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SHOCK LAYER BEHAVIOR FOR A QUASILINEAR BOUNDARY VALUE PROBLEM

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ABSTRACT. We provide sufficient conditions for both the existence and detailed approximations of shock layer solutions of the quasilinear problem

$$\varepsilon y'' = f(t, y)y' + g(t, y)$$
$$y = A, \qquad y(b) = B,$$

subject to relatively weak regularity requirements on the data ($\varepsilon > 0$ is a small positive parameter). Our approach is new and is based upon previous studies of this problem where detailed approximations have been given for solutions with boundary layer behavior at either the left- or right-hand endpoints. By joining boundary layer solutions together in an appropriate fashion, we are able to obtain the existence of solutions with shock layer behavior. Problems of this type arise in fluid dynamics.

1. Introduction. We consider here the singularly perturbed scalar boundary value problem

$$\begin{aligned} \varepsilon y'' &= f(t, y)y' + g(t, y), \qquad a < t < b, \\ y(a) &= A \quad \text{and} \quad y(b) = B, \end{aligned} \tag{1.1}$$

where $\varepsilon > 0$ is a small positive parameter, and where y, f, g, A, and B are real-valued quantities. Our goal is to provide sufficient conditions on the data so that there exists a solution of problem (1.1) exhibiting interior (shock) layer behavior at $t = T$ as $\varepsilon \to 0$, where T is a fixed constant in the interval $a < T < b$ to be determined below. Detailed approximations of solutions to (1.1) will also be obtained.

We approach this problem by considering the following two problems which have, under appropriate conditions such as those given below, boundary layer behavior at $t = \tau$, where $\tau = T + O(\varepsilon)$:

$$\begin{aligned} \varepsilon y_L'' &= f(t, y_L)y_L' + g(t, y_L), \qquad a < t < \tau, \\ y_L(a) &= A \quad \text{and} \quad y_L(\tau) = p, \end{aligned} \tag{1.2}$$

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and

$$\begin{aligned} &\varepsilon y_R'' = f(t, y_R)y_R' + g(t, y_R), \qquad \tau < t < b, \\ &y_R(\tau) = p \quad \text{and} \quad y_R(b) = B. \end{aligned} \tag{1.3}$$

p is a real valued constant to be discussed below. The idea is to let τ be a free parameter and to join the left solution, $y_L(t, \varepsilon)$, with the right solution, $y_R(t, \varepsilon)$, at some value of τ, say $\tau = \tau_0$, so that $y_L(\tau_0) = y_R(\tau_0)$ and $y_L'(\tau_0) = y_R'(\tau_0)$. Once this has been accomplished, we see from (1.2) and (1.3) that $y_L''(\tau_0) = y_R''(\tau_0)$, and hence that the function

$$y(t, \varepsilon) = \begin{cases} y_L(t, \varepsilon), & a \le t \le \tau_0, \\ y_R(t, \varepsilon), & \tau_0 \le t \le b, \end{cases}$$

is a solution of (1.1) with interior layer behavior. In Figure 1.1 the function passing through the point (τ_0, p) has a continuous derivative at that point and is a solution of (1.1). The functions passing through the points (τ_1, p) and (τ_2, p) do not have continuous derivatives at these points and consequently are not solutions to problem (1.1). As discussed below in our proof, it will be sufficient to vary the parameter τ in an interval $[\tau_1, \tau_2]$, where $\tau_2 - \tau_1 = O(\varepsilon)$, enabling detailed approximations for shock layer solutions to be obtained.

A number of papers have appeared in the literature throughout the last 30 years dealing with a wide variety of interior layer phenomena, such as in Haber and Levinson [**4**], Vasilèva [**12**], O'Malley [**10**], Boglaev [**1**], Fife [**3**], Howes [**5**], Lutz and Sari [**8**], Smith [**11**], and Jeffries [**6**]. It appears, however, that much remains that is unknown concerning interior layer behavior, for both the scalar as well as the vector boundary value problems. Most of the results to date have used rather strong assumptions on the data, or have not provided the detailed information, quantitatively speaking, for the solutions inside the interior layers. The work we present here has been highly motivated by the work of Howes [**5**] and O'Donnell [**9**]. Howes [**5**] provided shock layer results for the scalar problem (1.1), but neglected to include the appropriate "interior layer stability conditions," as they are automatically satisfied by certain types of problems, an example of which will be discussed in Section 4. O'Donnell [**9**] dealt with a vector analog of problem (1.1) and also did not include the necessary interior layer stability conditions. Using an asymptotic expansion and Green function technique of Smith [**11**], Jeffries [**6**] obtained an existence

FIGURE 1.1. The solid curves represent solutions to problem (1.3) and the dashed curves represent solutions to problem (1.2) for $\tau = \tau_1$, τ_0, and τ_2. $\tau_2 - \tau_1 = O(\varepsilon)$.

result providing detailed approximations of solutions of shock layer type for problem (1.1), although he required the functions f and g to satisfy strong smoothness conditions ($f, g, \in C^{(3)}$ w.r.t. y). The results in this section provide existence and fairly detailed approximations of solutions to (1.1) throughout the interval $[a, b]$, with the only lack of detailed information occurring inside the interior layer. Here our shock layer correction terms are just $O(\varepsilon)$ translations in the independent variable t away from providing uniformly valid detailed approximations (y_a) of the exact solutions (y), i.e., $|y(t, \varepsilon) - y_a(t, \varepsilon)| = O(\varepsilon)$ for t in $[a, b]$; these results are obtained under relatively weak conditions.

2. Preliminary results. In order to obtain our results, we make use of the following two theorems which provide existence and detailed approximations of solutions with boundary layer behavior at either the left or right endpoints. They are equivalent to Theorems 3.3.1 and 3.3.2 of Kirschvink [**7**] and are extensions of the results of Coddington and Levinson [**2**] and Howes [**5**]; the latter two papers do not provide the detailed approximation of solutions inside the boundary layers, which is crucial in obtaining the existence of shock layer solutions as is done in this paper. The two domains referred to in the theorems are defined in terms of lower and upper solutions as is done in Kirschvink [**7**], namely,

$$\mathcal{D}_L = \{(t,y) : a \le t \le \tau,\ \alpha_L(t,\varepsilon) \le y \le \mathcal{B}_L(t,\varepsilon)\}$$

and

$$\mathcal{D}_R = \{(t,y) : \tau \le t \le b,\ \alpha_R(t,\varepsilon) \le y \le \mathcal{B}_R(t,\varepsilon)\},$$

where $(\alpha_L, \mathcal{B}_L)$ and $(\alpha_R, \mathcal{B}_R)$ are appropriate bounding pairs corresponding to problems (1.2) and (1.3), respectively. A clear discussion of the domains used for the shock layer results will be discussed below.

Theorem 2.1. *Assume*

(1) *the reduced problem* $0 = f(t,u)u' + g(t,u)$, $u(a) = A$, *has a solution* $u_L = u_L(t)$ *of class* $C^{(2)}[a,\tau]$;

(2) f *and* g *are of class* $C^{(1)}$ *with respect to* t *and* y *for all* (t,y) *in* $\mathcal{D}_L$;

(3) *the reduced solution* $u_L = u_L(t)$ *is globally stable, that is,* $f(t,u_L(t)) \ge k > 0$ *for* t *in* $[a,\tau]$;

(4) *the inequality*

$$(u_L(\tau) - p) \cdot \int_n^{u_L(\tau)} f(\tau, s)\, ds > 0$$

holds for $p \le n < u_L(\tau)$ *if* $p < u_L(\tau)$, *or for* $u_L(\tau) < n \le p$ *if* $u_L(\tau) < p$.

Then there exists a solution $y_L = y_L(t,\varepsilon)$ *of problem* (1.2) *for each sufficiently small* $\varepsilon > 0$ *such that, for* t *in* $[a,\tau]$,

$$y_L(t,\varepsilon) = u_L(t) + w_L(t,\varepsilon) + O(\varepsilon) \tag{2.1}$$

and

$$y'_L(t,\varepsilon) = u'_L(t) + w'_L(t,\varepsilon) + O\left(e^{\frac{q_1(t-\tau)}{\varepsilon}}\right) + O(\varepsilon), \tag{2.2}$$

where $w_L(t,\varepsilon)$ *is the unique solution of the equation*

$$\varepsilon w''_L = f(\tau, u_L(\tau) + w_L)w'_L \tag{2.3}$$

satisfying $w_L(\tau) = p - u_L(\tau)$ *and* $\lim_{\varepsilon\to 0} w_L(t,\varepsilon) = 0$ *for each fixed* $t < \tau$. q_1 *is a positive constant.*

Theorem 2.2. *Assume that*

(1) *the reduced problem* $0 = f(t,u)u' + g(t,u)$, $u(b) = B$, *has a solution* $u_R = u_R(t)$ *of class* $C^{(2)}[\tau, b]$;

(2) f *and* g *are of class* $C^{(1)}$ *with respect to* t *and* y *for* (t,y) *in* $\mathcal{D}_R$;

(3) *the reduced solution* $u_R = u_R(t)$ *is globally stable, that is,* $f(t, u_R(t)) \le -k < 0$ *for* t *in* $[\tau, b]$;

(4) *the inequality*

$$(u_R(\tau) - p) \cdot \int_n^{u_R(\tau)} f(\tau, s)\, ds < 0$$

holds for $p \le n < u_R(\tau)$ *if* $p < u_R(\tau)$, *or for* $u_R(\tau) < n \le p$ *if* $u_R(\tau) < p$.

Then there exists a solution $y = y(t,\varepsilon)$ *of problem* (1.3) *for each sufficiently small* $\varepsilon > 0$ *such that, for* t *in* $[\tau, b]$,

$$y_R(t,\varepsilon) = u_R(t) + w_R(t,\varepsilon) + O(\varepsilon) \tag{2.4}$$

and

$$y'_R(t,\varepsilon) = u'_R(t) + w'_R(t,\varepsilon) + O\left(e^{\frac{-q_2(t-\tau)}{\varepsilon}}\right) + O(\varepsilon), \tag{2.5}$$

where $w_R(t,\varepsilon)$ *is the unique solution of the equation*

$$\varepsilon w''_R = f(\tau, u_R(\tau) + w_R)w'_R \tag{2.6}$$

satisfying $w_R(\tau,\varepsilon) = p - u_R(\tau)$ *and* $\lim_{\varepsilon\to 0} w_R(t,\varepsilon) = 0$ *for each fixed* $t > \tau$. q_2 *is a positive constant.*

3. Shock layer theory. In this section we present conditions that ensure the existence of solutions to problem (1.1) exhibiting shock layer behavior at some point $t = T$, $a < T < b$. A major assumption will be that the reduced equation

$$0 = f(t,y)y' + g(t,y) \tag{3.1}$$

has two solutions $u_L = u_L(t)$ and $u_R = u_R(t)$ of class $C^{(2)}$ on $[a, t_L]$ and $[t_R, b]$, respectively, with $a \le t_R < t_L \le b$; moreover, $u_L(a) = A$ and $u_R(b) = B$. Another condition of importance will be that there exists a point T in (t_R, t_L) such that $J(T) = 0$ and $J'(T) \ne 0$, where

$$J(t) = \int_{u_L(t)}^{u_R(t)} f(t,s)\,ds \quad \text{for } t_R < t < t_L. \tag{3.2}$$

We also assume that $u_L(T) \ne u_R(T)$, since equality would preclude the possibility of a shock layer. If $u_L(T) = u_R(T)$, our results would still imply the existence of a solution, but the "shock layer jump" would either be nonexistent or extremely small in magnitude, i.e., $O(\varepsilon)$.

It will be necessary to impose smoothness requirements on the functions $f(t,y)$ and $g(t,y)$ in certain regions, so we now define a domain $\mathcal{D}$ which will be referred to in Theorem 3.1 below. We first mention that the domain $\mathcal{D}$ can have two basic shapes, depending on whether $u_L(T) > u_R(T)$ or $u_L(T) < u_R(T)$. Assuming $u_L(T) > u_R(T)$, we define the domain $\mathcal{D}$ as follows. Let

$$\mathcal{D}_1 = \{(t,y) : |y - u_L(t)| \le O(\varepsilon),\ a \le t \le t_R\},$$
$$\mathcal{D}_2 = \{(t,y) : u_R(t) + O(\varepsilon) \le y \le u_L(t) + O(\varepsilon),\ t_R \le t \le t_L\},$$

and

$$\mathcal{D}_3 = \{(t,y) : |y - u_R(t)| \le O(\varepsilon),\ t_L \le t \le b\}.$$

Then $\mathcal{D}$ is defined as the union of these three domains, namely,

$$\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3.$$

FIGURE 3.1. The domain $\mathcal{D}$ for the case $u_L(T) > u_R(T)$ is shown. $t_R - t_L = O(\varepsilon \ln \varepsilon)$.

If $u_L(T) < u_R(T)$, one only interchanges the placement of $u_R(t)$ and $u_L(t)$ in $\mathcal{D}_2$ to define $\mathcal{D}$. A diagram of $\mathcal{D}$ for the case $u_L(T) > u_R(T)$ is shown in Figure 3.1. The position of t_R and t_L may be chosen such that $t_R - t_L = O(\varepsilon \ln \varepsilon)$. We note that with this choice of t_R and t_L, and sufficiently small ε, $u_L(T) > u_R(T)$ implies that $u_L(t) > u_R(t)$ for t in the interval $[t_R, t_L]$. Reducing the size of $\mathcal{D}$ seems to be difficult, since having $t_L - t_R$ smaller than $O(\varepsilon \ln \varepsilon)$ would not ensure that the exponentially decaying left and right solutions, y_L and y_R, defined by (1.2) and (1.3), respectively, would lie entirely in the domain $\mathcal{D}$, which is crucial in the proof of Theorem 3.1 given below.

Theorem 3.1. *Assume that*

(1) *the reduced equation* (3.1) *has two solutions* $u_L = u_L(t)$ *and* $u_R = u_R(t)$ *as described above*;

(2) *the functions* f *and* g *are of class* $C^{(1)}$ *with respect to* t *and* y *in the domain* $\mathcal{D}$;

(3) *the function* $u_L(t)$ *is stable on* $[a, t_L]$, *that is,* $f(t, u_L(t)) > 0$, *and* u_R *is stable on* $[t_R, b]$, *that is,* $f(t, u_R(t)) < 0$;

(4) *there is a point* T *in* (t_R, t_L) *such that* $J(T) = 0$ *and* $J'(T) \neq 0$, *where* $J(t)$ *is defined in* (3.2);

(5) *the inequality*

$$(u_R(T) - u_L(T)) \cdot \int_n^{u_R(T)} f(T, s)\, ds < 0$$

holds for $p \leq n < u_R(T)$ *if* $u_L(T) < u_R(T)$, *or for* $u_R(T) < n \leq p$ *if* $u_L(T) > u_R(T)$, *where* p *is an appropriately chosen number between* $u_L(T)$ *and* $u_R(T)$;

(6) *the inequality*

$$(u_R(T) - u_L(T)) \cdot \int_{u_L(T)}^{n} f(T, s)\, ds > 0$$

holds for $u_L(T) < n \leq p$ *if* $u_L(T) < u_R(T)$, *or for* $p \leq n < u_L(T)$ *if* $u_L(T) > u_R(T)$, *where* p *is as defined in assumption* (5).

Then there exists a solution $y = y(t, \varepsilon)$ *of problem* (1.1) *for each sufficiently small* $\varepsilon > 0$ *such that*

$$y(t, \varepsilon) = \begin{cases} u_L(t) + w_L(t + O(\varepsilon), \varepsilon) + O(\varepsilon) & \text{for } a \leq t \leq T, \\ u_R(t) + w_R(t + O(\varepsilon), \varepsilon) + O(\varepsilon) & \text{for } T \leq t \leq b, \end{cases} \tag{3.3}$$

and

$$y'(t, \varepsilon) = \begin{cases} u_L'(t) + w_L'(t, \varepsilon) + O(e^{q_1(t-T)}) + O(\varepsilon) & \text{for } a \leq t \leq T, \\ u_R'(t) + w_R'(t, \varepsilon) + O(e^{-q_2(t-T)}) + O(\varepsilon) & \text{for } T \leq t \leq b, \end{cases} \tag{3.4}$$

where w_L *and* w_R *are the solutions to problems* (2.3) *and* (2.6), *respectively, for* $\tau = T$. q_1 *and* q_2 *are positive constants.*

Before proving this theorem, we note that the estimates in (3.3) do not provide the $O(\varepsilon)$ accuracy for $y(t,\varepsilon)$ as in the estimates (2.1) and (2.4), even though the estimates for $y'(t,\varepsilon)$ given in (3.4) are essentially the same as the estimates of (2.2) and (2.5). This difficulty arises because of our inability to precisely pin down the position of points on the solution inside the shock layer. Since the derivative of the solution, $y'(t,\varepsilon)$, is $O(1/\varepsilon)$ inside the shock layer, we would need to know the t-coordinates of points inside the shock layer to within $O(\varepsilon^2)$ in order to obtain $O(\varepsilon)$ accuracy in the solution. However, useful information is contained in (3.3) since $w_L(t+O(\varepsilon),\varepsilon)$ and $w_R(t+O(\varepsilon),\varepsilon)$ are just $O(\varepsilon)$ translations away from $w_L(t,\varepsilon)$ and $w_R(t,\varepsilon)$, respectively.

Proof. We let τ be a free parameter in problems (1.2) and (1.3) as discussed in Section 1. In order that problems (1.2) and (1.3) have solutions exhibiting boundary layer behavior at $t=\tau$, condition (4) in each of Theorems 2.1 and 2.2 must be satisfied. The first three conditions in Theorems 2.1 and 2.2 follow directly from the conditions as stated in Theorem 3.1. Since f is a continuous function, however, conditions (3), (5), and (6) of Theorem 3.1 imply the validity of condition (4) in each of the Theorems 2.1 and 2.2 provided that τ is sufficiently close to T. In this analysis we set $\tau = T + r\varepsilon$, where r is an $O(1)$ parameter, and, hence, for sufficiently small ε, problems (1.2) and (1.3) have solutions with boundary layer behavior at $t=\tau$ for r in any interval $[r_1, r_2]$, where r_1 and r_2 are fixed constants. The constants q_1 and q_2 in estimates (2.2) and (2.5) actually become functions of τ, namely, $q_1 = q_1(\tau)$ and $q_2 = q_2(\tau)$, if τ is allowed to vary. We thus hasten to mention that the quantities

$$O\left(e^{\frac{q_1(\tau)(t-\tau)}{\varepsilon}}\right) \quad \text{for } a \le t \le \tau$$

and

$$O\left(e^{\frac{-q_2(\tau)(t-\tau)}{\varepsilon}}\right) \quad \text{for } \tau \le t \le b,$$

found in equations (2.2) and (2.5), respectively, are bounded above and below by suitable constants for r in the interval $[r_1, r_2]$.

Integrating equation (2.3) for w_L from $-\infty$ to t and equation (2.6) for w_R from ∞ to t, we obtain, after appropriate integral substitutions,

$$\varepsilon w_L'(t,\varepsilon) = \int_{u_L(\tau)}^{u_L(\tau)+w_L(t,\varepsilon)} f(\tau, s)\, ds \tag{3.5}$$

and

$$\varepsilon w_R'(t,\varepsilon) = \int_{u_R(\tau)}^{u_R(\tau)+w_R(t,\varepsilon)} f(\tau,s)\,ds. \tag{3.6}$$

Evaluating (3.5) and (3.6) at $t=\tau$ and subtracting, we have

$$\begin{aligned} &\varepsilon[w_L'(\tau,\varepsilon) - w_R'(\tau,\varepsilon)] \\ &\qquad = \int_{u_L(\tau)}^{p} f(\tau,s)\,ds + \int_{p}^{u_R(\tau)} f(\tau,s)\,ds \\ &\qquad = \int_{u_L(\tau)}^{u_R(\tau)} f(\tau,s)\,ds = J(\tau). \end{aligned} \tag{3.7}$$

Expanding $J(\tau)$ in a Taylor series about T, we have $J(\tau) = J(T+r\varepsilon) = J(T) + J'(n)\cdot r\varepsilon$, where $n = T+\theta r\varepsilon$ for $0<\theta<1$. But $J(T)=0$, so that we get, from (3.7),

$$w_L'(\tau,\varepsilon) - w_R'(\tau,\varepsilon) = J'(n)\cdot r. \tag{3.8}$$

Since $J'(T)\neq 0$, we must have, for some positive constant c, either $J'(n)>c$ or $J'(n)<-c$ for each sufficiently small ε with r in $[r_1,r_2]$. We write $y_L = y_L(t,\varepsilon,r)$ and $y_R(t,\varepsilon,r)$ to emphasize the dependence of y_L and y_R on the parameter r. One can see from equations (2.2), (2.5) and (3.8) that there exist two values of r, namely r_3 and r_4, such that

$$y_L'(\tau_3,\varepsilon,r_3) > y_R'(\tau_3,\varepsilon,r_3)$$

and

$$y_L'(\tau_4,\varepsilon,r_4) < y_R'(\tau_4,\varepsilon,r_4),$$

where r_3 and r_4 both lie in $[r_1,r_2]$ and where τ_3 and τ_4 are the respective values of τ corresponding to r_3 and r_4. It follows by standard arguments that the solutions y_L and y_R are continuous functions of r, and we may conclude that there must be an $r=r_0$ such that

$$y_L'(\tau_0,\varepsilon,r_0) = y_R'(\tau_0,\varepsilon,r_0), \quad \text{where } \tau_0 = T+\varepsilon r_0,$$

and, hence, the problem (1.1) has a solution with shock layer behavior.

The estimates (3.3) and (3.4) for $a \le t \le T$ can be obtained by showing that the solutions of the two problems

$$\begin{aligned} \varepsilon w_L'' &= f(T, u_L(T) + w_L) w_L', \qquad a < t < T, \\ w_L(T) &= p - u_L(T) \quad \text{and} \quad w_L(-\infty) = 0 \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \varepsilon w_{L1}'' &= f(\tau_0, u_L(\tau_0) + w_{L1}) w_{L1}', \qquad a < t < T, \\ w_{L1}(T) &= p - u_L(\tau_0) \quad \text{and} \quad w_{L1}(-\infty) = 0, \end{aligned} \tag{3.10}$$

where $\tau_0 = T + \varepsilon r_0$, satisfy $|w_L(t) - w_{L1}(t)| = O(\varepsilon)$ and

$$w_L' - w_{L1}' = O\left(e^{\frac{q(t-T)}{\varepsilon}}\right)$$

for t in $[a, T]$. Equation (3.9) is obtained by setting $\tau = T$ in equation (2.3), that is, w_L is as given in the conclusions of Theorem 3.1. Equation (3.10) is obtained by setting $\tau = \tau_0$ in equation (2.3) and translating the independent variable by $t' = t + (T - \tau_0)$ and then rewriting t' as t. Similar results hold on the interval $T \le t \le b$ for the solutions involving the right-hand equations. The details are very similar to the analysis (Gronwall-type arguments) given in Chapter 3 of Kirschvink [**7**] and are therefore omitted. □

4. Examples and remarks. We begin here with an example illustrating the use of Theorem 3.1 in predicting shock layer behavior and approximating the solutions.

Example 4.1. Consider the following problem

$$\begin{aligned} \varepsilon y'' &= -y^{\frac{5}{3}} \cdot y' + y^{\frac{5}{3}}, \qquad 0 < t < 1, \\ y(0) &= -2, \qquad y(1) = \frac{3}{2}, \end{aligned} \tag{4.1}$$

which is of the form of the quasilinear problem (1.1) with $f = -y^{5/3}$ and $g = y^{5/3}$. The functions f and g are only of class $C^{(1)}$ with respect to y since $f''y$ and $g''y$ approach infinity as $y \to 0$, and to the author's knowledge, there is no result in the literature to date that will provide the existence of a shock layer solution to (4.1) as well as the detailed approximations given below. The left and right reduced solutions turn out to be $u_L(t) = t - 2$ and $u_R(t) = t + 1/2$, respectively, which can be used to obtain the function $J(t)$, namely,

$$J(t) = \int_{t-2}^{t+\frac{1}{2}} -s^{\frac{5}{3}}\, ds = \frac{3}{8}\left[(t-2)^{\frac{8}{3}} - (t+\frac{1}{2})^{\frac{8}{3}}\right].$$

Since $J(t)$ has a zero at $T = 3/4$, and all the conditions of Theorem 3.1 are satisfied, problem (4.1) has a solution with a shock layer at $T = 3/4$ as shown in Figure 4.1. The estimates given in (3.3) provide us with the approximate solution

$$y_a(t,\varepsilon) = \begin{cases} t - 2 + w_L(t,\varepsilon) & \text{for } 0 \le t \le \frac{3}{4}, \\ t + \frac{1}{2} + w_R(t,\varepsilon) & \text{for } \frac{3}{4} \le t \le 1, \end{cases} \tag{4.2}$$

where $w_L(t,\varepsilon)$ and $w_R(t,\varepsilon)$ are the respective solutions of the boundary value problems

$$\begin{aligned} \varepsilon w_L'' &= -(-\frac{5}{4} + w_L)^{\frac{5}{3}} \cdot w_L' \\ w_L(\frac{3}{4}) &= p + \frac{5}{4}, \qquad w_L(-\infty) = 0 \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} \varepsilon w_R'' &= -(\frac{5}{4} + w_R)^{\frac{5}{3}} \cdot w_R' \\ w_R(\frac{3}{4}) &= p - \frac{5}{4}, \qquad w_R(\infty) = 0. \end{aligned} \tag{4.4}$$

A convenient choice of p in this example is $p = 0$. Although problems (4.3) and (4.4) appear rather complicated, they are much easier to deal with than the original boundary value problem (4.1); in fact, with the boundary conditions at infinity, they can be integrated and reduced to first order initial value problems which are much easier to solve numerically on a computer than problem (4.1).

Example 4.1 is a special case of a general class of problems which automatically satisfy conditions (5) and (6) of Theorem 3.1, provided that assumptions (1) through (4) are satisfied. For example, functions

FIGURE 4.1. A solution $y = y(t, \varepsilon)$ of problem (4.1) with interior layer behavior at $T = 3/4$ is shown.

$f(t, y)$ such that $f(T, y)$ has a unique zero between $u_L(T)$ and $u_R(T)$ automatically satisfy the shock layer stability conditions (5) and (6). This can be easily seen by choosing the number p described in Theorem 3.1 to be the unique root of $f(T, y)$, namely, $f(T, p) = 0$, and then observing a sketch of $f(T, y)$ such as in Figure 4.2, where for definiteness we have assumed $u_L(T) < u_R(T)$. One can easily see that

$$\int_{u_L(T)}^{n} f(T, s)\, ds > 0 \quad \text{for } u_L(T) < n \leq p,$$

FIGURE 4.2. The function $f(T, y)$ has a unique zero between $u_L(T)$ and $u_R(T)$ and the shock layer stability conditions of Theorem 3.1 are satisfied.

and also that

$$\int_n^{u_R(T)} f(T, s)\, ds < 0 \quad \text{for } p \leq n < u_R(T).$$

The following problem is similar to an interesting example of Smith [**11**].

Example 4.2. Consider the problem

$$\varepsilon y'' = (1 + t) y y' - (t^2 + t) y^3$$

FIGURE 4.3. The graph of $f(T, y)$ is symmetric about the point $(y_0, 0)$ for $u_L(T) \leq y \leq u_R(T)$.

$$y(0) = \frac{1}{2}, \qquad y(1) = -\frac{1}{2},$$

where the function

$$u_R = -\frac{2}{3+t^2}$$
$$J(t) = \int (1+t)s\, ds$$
$$u_L = \frac{2}{4-t^2}$$

has a zero at $T = 1/\sqrt{2} \approx 0.71$. One can show that all the conditions of Theorem 3.1 are satisfied, and we are guaranteed the existence of a solution with a shock layer at $T = 1/\sqrt{2}$. The estimates given in (3.3), with $p = 0$, can be used to write this solution as

$$y(t,\varepsilon) = \begin{cases} \frac{2}{4-t^2} + w_L(t + O(\varepsilon), \varepsilon) + O(\varepsilon) & \text{for } 0 \le t \le \frac{1}{\sqrt{2}}, \\ \frac{-2}{3+t^2} + w_R(t + O(\varepsilon), \varepsilon) + O(\varepsilon) & \text{for } \frac{1}{\sqrt{2}} \le t \le 1, \end{cases}$$

where $w_L(t,\varepsilon)$ and $w_R(t,\varepsilon)$ are the respective solutions of the problems

$$\text{(4.5)} \qquad \begin{aligned} \varepsilon w_L'' &= (1 + \frac{1}{\sqrt{2}}) \cdot [\frac{4}{7} + w_L] w_L' \\ w_L(\frac{1}{\sqrt{2}}) &= -\frac{4}{7}, \qquad w_L(-\infty) = 0 \end{aligned}$$

and

$$\text{(4.6)} \qquad \begin{aligned} \varepsilon w_R'' &= (1 + \frac{1}{\sqrt{2}}) \cdot [\frac{-4}{7} + w_R] w_R' \\ w_R(\frac{1}{\sqrt{2}}) &= \frac{4}{7}, \qquad w_R(\infty) = 0, \end{aligned}$$

which can be solved exactly using elementary techniques. We note in passing that Theorems 2.1 and 2.2 imply that this example has at least two other solutions, one with a boundary layer at $t = 0$ and one with a boundary layer at $t = 1$.

The shock layer solutions of Examples 4.1 and 4.2 each have an inflection point midway between $u_L(T)$ and $u_R(T)$, namely, $y = 0$ in each case, and their graphs have a symmetric appearance near the shock layer. This will usually be the case when the graph of $f(T, y)$ is symmetric about a point $(y_0, 0)$ such that $f(T, y_0) = 0$, i.e., $f(T, y)$ must satisfy $f(T, y_0 + c) = -f(T, y_0 - c)$, where c is any number such that both $y_0 - c$ and $y_0 + c$ lie between $u_L(T)$ and $u_R(T)$. This is illustrated in Figure 4.3 where the function $f(T, y)$ has three zeros between $u_L(T)$ and $u_R(T)$. It seems that $f(T, y)$ must have an odd number of zeros between $u_L(T)$ and $u_R(T)$ for the possibility of a solution to exist with a symmetric shock layer.

Finally, we discuss an example which has a solution with asymmetric shock layer behavior.

Example 4.3. Consider the problem

$$\begin{aligned} \varepsilon y'' &= y(y+4)y' - y(y+4) \\ y(0) = 1, &\qquad y(1) = -2, \end{aligned}$$

where $u_L(t) = t+1$, $u_R(t) = t-3$, and $J(t) = \int_{t+1}^{t-3} s(s+4)\,ds$. One can readily show that the conditions of Theorem 3.1 are satisfied, and we are guaranteed the existence of a solution with a shock layer at the following root of $J(t)$, namely, $T = 2\sqrt{6}/3 - 1 \approx 0.633$. Note that $u_L(T) \approx 1.63$ and $u_R(T) \approx -2.37$, while the inflection point occurs at $y = 0$, which is not midway between $u_L(T)$ and $u_R(T)$. This asymmetry should not be surprising since the graph of $f(y) = y(y+4)$ is not symmetric about the point $(0,0)$.

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