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ON THE ZEROS OF POLYNOMIALS AND SOME RELATED FUNCTIONS

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ABSTRACT. We consider the zeros of a polynomial $P(z)$ together with those of $F(z) \equiv (z-a)P'(z) - \nu P(z)$ where a and ν are arbitrary complex constants, and we extend some results obtained by Obrechkoff and Weisner on the relations between these sets of zeros. These results are applied to the zeros of certain quasi-trigonometric polynomials.

1. Introduction. Let $P(z)$ be a polynomial of n-th degree with zeros at $z_1, z_2, \dots, z_n$. For arbitrary ν and a set

$$F(z) = (z-a)P'(z) - \nu P(z). \tag{1.1}$$

We observe that if $\nu = 0$ then $F(z) = (z-a)P'(z)$, and if $\nu = n$ then $F(z)$ is the negative of the derivative of $P(z)$ with respect to the point a, see [**3**; Vol. 2, pp. 61-63]. Since much is known about the zeros of $F(z)$ in these two special cases, we may assume henceforth that $\nu \neq 0$ and $\nu \neq n$.

Theorem A. *In* (1.1) *set* $\nu = n/2$. *If all the zeros of* $P(z)$ *lie inside* (*on, outside*) *a circle* $|z-a| = r$, *then all the zeros of* $F(z)$ *lie inside* (*on, outside*) *the same circle.*

This Theorem was proved by Obrechkoff [**2**] and later independently by Weisner [**4**]. In fact, both [**2**] and [**4**] prove far more general theorems which contain Theorem A as a special case.

In [**1**] Theorem A was extended to include arbitrary ν to obtain

Theorem B. *Let* $P(z)$ *be an* n*-th degree polynomial, and let* $F(z)$ *be defined by* (1.1). *If* $\operatorname{Re}\nu \geq n/2$, *let* G *be the region* $|z-a| \geq r$. *If*

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$\operatorname{Re}\nu \leq n/2$, *let* G *be the disk* $|z-a| \leq r$. *If* $P(z)$ *has all its zeros in* G, *then* $F(z)$ *has all its zeros in the same region.*

This result is then applied in [**1**] in a very clever way to obtain some interesting theorems about rational functions and trigonometric polynomials. Our purpose is to extend Theorem B to include some cases which that theorem omits. For example, the case $\operatorname{Re}\nu < n/2$ and G a disk $|z-a| \geq r$ is missing in Theorem B.

All of our theorems have the following form: $P(z)$ is a function from a given set. If all the zeros of $P(z)$ lie in a closed circular region G, then all of the zeros z^* of $F(z)$, defined by (1.1), lie in a second closed circular region G^*. For brevity, we omit the excess words and merely list the conditions on $P(z), \nu, G$ and G^*. Our method of proof is essentially that used in [**1**], and our results contain Theorems A and B as special cases.

2. Polynomials and rational functions. We have

Theorem 1. *Let* $P(z)$ *be a polynomial of degree* n. *For brevity set* $b = \operatorname{Re}\nu$.

(I) *If* $b \geq n/2$ *and* $G : |z-a| \geq r$, *then* $G^* : |z-a| \geq r$.

(II) *If* $0 < b < n/2$ *and* $G : |z-a| \geq r$, *then* $G^* : |z-a| \geq \frac{b}{n-b} r$.

(III) *If* $b < 0$ *and* $G : |z-a| \geq r$, *then* $G^* : |z-a| \geq \frac{-b}{n-b} r$.

(IV) *If* $b \leq n/2$ *and* $G : |z-a| \leq r$, *then* $G^* : |z-a| \leq r$.

(V) *If* $n/2 < b < n$ *and* $G : |z-a| \leq r$, *then* $G^* : |z-a| \leq \frac{b}{n-b} r$.

(VI) *If* $n < b$ *and* $G : |z-a| \leq r$, *then* $G^* : |z-a| \leq \frac{b}{b-n} r$.

In each case the region G^* *for the zeros* z^* *of* $F(z)$ *is best possible.*

Clearly, (I) and (IV) together give Theorem B. The other parts complete the set of all possible values for ν.

Proof. If z^* is a zero of $F(z)$ and $z_1, z_2, \ldots, z_n$ are the zeros of $P(z)$, then a modest computation with (1.1) will give

$$S \equiv \sum_{k=1}^{n} \frac{1}{1 - \frac{z_k - a}{z^* - a}} = \nu. \tag{2.1}$$

We consider the Möbius transformation $W = 1/(1-Q)$ where $Q_k = (z_k - a)/(z^* - a)$.

(I). Assume $F(z)$ has a zero z^* such that $|z^* - a| < r$. Then $|Q_k| > 1$ and $\operatorname{Re}(1/(1-Q_k)) < 1/2$. Then $\operatorname{Re} S < n/2 \le \operatorname{Re}\nu$. This contradiction proves that $|z^* - a| \ge r$.

(II). Assume that $|z^* - a| < br/(n-b)$. Then $|Q_k| = |(z_k - a)/(z^* - a)| > (n-b)/b > 1$. Now $W = 1/(1-Q)$ maps this circular region onto a disk with diameter end points $-b/(n-2b)$ and b/n. Then $\operatorname{Re} S < b = \operatorname{Re}\nu$. Hence, (2.1) is impossible. Therefore, $|z^* - a| \ge br/(n-b)$.

(III). If $|z^* - a| < -br/(n-b)$, then $|Q_k| > 1 - n/b > 1$. Now $W = 1/(1-Q)$ maps this region onto a circular disk with end points $b/n < 0$ and $b/(2b-n) > 0$. Thus, $\operatorname{Re} S > n(b/n) = b = \operatorname{Re}\nu$. Again, (2.1) is impossible, so $|z^* - a| \ge -br/(n-b)$, a positive number.

(IV). If $|z^* - a| > r$, then $|Q_k| < 1$ and $\operatorname{Re}(1/(1-Q_k)) > 1/2$. thus $\operatorname{Re} S > n/2 \ge b = \operatorname{Re}\nu$. Hence, $|z^* - a| \le r$.

(V). If $|z^* - a| > br/(n-b)$, then $|Q_k| < (n-b)/b$. Now $W = 1/(1-Q)$ maps this disk onto a disk, with diameter end points b/n and $b/(2b-n)$. Hence, $\operatorname{Re} S > n(b/n) = b = \operatorname{Re}\nu$. Consequently, $|z^* - a| \le br/(n-b)$.

(VI). If $|z^* - a| > br/(b-n)$, then $|Q_k| < (b-n)/b$. Now $W = 1/(1-Q)$ maps this disk onto a disk with diameter end points $b/(2b-n)$ and b/n. Then $b = \operatorname{Re}\nu = \operatorname{Re} S < n(b/n) = b$, a contradiction. Hence, $|z^* - a| \le br/(b-n)$.

The example function $P(z) = (z - a + \mathrm{re}^{i\alpha})^n$ shows that each bound is sharp. For (1) and (IV) the result is trivial. For the others $(F(z)$ has a zero at $z^* = a + \nu \mathrm{re}^{i\alpha}/(n-\nu)$. □

Suppose that $P(z)$ is an n-th degree polynomial and

$$R(z) \equiv \frac{P(z)}{(z-a)^{\nu}}. \tag{2.2}$$

If ν is an integer, then $R(z)$ is a rational function or a polynomial. Otherwise, $(z-a)^\nu \equiv \exp(\nu \ln(z-a))$ and $R(z)$ is defined on the Riemann surface for $\ln(z-a)$. Now

$$R'(z) = (z-a)^{-\nu-1}[(z-a)P'(z) - \nu P(z)]. \tag{2.3}$$

Thus, except for the possible zero of $R'(z)$ at $z = a$, the zeros of $R'(z)$ coincide with those of $F(z)$ defined by (1.1). This gives

Theorem 2. *Let $P(z)$ be an n-th degree polynomial, and let $R(z)$ be defined by* (2.2). *Then the zeros of the derivative $R'(z)$ satisfy the six assertions of Theorem* 1, *and in each case the region G^* is best possible.*

For our next application we look at functions of the form

$$\begin{aligned} R(z) &= \frac{1}{(z-a)^\nu} \sum_{k=-n}^{n} a_k (z-a)^k \\ &\equiv \frac{P(z)}{(z-a)^{n+\nu}}, \qquad a_n a_{-n} \neq 0. \end{aligned} \tag{2.4}$$

Now $P(z)$ is a polynomial of degree $2n$. We can apply Theorem 2 to $R'(z)$ by making the changes: $n \to 2n$ and $\nu \to n + \nu$. After a little labor we obtain

Theorem 3. *Let G contain all the zeros of $P(z)$, a polynomial of degree $2n$ where $P(a) \neq 0$. Let $R(z)$ be defined by* (2.4), *and let G^* be a region that contains all the zeros of $R'(z)$. Finally, set $b = \operatorname{Re} \nu$.*

(I). *If $b \geq 0$ and $G : |z-a| \geq r$, then $G^* : |z-a| \geq r$.*

(II). *If $-n < b < 0$ and $G : |z-a| \geq r$, then $G^* : |z-a| \geq ((n+b)/(n-b))r$.*

(III). *If $b < -n$ and $G : |z-a| \geq r$, then $G^* : |z-a| \geq ((|b|-n)/(|b|+n))r$.*

(IV). *If $b \leq 0$ and $G : |z-a| \leq r$, then $G^* : |z-a| \leq r$.*

(V). *If $0 < b < n$ and $G : |z-a| \leq r$, then $G^* : |z-a| \leq ((n+b)/(n-b))r$.*

(VI). *If* $n < b$ *and* $G : |z-a| \le r$, *then* $G^* : |z-a| \le ((b+n)/(b-n))r$.

In each case the region G^* for the zeros of $R'(z)$ is best possible.

Genchev [**1**] proved (I) and (IV) when $\operatorname{Re}\nu = b = 0$.

3. Quasi-trigonometric polynomials. By a quasi-trigonometric polynomial of degree n we mean any function of the form

$$T(z) = e^{-i\nu z} \sum_{k=-n}^{n} a_k e^{ikz}, \tag{3.1}$$

where $a_n a_{-n} \neq 0$ and ν is an arbitrary complex number.

We convert Theorem 3 to a theorem on quasi-trigonometric polynomials as follows. First set $a = 0$ and replace z by w in $R(z)$. Then the substitution $w = e^{iz}$ changes $R(w)$ into $T(z)$ as given by (3.1). Further, $R'(w)$ and $T'(z)$ differ by a trivial nonzero factor. Finally, the condition $|w| \le r$ becomes $\operatorname{Im} z \ge -\ln r \equiv A$. With these changes, Theorem 3 gives

Theorem 4. *Let* $T(z)$ *be a quasi-trigonmetric polynomial* (3.1) *with* $a_n a_{-n} \neq 0$. *Let* G *be a region that contains all the zeros of* $T(z)$ *and* G^* *be a region that contains all the zeros of* $T'(z)$.

(I). *If* $b \ge 0$ *and* $G : \operatorname{Im} z \le A$, *then* $G^* : \operatorname{Im} z \le A$.

(II). *If* $-n < b < 0$ *and* $G : \operatorname{Im} z \le A$, *then* $G^* : \operatorname{Im} z \le A - \ln((n+b)/(n-b))$.

(III). *If* $b < -n$ *and* $G : \operatorname{Im} z \le A$, *then* $G^* : \operatorname{Im} z \le A - \ln((|b| - n)/(|b| + n))$.

(IV). *If* $b \le 0$ *and* $G : \operatorname{Im} z \ge A$, *then* $G^* : \operatorname{Im} z \ge A$.

(V). *If* $0 < b < n$ *and* $G : \operatorname{Im} z \ge A$, *then* $G^* : \operatorname{Im} z \ge A - \ln((n+b)/(n-b))$.

(VI). *If* $n < b$ *and* $G : \operatorname{Im} z \ge A$, *then* $G^* : \operatorname{Im} z \ge A - \ln((b+n)/(b-n))$.

In each case the region G^* *for the zeros of* $T'(z)$ *is best possible.*

If $T(z) = e^{-i(n+\nu)z}(e^{iz} - \mathrm{re}^{\,i\alpha})^{2n}$, then $T'(z)$ has zeros z^* for which

$$\text{(3.2)} \qquad \operatorname{Im} z^* = -\ln r \quad \text{and} \quad \operatorname{Im} z^* = -\ln r - \operatorname{Re}\ln\frac{n+\nu}{n-\nu}.$$

With a little labor these zeros will show that the region G^* in Theorem 4 is best possible in every case.

Genchev [**1**] proved (I) and (IV) when $\operatorname{Re}\nu = b = 0$. As he observed, this special case $\nu = 0$ gives a Gauss–Lucas type theorem for trigonometric polynomials, namely,

Theorem C. *If all the zeros of the trigonometric polynomial $T(z)$ lie in the strip $A \le \operatorname{Im} z \le B$, then the zeros of $T'(z)$ lie in the same strip.*

Various combinations of items (I)–(VI) of Theorem 4 will give a similar type theorem, but the results are not as pretty as Theorem C. For example, if we combine items (II) and (IV), we obtain

Theorem 5. *Let $T(z)$ be a quasi-trigonometric polynomial of the form* (3.1). *Suppose that*

$$\text{(3.3)} \qquad -n < b = \operatorname{Re}\nu < 0,$$

and that all the zeros of $T(z)$ lie in the strip $A \le \operatorname{Im} z \le B$. Then all the zeros of $T'(z)$ will lie in the strip

$$\text{(3.4)} \qquad A \le \operatorname{Im} z \le B - \ln\frac{n+b}{n-b} = B + \ln\frac{n+|b|}{n-|b|}.$$

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