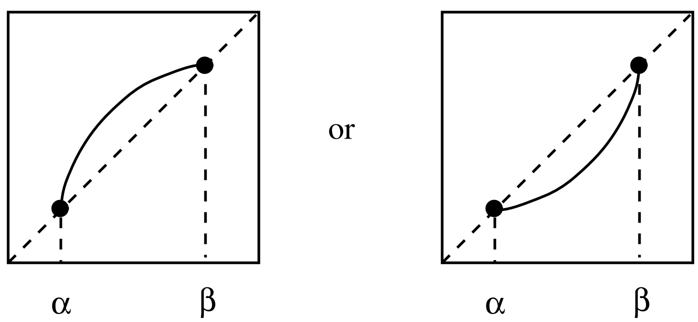
(d) *f has a nondense orbit*

(e) *f is not conjugate to an irrational rotation.*

Proof of (a). Proceed as in the proof of (3.2) letting $M = [0,1]$ in Case 1. In order to make the correspondence between $[0,1]$ and S^1 function in the proof, it is necessary to restrict to a proper subinterval of $[0,1]$. But $\langle \alpha, \beta \rangle \neq [0,1]$ since by hypothesis f has a periodic point. Of course, one must also use Lemma (6.1) rather than Lemma (3.1) whose proof does not adapt to S^1 but was included by virtue of its brevity on the line. □

Proof of (b). First we show that if f is chaotic then f is not one-to-one or, which is the same, we will show that if f is one-to-one and has at least two periodic points then there is no dense orbit.

If f has two periodic points $0 \le a < b < 1$, then a and b are fixed points for f^n where n is the product of the periods. If the set of fixed points for f^n is dense in (a, b), then $f^n(x) = x$ on $[a, b]$ and thus there cannot be a dense orbit (any dense orbit must land in (a, b) and hence becomes a periodic point). So in order to continue there must exist a maximum open interval free of fixed points in (a, b), say (α, β) with α and β fixed points of f^n. If f is orientation-preserving the graph of the homeomorphism f^n looks like the following figure.



Without loss of generality, we assume the left picture which implies that if $x_0 \in (\alpha, \beta)$ then $x_0, x_n, x_{2n} \ldots \nearrow \beta$, and the same argument as at the end of the proof of (3.2) shows that a tail of $\{x_j\}$ is arbitrarily close to $\{\beta, f(\beta), \ldots, f^{n-1}(\beta)\}$ so $\{x_j\}$ is not dense. Thus, f has no dense orbit.

If f is orientation-reversing, then it automatically has two fixed points and hence so does f^2 which is orientation-preserving.

We now show that if f is not one-to-one then f is chaotic. Let $f(a) = f(b)$, $a \ne b$ and with no loss of generality, we may also let $a = 0$ and $b < 1$. Let F be that lift of f with $0 \le F(0) < 1$.

If we can show that f has a periodic point we will be done by part (a). So assume not. Thus, $0 < F(0) < 1$ and $F(b) = F(0) + k$. We will now find two points where F has the same value.

Case 1. If $F(0) = b$, then $F(b) = b + k$ implies that $f(b) = b$ which is a periodic point.

Case 2. If $F(0) > b$, then $k = 0$. For if $k < 0$ there exists $x \in (0, b)$ with $F(x) = x$ (draw a picture) and thus x is a fixed point for f. If $k > 0$, then $F(b) \geq F(0) + 1 > b + 1$ implies that there exists $x \in (0, b)$ such that $F(x) = x + 1$ which is again a fixed point of f. Hence, $F(0) = F(b)$.

Case 3. If $F(0) < b$, then $k = 1$. If $k \leq 0$, then there exists $x \in (0, b)$ with $F(x) = x$ so $f(x) = x$. If $k \geq 2$, then $F(b) > 2 > b + 1$ and there exists $x \in (0, b)$ with $F(x) = x + 1$ so $f(x) = x$. Since f is continuous, $F(1) = F(0) + m$ and $m = 1$ for the same reason that $k = 1$. Hence, $F(b) = F(1)$.

So in any case there exists $0 \leq c < d \leq 1$, with not both $c = 0$ and $d = 1$, and $F(c) = F(d)$.

Now use Lemma (6.2) to obtain a periodic point for f. □

It should be pointed out that by modifying an example due to Denjoy [**8**, p. 107] one can find continuous maps on the circle that are not one-to-one and have no periodic points.

Proof of (c). We first note the following combinatorial fact: Let

$$I_n = \{0, 1, \dots n - 1\}, \qquad S \subset I_n \times I_n.$$

If $(j, I_n) \cap S \neq \varnothing$ for all j and whenever $(x, y), (y, z) \in S$, then $(x, z) \in S$, then there exists a point (x, x) in S.

Assume otherwise. Then S must contain

$$\begin{gathered}(0, n_1) \text{ with } n_1 \notin \{0\}\\ (n_1, n_2) \text{ with } n_2 \notin \{0, n_1\}\\ (n_2, n_3) \text{ with } n_3 \notin \{0, n_1, n_2\}\\ \vdots\\ (n_{n-1}, n_n) \text{ with } n_n \notin \{0, n_1, n_2, \dots, n_{n-1}\} = I_n,\end{gathered}$$

which is a contradiction. Thus the fact is established.

Now f being sensitive to initial conditions implies that there exists $\delta > 0$ such that for x and $N(x)$ there is $y \in N(x)$ and $n = n_{x,y}$ such that

$$|f^n(x) - f^n(y)| > \delta.$$

Choose $m \in \mathbf{Z}^+$ so that $\delta > 2/m$ and set

$$x_i = (i/m) + (1/2m), \qquad i = 0, 1, \dots, m-1.$$

Let

$$N(x_i) = (x_i - 1/2m, x_i + 1/2m) = (i/m, (i+1)/m).$$

There exists $y_i \in N(x_i)$ and n_i such that

$$|f^{n_i}(x_i) - f^{n_i}(y_i)| \geq \delta.$$

This implies there exists $k \in I_m$ such that

$$f^{n_i}([i/m, (i+1)/m]) \supset [k/m, (k+1)/m].$$

Let

$$M_i = \{k | f^s([i/m, (i+1)/m]) \supset [k/m, (k+1)/m] \text{ for some } s\},$$

so that M_i is nonempty.

Let $S = \{(i,k) | k \in M_i, i = 0, 1, \dots, m-1\}$. Then $(i, I_m) \cap S \neq \varnothing$ for all $i \in I_m$. Also, if (i,j) and (j,k) are in S then

$$f^{n_i+n_j}([i/m, (i+1)/m]) \supset [k/m, (k+1)/m]$$

so (i,k) is in S. Hence, by the combinatorial fact above we have k such that $(k,k) \in S$, i.e., there exists s with

$$f^s([k/m, (k+1)/m]) \supset [k/m, (k+1)/m]).$$

Now either

(1) f^s has a fixed point in $[k/m, (k+1)/m]$, or

(2) f^s has no fixed point in $[k/m, (k+1)/m]$.

If (1), then f has a periodic point and thus by part (a) f is chaotic. If (2), then $f^s([k/m,(k+1)/m])$ must be equal to S^1 to avoid a fixed point (see Section 6). Now f^s cannot be a homeomorphism for that would say that the closed interval and the circle are homeomorphic which is false. But if f^s is not a homeomorphism then neither is f which implies that f is chaotic by part (b). □

Henceforth α will denote an irrational number.

Proof of (e). We know that a rotation, R_α, where $R_\alpha(x) = x + \alpha \bmod 1$, has a dense orbit and no periodic points so the same is true for any conjugate of R_α. So what remains to be shown is that if f is not chaotic then $f \sim R_\alpha$.

However, if f is not chaotic, it is a homeomorphism (part (b)) with no periodic points (part (a)). So we will be done with part (e) if we can show

Proposition (7.2). *A homeomorphism with a dense orbit and no periodic points is conjugate to R_α.*

This result was known to Poincaré [**15**]. Improved proofs were given by Kneser [**11**], Nielson [**13**], and van Kampen [**17**]. A number of more modern books contain versions of this result, e.g., [**5,6,10,11,18**]. For completeness we will sketch a proof of (7.2).

Since an orientation-reversing homeomorphism has a periodic point we know that the map f is orientation-preserving. The rotation number of $f, \rho(f)$, is defined as follows. If F is a lift of f, then

$$\rho(F) = \lim_{n\to\infty} \frac{F^n(x)}{n}$$

exists, is independent of x, and is irrational if and only if f has no periodic points. Then $\rho(f) =$ the fractional part of $\rho(F) = \alpha$.

To see why $f \sim R_\alpha$, let $x_0 \in S^1$ have dense orbit and let $F^n(x_0) = X_n$. Now let

$$A = \{n\alpha + m | n, m \in \mathbf{Z}\}, \qquad B = \{X_n + m | n, m \in \mathbf{Z}\},$$

and define $H : A \to B$ by

$$H(n\alpha + m) = X_n + m.$$

It is not difficult to show that A and B are dense in $\mathbf{R}$ and H is one-to-one. More difficult is that H is monotone increasing (this uses the definition of $\rho(f)$). Thus, H has a unique extension $\bar{H}$ to $\mathbf{R}$ that is an increasing homeomorphism. Because

$$\overline{H}(n\alpha + m + 1) = X_n + m + 1 = \overline{H}(n\alpha + m) + 1$$

we have

$$\overline{H}(y + 1) = \overline{H}(y) + 1.$$

Thus, we can *unlift* $\overline{H}$ to obtain a homeomorphism h on S^1.

Also,

$$\begin{aligned}\overline{H}^{-1}(F(X_n + m)) &= \overline{H}^{-1}(X_{n+1} + m) = (n+1)\alpha + m \\ &= n\alpha + m + \alpha = \overline{H}^{-1}(X_n + m) + \alpha\end{aligned}$$

implies $H^{-1}(F(y)) = H^{-1}(y) + \alpha$ and hence

$$h^{-1}fh(y) = y + \alpha, \quad \text{i.e., } f \sim R_\alpha. \qquad \square$$

Proof of (d). If f has a nondense orbit, it cannot be a conjugate of an irrational rotation R_α which has nothing but dense orbits. Thus, by part (e) f is chaotic. If all orbits of f are dense, then f is not chaotic, so $f \sim R_\alpha$. $\square$

8. Further observations on $\mathbf{S^1}$ and applications of (7.1). In the literature cited above a more common way of expressing (7.2) is:

(8.1) Let g be a homeomorphism with $\rho(g) = \alpha$. If g has a dense orbit, then $g \sim R_\alpha$.

Along with this one finds:

(8.2) If g is a homeomorphism with $\rho(g) = \alpha$ and $E = E_x =$ the set of limit points of the orbit of x, then

(1) E is independent of x and invariant, i.e., $g(E) = E$.

(2) $E = S^1$ (ergodic, transitive, or minimal case) or

(3) E is perfect and nowhere dense.

Thus one sees

(8.1)′ Let g be a homeomorphism with $\rho(g) = \alpha$. If g is ergodic, then $g \sim R_\alpha$.

From (7.1)(a) a map with a dense orbit and a periodic point is chaotic, but (7.1)(b) says a homeomorphism is not chaotic so

(8.3) A homeomorphism with a periodic point cannot have a dense orbit.

Also (7.1)(a),(b), and (e) give stronger versions of (7.2).

(8.4) A map with a dense orbit and no periodic points is conjugate to R_α.

(8.5) A homeomorphism with a dense orbit is conjugate to R_α which implies

(8.6) A homeomorphism cannot have both dense and nondense orbits.

In the spirit of this section, we must include a statement of Denjoy's [**7**] elegant sufficient condition guaranteeing the ergodic case in (8.2).

(8.7) If f is a C^2 diffeomorphism (f and its inverse are twice continuously differentiable) with no periodic points, then $f \sim R_\alpha$.

9. Higher dimensions. A few examples due to Ray Mayer will show that no results as striking as those of Sections 3 and 7 will be revealed for dimensions larger than 1.

Let f be chaotic on $[0,1]$ and r_0 have a dense orbit. Let

$$D = \{(r,\theta) | r \in [0,1], \theta \in S^1\}$$

be the unit disc and $G : D \to D$ be defined by

$$G(r,\theta) = (f(r), R_\alpha(\theta)), \quad \alpha \text{ irrational}.$$

Then $(r_0, 0)$ has a dense orbit, $(0,0)$ is the only periodic point and G is sensitive to initial conditions.

The latter two statements are clear enough. To show that $(r_0, 0)$ has a dense orbit let $y \in [0,1]$ be periodic with respect to f with period Y and let ε be positive. Now let K be large enough to guarantee that the set

$$\{R_\alpha^{kY}(0)\}_{k=0}^K = \{kY\alpha \bmod 1\}_{k=0}^K$$

is within $\varepsilon/2$ of any point in S^1 (notice that $Y\alpha$ is just another irrational number).

There is a neighborhood of y, $N(y) \subset [0,1]$, such that $z \in N(y)$ implies that

$$|f^n(z) - f^n(y)| < \varepsilon/2$$

for $0 \le n \le KY$. Now choose m so that $r_m \in N(y)$. Hence the set

$$\{G^{m+n}(r_0, 0) | 0 \le n \le KY\}$$

is within ε of any point in $\{(y,\theta)|\theta \in S^1\}$. But the set $\{(y,\theta)|y$ is periodic, $\theta \in S^1\}$ is dense in D so $(r_0, 0)$ has dense orbit.

Lest one may wonder that demanding a few more periodic points along with a dense orbit might imply chaos in two dimensions, the next example has a continuum of periodic points and a dense orbit and does not satisfy **DPP**.

Again, let D be the unit disc and $H : D \to D$ by

$$H(r,\theta) = (4r(1-r), \theta + \alpha(r - 3/4)^2).$$

The set of periodic points of H is $\{(0,0)\} \cup \{(3/4,\theta)|\theta \in S^1\}$, and H has a dense orbit for reasons similar to the above argument for G.

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3402 S.W. Corbett Ave., Portland, OR 97201