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CLASSIFYING GENERIC ALGEBRAS

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ABSTRACT. This paper proposes a program for the inductive classification of the generic unitary associative algebras of dimension n, based on previous work in the field and on two new results pertaining to different aspects of the problem. The first is a diagonalization theorem, showing that the nonnilpotent sections of an idempotent-creating deformation can be chosen from within the direct sum of the local rings at the newly created idempotents. The second gives sufficient conditions for a "loopless" basis graph with a specified radical flag structure to determine a unique component of the structure-constant scheme Alg_n. The procedure for classifying generic algebras is then described and illustrated by determining the generic algebras of dimension six.

1. Introduction. The classical problem of classifying n-dimensional algebras suffers from being too easy. Once the ground rules are explained, a competent algebraist with time and patience can sit down and generate multiplication tables for associative algebras, but the activity becomes unilluminating around dimension six and has not been carried much further. Such calculations flourished for a while at the end of the last century [**12**] but more or less died out in the face of more general structure theorems, particularly the Wedderburn theorems.

The subject became more interesting when it was broadened to include determining not only the algebras themselves but also the partial ordering of the algebras by the relation of specialization. Of particular interest from this point of view are the generic algebras, the maximal algebras or families of algebras with regard to this partial ordering.

An n-dimensional algebra is generally defined by fixing a K-basis $v_1, \dots, v_n$ and giving the multiplication structure

$$v_i v_j = \sum_k a_{ij}^k v_k. \tag{1}$$

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The coefficients $\{a_{ij}^k\}$ are called the *structure constants.* We can define an algebraic scheme Alg_n in affine n^3-space by requiring the multiplication structure to be associative and unitary. This scheme will be called the *structure constant scheme.* Since it is an algebraic scheme it has a finite number of irreducible components.

Basis change gives an algebraic group action on Alg_n. If an algebra or family A lies in the closure of the orbit of a different algebra or family A', then we will say that A is a specialization of A' or that A' is a deformation of A.

Definition. We will call an algebra structure, perhaps depending on continuous parameters, *generic*, if the orbits under basis change of the algebras with this structure form an open dense subset of an irreducible component of the structure constant scheme Alg_n.

The particular subject of this paper is a procedure for determining all the generic algebras of a given dimension. We will build on two essentially independent lines of research: that of M. Gerstenhaber [**4**] and F. Flanigan [**1,2**] on formal deformations, and that of P. Gabriel [**3**] and the European representation theorists on classifying algebras. Our long-range aim, part of which is being implemented in the first author's Ph.D. thesis, is to try to carry out the classification at least up to dimension eight, in such a way that the results will be graspable and will highlight examples of mathematical interest. In order to do so, at least at first, we will abandon the requirement of completeness. The goal of our program is to find *all* irreducible components, *most* algebras, and *some* of the deformations among them. Our efforts will be concentrated on clearing away those algebras and deformations which can be determined inductively from algebras and deformations of lower dimensions, in order to spotlight the essentially new phenomena which first arise in the new dimension.

In a first paper on the subject [**8**], the second author gave a computer-implemented algorithm which included a calculation of the cohomology module which determines that an algebra is *rigid*, i.e., that its orbit under the action of the group of basis changes is dense in an irreducible component of Alg_n. It can also be used to determine that a family of algebras is *semirigid*, i.e., that the union of the orbits of the algebras in this family is dense in an irreducible component of Alg_n. Thus the problem of finding all generic algebras is basically a problem of finding a complete and not unnecessarily unwieldy list of candidates to propose.

In Section 2 we define an interleaving to be a set of generators for the radical elements created by the collapse of idempotents and show that in many common cases these can be diagonalized. We then give two applications for such interleavings. One is to show that certain candidates for local generic algebras are not in fact rigid, by constructing an interleaving which gives an idempotent-splitting deformation of the algebra. The second, more useful for nonlocal algebras, is a sufficiency condition for the existence of a certain kind of standard deformation for an algebra which is a fiber product with a truncated polynomial algebra. This considerably reduces the number of candidates for nonlocal generic algebras.

In low dimensions, most rigid algebras are built on a loopless basis graph weighted by the dimensions of the Peirce components of the radical flag, and we give a thorough treatment of such algebras in Section 3. After a certain amount of computer experimentation, we were able to reduce the determination of the loopless rigid algebras in dimension < 10 to a combinatorial problem which can easily be solved by hand.

In Section 4 we give two improvements to the computer algorithm. Finally, in Section 5 we describe the procedure for classifying generic algebras and give the results up through dimension six.

2. Idempotent splitting deformations. Following Happel [**5**] we denote by $E(m)$ the $m+1$-dimensional truncated polynomial algebra, isomorphic to $K[Z]/Z^{m+1}$. The most elementary deformation splitting one idempotent into two is the deformation of $E(m+n+1)$ to $E(m) \times E(n)$. What we show in this section is that many idempotent-splitting deformations are built in a similar way. However, we first establish some notational conventions and, in particular, fix the definition of a deformation. In what follows K denotes an algebraically-closed field.

For following all of the examples and most of the exposition in this paper, it suffices to consider the case of deformations over the affine line C, whose closed points t are in one-to-one correspondence with elements of K. We actually consider the pair (C, t_0), where t_0 is a distinguished point of C which can usually, but not always, be taken to be the point corresponding to the zero element of K. Let R be the polynomial ring $K[T]$ in one variable. A deformation B of a K-algebra B_0 over (C, t_0) is then an R-algebra whose structure constants are polynomials in T, such that substituting the value of the distinguished point for T gives the structure constants of B_0.

Substituting the value corresponding to a point t in place of the variable T then gives the fiber B_t. An element z of B can be thought of as determining a mapping of C into the family B_t, sending each t to the evaluation $z(t) \in B_t$. For this reason an element z of B will often be referred to as a *section* of the deformation, since it represents a sort of cross-section of the fibration of B over C.

Unfortunately, for theoretical reasons, we cannot restrict ourselves solely to deformations over rational curves, so it is necessary to consider more general deformations. To avoid cluttering the body of the text with technicalities, we have placed the exact definitions of such terms as *algebraic family*, *deformation*, *restriction*, *evaluation*, *etale neighborhood*, and *sober idempotent* in an appendix at the end of this section.

Definition. Let B be a deformation over a curve (C, t_0) and let J be the unique two-sided ideal of B whose fiber J_t is equal to the radical of B_t in almost every fiber [**9**, Lemma 1]. A set of elements of B which, together with a complete idempotent set, generate B_t/J_t for each fiber in an open neighborhood of t_0 will be called an *interleaving*. If each element of the interleaving can be diagonalized so that it is contained in the direct sum of the local algebras at each idempotent, then the interleaving will be called a *diagonal* interleaving.

Remark . In the range of dimensions up to five for which the complete deformation chart is known, every idempotent-splitting deformation of a sober algebra factors through a diagonal interleaving.

Before proving the main theorem of this section, we will first give one simple example, the specialization of two truncated polynomial algebras $E(n)$ and $E(m)$ into a single truncated polynomial algebra $E(n+m+1)$.

Example 1. Let $\tilde{B}$ be a family over $U = C - \{t_0\}$ with two sober idempotents e' and e'' whose sum is the identity, and suppose that $\tilde{B}$ is decomposable into the direct sum of two local blocks

$$\begin{aligned} e'\tilde{B}e' &\approx K[w']/(w')^{r'+1}, \\ e''\tilde{B}e'' &\approx K[w'']/(w'')^{r''+1}. \end{aligned} \tag{2}$$

Let b be any element of the maximal ideal μ_0 defining t_0. Let s' be any element of R, and let $s'' = 1 - s'$. Set

$$z = bs'e' - bs''e'' + w' + w''. \tag{3}$$

Let $e = e' + e''$, and let $r = r' + r'' + 1$. Then, as we will prove below, $e, z, z^2, \dots, z^r$ are independent over U and z satisfies a homogeneous equation of degree $r + 1$ in b and z given by

$$f(z) = (z - bs'e)^{r'+1}(z + bs''e)^{r''+1} = 0. \tag{4}$$

Thus, we can identify $\tilde{B}$ with restriction to U of the R-algebra $B = R[Z]/f(Z)$, which is well defined also at t_0. At t_0, the element b vanishes, and if we let z_0 represent $z(t_0)$, this equation becomes

$$z_0^{r+1} = 0. \tag{5}$$

Thus, B represents a deformation of the truncated polynomial algebra $E(r)$ to a direct product $E(r') \times E(r'')$.

With this example as motivation, we now prove the main theorem of this section:

The Diagonalization Theorem. *Let B be a deformation of sober K-algebras over an algebraic curve (C, t_0), with K an algebraically-closed field. Let J be the flat family of two-sided ideals such that J equals the radical in the general fiber. Passing to an etale neighborhood of (C, t_0) as necessary, we assume that every idempotent of the special fiber B_0 lifts to an idempotent section e which splits completely into primitive orthogonal idempotent sections $f_1, \dots, f_d$ over every closed point t of C not equal to t_0. Then, for each idempotent section e such that $e(B/J)e$ is a truncated polynomial ring, we can choose the idempotent sections f_i in such a way that there is a section z such that*

$$z|_U = \Sigma b_i f_i + \Sigma w_i, \qquad w_i \in f_i(J|_U)f_i, \tag{6}$$

and powers of z *are a diagonal interleaving for* $e(B/J)e$*. Thus, if* B/J *is a product of truncated polynomial rings,* B/J *has a basis consisting of elements which are generically diagonal with respect to a set of orthogonal idempotent sections.*

Proof. By Corollary 1.1 of [**9**], for each e for which the special fiber at e is a truncated polynomial ring of dimension d, we may choose a section z lifting this radical generator and lying in eBe. The sections $e, z, z^2, \dots, z^{d-1}$ form a basis for $e(B/J)e$ at the special fiber and, thus, at almost every fiber over C. Modulo the ideal J, which is unique by Lemma 1 of [**9**], we have a commutative deformation splitting the identity of the algebra $K[Z]/Z^d$ into d distinct idempotents. If we let μ_0 be the maximal ideal of R at t_0, we then know that such a deformation must have the form

$$R[Z]/f(Z), \qquad f(Z) = (Z - b_1)\cdots(Z - b_d), \tag{7}$$

with $b_i \in \mu_0$ and all b_i distinct in the fiber over a closed point t different from t_0. Each linear factor $Z - b_i$ vanishes on one of the d distinct points in $\operatorname{Spec}(K[Z]/f(Z))$, and we may presume that we have chosen the numbering so that $Z - b_i$ vanishes on the point corresponding to the idempotent f_i. Since Z is the image of z under the homomorphism sending eBe to $R[Z]/f(Z)$, we conclude that z is congruent to $b_i f_i$ modulo the ideal generated by eJe and the remaining idempotents f_j, so z must be of the form

$$z|_U = \Sigma b_i f_i + v, \qquad v \in J. \tag{8}$$

Furthermore, the b_i belong to μ_0 and are distinct over each point of U, so $b_i - b_j$ is invertible over U.

We now proceed to diagonalize the section z by induction on the depth m of v in the radical flag, that is to say, on the highest power of J containing v. We apply the following lemma with $A = B|_U$.

Lemma 1. *Let* A *be a finite rank algebra over an affine ring* S*, such that* A *has a complete set of sober orthogonal idempotents* $f_1, \dots, f_d$*. Let* z *be an element of* A *of the form* $\Sigma b_i f_i + v$*, where* v *is an element of the radical* J *of* A*, and the* $b_i - b_j$ *are invertible. Then there is a* *unit section u in A such that z is diagonal with respect to the set of idempotents obtained from $f_1, \dots, f_d$ by conjugation by u.*

Proof. Since we can replace u by u^{-1}, the conclusion of the lemma will hold if we can find a u such that uzu^{-1} is diagonal with respect to the original set of idempotents, that is to say, such that uzu^{-1} lies entirely in the direct sum of the Peirce blocks $f_1Af_1, \dots, f_dAf_d$. Let J be the radical of A. Since it is nilpotent, we can proceed by induction, assuming that z has already been diagonalized modulo J^m. Write

$$z = \Sigma b_i f_i + \Sigma w_{jk} \tag{9}$$

with $w_{jk} \in J \cap f_jAf_k$. By hypothesis, each $b_j - b_k$ is nonzero. We assume that w_{jk} lies in J^m whenever j and k are distinct. Then if we set

$$u = 1 + \Sigma(b_j - b_k)^{-1}(w_{jk} - w_{kj}), \tag{10}$$

with the sum taken over all pairs of distinct j and k, a straightforward computation shows that uzu^{-1} is diagonal modulo J^{m+1}. □

Adjusting the idempotents as required by the lemma, we now have z in the proper form, and its powers give the desired basis for eBe, thereby demonstrating the theorem. □

Remark . The quotient B/J will always be a product of truncated polynomial rings if $d = 1$ or 2 at every idempotent or if B has a distributive ideal lattice, two cases which are quite important in practice.

As a first application of the theory of diagonal interleavings, we consider a situation which arises frequently in the classification of generic local algebras. In testing a family of local algebras to see if it is generic, the computer output indicates that the algebra has infinitesimal idempotent splitting deformations. This is a strong indication that the algebra is not generic but does not constitute proof, since the algebra might correspond to the closed point in a nonreduced component.

From the computer output and the structure of the given algebra B_0, it is possible to deduce roughly the structure of one of the deformed algebras, and usually it can be chosen to have two idempotents. To verify that there is an actual algebraic deformation B_0 with this general fiber, one has to propose an interleaving z and show that its products with the remaining elements of J give the structure constants of the special algebra B_0 at the distinguished point t_0. We now consider in more detail this common case of a deformation which collapses a single pair of idempotents e' and e'' into their sum e.

In the case of two idempotents collapsing, the quotient B/J will be the product of copies of the field K with a two-dimensional K-algebra, generated by the idempotent e and an interleaving z. Since z lies in the radical of the fiber over the closed point t_0, the trace of z in the left multiplication representation is zero over t_0. Thus, we may replace z by $z - (\mathrm{Tr}(z)/\mathrm{Tr}(e))$, and it will still generate the radical in the special fiber.

Suppose that z has the form $b'e' + b''e'' + w' + w''$, with $w' \in e'Je'$ and $w'' \in e''Je''$. Set

$$s' = \mathrm{Tr}(e'')/\mathrm{Tr}(e) \tag{11}$$

$$s'' = \mathrm{Tr}(e')/\mathrm{Tr}(e)$$

$$b = b'/s'. \tag{12}$$

Since $0 = (\mathrm{Tr}(z)/\mathrm{Tr}(e)) = b's'' + b''s'$, we conclude that b can also be represented as $-b''/s''$, and thus z may be rewritten as

$$z = bs'e' - bs''e'' + w' + w''. \tag{13}$$

Let r' be the highest power of w' which is nonzero, and let r'' be the highest power of w'' which is nonzero. Consider the set of polynomials in z,

$$g_{ij}(z) = (z - bs'e)^i (z + bs''e)^j, \tag{14}$$

for $0 \leq i \leq r' + 1$ and for $0 \leq j \leq r'' + 1$.

These polynomials have two uses. One use, which we will need in the proof of Proposition 1 below, is to express the powers of w' and w'' as functions of z. In general we have

$$\begin{aligned} g_{ij}(z) &= (z - bs'e)^i (z + bs''e)^j \\ &= (-be'' + w' + w'')^i (be' + w' + w'')^j \\ &= (be' + w')^j w'^i + (-be'' + w'')^i w''^j. \end{aligned} \tag{15}$$

When $i = r'+1$, $j = 1, \dots, r''$, the w' part vanishes, and the coefficients of the powers of w'' form an upper triangular matrix which can be solved for different powers of w''. Similarly, the g_{ij} for $j = r''+1$ can be solved for the powers of w', as required.

The other use is to determine products of z with sections of the ideal J by considering products of elements of J with

$$\begin{aligned} g_{01}(z) &= (z + bs''e) = be' + w' + w'' \\ g_{10}(z) &= (z - bs'e) = be'' + w' + w''. \end{aligned} \tag{16}$$

Thus, for example, if q is a section of J which lies generically in $e'Be'$, we can multiply by $g_{10}(z)$ and conclude that

$$z_0 q(t_0) = w' q(t_0). \tag{17}$$

Such calculations are crucial to the construction of interleavings which show that algebras are not rigid.

For a second application of interleavings, we prove a sufficient condition for the existence of deformations. This will allow us to eliminate many algebras as possible candidates for being generic and gives us automatically many deformations on a deformation chart of n-dimensional algebras. This theorem applies to algebras, one of whose local rings eBe is a fiber product with a truncated polynomial algebra. This situation arises most frequently in the case of nonlocal connected algebras with nontrivial local Peirce components.

Definition. Let B be a deformation of a K-algebra B_0 over a scheme (C, t_0), with $C = \mathrm{Spec}(R)$, and let N be the maximal ideal at a sober idempotent e. Let A be a local K-algebra with radical J. An *attachment* of a local K-algebra A to B along e is a flat R-algebra $\hat{B}$ whose underlying R-module is the direct sum of B with $J \otimes R$ such that

(i) the internal multiplication within each factor is just the induced multiplication;

(ii) the idempotent section e multiplies $J \otimes R$ like an identity section; and

(iii) right or left multiplication by an element of $J \otimes R$ maps N into itself.

The special case $R = K$, which is just a fiber product of A with B, will also be called an attachment.

Proposition 1. *Let B be a deformation of a K-algebra B_0 over a curve (C, t_0), let r' and r'' be natural numbers, and let $\hat{B}$ be the attachment of $E(r')$ to B along a sober idempotent section e'. Then there is an idempotent-splitting deformation of $B_0 \times_k E(r' + r'' + 1)$ whose general fiber is isomorphic to the general fiber of $\hat{B} \times E(r'')$, where the R-module structure is induced from the first factor.*

Proof. Let w' be the section of $\hat{B}$ which generates the radical of $E(r')$, let w'' be the generator of the radical of $E(r'')$ and let e'' be the identity in $E(r'')$. Let A be a generator of the maximal ideal at t_0. Set

$$z = bs'e' - bs''e'' + w' + w'', \tag{18}$$

where s' and s'' are defined as in (11) above, and let $e = e' + e''$. We can recover the powers of w' and w'' from the polynomials $g_{ij}(z)$ for $i = r' + 1$ or $j = r'' + 1$. Thus, $e, z, \ldots, z^{r'+r''+1}$ are linearly independent. The left or right product of z with any element of N is well defined at t_0 and is equal to zero, as is $z^{r'+r''+2}$. Thus, by combining the powers of z given above with a basis of N, we get a multiplication structure which extends to the special fiber and gives the desired algebra $B_0 \times_k E(r' + r'' + 1)$ as the fiber over t_0. □

We illustrate both the construction of an interleaving and the use of the special functions g_{ij} with examples of diagonal interleavings. We will actually give three examples with the same general fiber, of which the first two can be obtained as in Proposition 1 above and the third cannot. The third example is a generalization of the "most interesting" deformation in dimension five, the deformation of a noncommutative algebra to a semirigid family in such a way that the orbit of the special fiber in the structure-constant scheme has the same dimension as the orbit of each of the general fibers, yet the special fiber is not a natural member of the family. In Happel's list [**5**] this is the specialization of $F_s^4 \times K$ to D^5, and in Mazzola's paper [**7**] it is the specialization of the family $13(\tau)$ to (24), which is discussed there in some length.

Example 2. (Interleavings). Let $R = K[s]$ be a polynomial ring in one indeterminate. Let A'' be $R[w'']/w''^2$, and let

$$A'_s = K[s]\langle x, y\rangle/(x^2, y^2, xy - syx). \tag{19}$$

Set

$$\begin{aligned} u &= x + y, \\ w &= x - y. \end{aligned} \tag{20}$$

If we let $\sigma = (1 - s)/(1 + s)$, then these new basis elements satisfy the equations

$$\begin{aligned} w^2 &= -u^2, \\ uw &= \sigma u^2, \\ wu &= -\sigma u^2. \end{aligned} \tag{21}$$

Let C be the affine line $\mathrm{Spec}(R)$, $R = K[s]$, and let t_0 be the point $s = 1$, with maximal ideal $(s-1)R$. Let $U = C - \{t_0\}$, and let $\hat{B}$ be the family over U whose closed fiber is $A'_s \times A''$. Let e' be the idempotent in the first factor and e'' the idempotent in the second. Then if J is the radical of A'_s, the attachment $\hat{B}$ of $K[w'']/w''^2$ to A'_s will have

$$A'_s[w'']/(w''^2, w''J) \tag{22}$$

as its closed fiber. This is the five-dimensional family denoted by B^8_s in Happel's list [**5**]. We now consider three different deformations whose restriction to U has closed fiber $A'_s \times A''$.

Example 2.1. (A trivial interleaving). Let $b = s - 1$, and set

$$\tilde{z} = (b/3)e' - (2b/3)e''. \tag{23}$$

Any element of the radical I of $\hat{B}$ can be written in the form $v = q + cw''$, with $q \in J$ and $c \in R$. Letting $\tilde{v} = v|_U$, we see that

$$\tilde{v}\tilde{z} = (b/3)q - c(2b/3)w''. \tag{24}$$

This is a well-defined element of $\hat{B}$ and it is equal to 0 in the fiber over t_0. Similarly, $\tilde{z}\tilde{v}$ determines a well-defined element of $\hat{B}$ which vanishes in the fiber over t_0. We conclude that the closed fiber is isomorphic to the algebra obtained from A_1' by attaching two trivial radical elements w_0'' and z_0.

Example 2.2. (An attachment). This time, instead of an interleaving which is independent of the radicals in the two components of the general fiber, we will give an interleaving involving the radical of A''. As before, we let $b = s - 1$. We set

$$\tilde{z} = (b/3)e' - (2b/3)e'' + w'', \tag{25}$$

which satisfies the equation

$$g_{12}(\tilde{z}) = (\tilde{z} - (b/3))(\tilde{z} + (2b/3))^2 = 0, \tag{26}$$

and since this is an equation homogeneous in $\tilde{z}$ and b, we get $z_0^3 = 0$ when we extend it to t_0. An analysis similar to that in Example 2.1 shows that z_0 multiplies every other radical element trivially, showing that the closed fiber B_0 is the fiber product of A_1' with $E(3)$.

Example 2.3. (A nontrivial interleaving). The final example is the one toward which we have been working, in which the radicals of the two components of the general fibers are integrally combined in the special fiber. We must first remove from the affine line C the point at which $s = -1$. We now define

$$\begin{aligned} b &= (1-s)/(1+s) = \sigma \\ w' &= b^{-1}u \end{aligned} \tag{27}$$

and set

$$\tilde{z} = (b/3)e' + w' - (2b/3)e'' + w''. \tag{28}$$

We now calculate

$$
\begin{aligned}
g_{22}(\tilde{z}) &= (\tilde{z}-(b/3))^2(\tilde{z}+(2b/3))^2 \\
&= w'^2(b+w')^2 \\
&= b^2w'^2 \\
&= u^2, \\
(\tilde{z}-(b/3))w &= (w'-be''+w'')w \\
&= b^{-1}uw \\
&= b^{-1}(\sigma u^2) \\
&= u^2, \\
w(\tilde{z}-(b/3)) &= -u^2.
\end{aligned} \tag{29}
$$

We already knew that

$$
w^3 = 0. \tag{30}
$$

Taken together, equations (29–30) give a set of defining equations for a six-dimensional algebra B with basis $1, z, z^2, z^3, z^4$, and w. It is associative and unitary because $B|_U \approx \tilde{B}$. Suppressing the zero subscripts to shorten the formulae, we find that in the fiber over t_0 the ideal is given by

$$
(z^4+w^2, zw+w^2, wz-w^2, w^3). \tag{31}
$$

The ideal contains z^5 as a combination of the generators. Note that, unlike the Examples 2.1 and 2.2, in this case the special fiber is not obtained from A_1' by an attachment, and we cannot obtain this deformation by applying Proposition 1. This example is actually used below in the classification of the generic algebras in dimension six.

Appendix

Definitions Involving Algebraic Geometry

Definition. Let C be an affine algebraic scheme over an algebraically closed field, K, i.e., $C = \mathrm{Spec}(R)$, with R a finitely generated K-algebra. An R-algebra B which is flat as an R-module is called a *flat family of K-algebras.* An ideal J in B which is flat as an R-module is called a *flat family of ideals.*

Remark . Since C is Noetherian, the flatness of B implies that B is locally free over C, and for any t we can always choose an open neighborhood of t in C over which B is a free R-module.

Definition. Let R be a finitely-generated K-algebra, and let t_0 be a closed point of $C = \operatorname{Spec}(R)$ corresponding to a maximal ideal μ_0 in R. Let B_0 be a finite dimensional K-algebra. A *deformation* B of B_0 over (C, t_0) is an algebraic family B over C together with a fixed isomorphism

$$B_0 \approx B \otimes_R R/\mu_0. \tag{32}$$

If t is the closed point of R corresponding to a maximal ideal μ, then

$$B_t \approx B \otimes_R R/\mu \tag{33}$$

is called the *fiber* over t.

Definition. Let V be an n-dimensional vector space over K with basis $v_1, \ldots, v_n$. If B is a free R-module, we identify each element of B with a *section* of C into $C \times V$. If $z \in B$ and t is a closed point of C, then the *evaluation* $z(t)$ of z at t will denote the closed point in the fiber B_t obtained by reducing the coefficients of the v_i modulo the maximal ideal μ corresponding to t. An idempotent e of B will be called an *idempotent section* and will satisfy $e(t)^2 = e(t)$ in each closed fiber.

Definition. An idempotent e in an algebra A over an affine ring R is called *sober* if there is a two-sided ideal N in A such that $A/N \approx R$ and $1 - e \in N$. An algebra A over an algebraically closed field K is called *sober* if $A/\operatorname{Rad}(A)$ is a product of copies of K.

Definition. Let B be a deformation of B_0 over (C, t_0). Let $U = C - \{t_0\}$. We denote the *restriction* of the algebraic family B to U by $B|_U$.

Definition. An *etale neighborhood* of a pair (C, t_0) is a pair (C', t_0') such that $C' = \operatorname{Spec}(R')$ and $t_0' \in C'$, together with an etale morphism of schemes $p : C' \to C$ such that $p(t_0') = t_0$.

Definition. If C is a nonsingular curve, then the maximal ideal μ_0 can be generated by a single element τ. Let S be the multiplicatively closed set consisting of the powers of τ. Since R is a domain, there is a canonical injection of R into the ring $S^{-1}R$. We will say that an element of $S^{-1}R$ is *well-defined* at t_0 if it lies in the image of R.

3. Loopless basis graphs. In an earlier work [**9**] the second author considered the following directed graph associated to any finite dimensional algebra: The number of vertices is the number of idempotents in a set of primitive orthogonal idempotents. Choose such a complete primitive set $e_1, \dots, e_r$ of idempotents and label each vertex by one idempotent. For i different from j, the number of arrows from e_j to e_i is the number $\dim_k e_i A e_j$. The number of loops from e_i to e_i is $(\dim_K e_i A e_i) - 1$. (The -1 occurs because the element e_i of a basis for $e_i A e_i$ is already represented by the vertex.) To distinguish this graph from the quiver, which includes arrows only for elements of $A/\mathrm{Rad}(A)^2$, we call this a *basis graph.* If $\mathrm{Rad}(A)^2 = 0$, and $A/\mathrm{Rad}(A)$ is a product of fields, then the basis graph corresponds to the ordinary quiver. The radical-squared zero algebra associated with a given basis graph will be called its *basis graph algebra*.

We showed in [**9**] that to every irreducible component, and thus to every generic algebra, we can associate a unique generic basis graph, and for most irreducible components this basis graph is *loopless*, i.e., has no arrows which begin and end at the same vertex. The main result of this section is, in dimensions < 10, to reduce the problem of finding all generic algebras with given loopless basis graph to a combinatorial problem.

We can always choose a basis for the algebra which respects the Peirce decomposition of A into a direct sum of vector subspaces $A_{ij} = e_i A e_j$. We may also assume that this basis contains bases for the various powers of the radical J. In [**9**] we defined the *weighted basis graph* associated with an algebra A to be the basis graph of A with the arrows weighted by natural numbers or infinity according to the positions of the basis elements in the chain of ideals formed by powers of the radical. Let J be the radical of A. The number of arrows from j to i with weight k is given by

$$H_{ij}^k = \dim e_i (J^k / J^{k+1}) e_j. \tag{34}$$

FIGURE 1

We proved that this is an upper semicontinuous function whenever we have a deformation of finite dimensional algebras in which the basis graph and the radical dimension remain fixed over the entire deformation.

The weight of an arrow is represented by the number of "barbs" drawn on the arrow. Matrix units have a weight of infinity and are represented by a solid triangular barb. For example, the first weighted basis graph in Figure 1 represents the 3×3 matrices. The second represents the upper triangular 3×3 matrices. We define a partial ordering on these weightings.

Definition. Suppose Φ and Φ' are two weightings on the same basis graph Q. Then $\Phi \leq \Phi'$ if, for each pair i, j and for each k, the number of arrows from i to j of weight less than or equal to k in Φ is greater than or equal to the corresponding number of arrows in Φ'.

Definition. An arrow x will be called *reducible* if there are two arrows, not including x, forming a path from the initial point of x to its end point. Other arrows will be called irreducible. An arrow through which a reducible arrow can factor will be called *reducing*.

Remark . Only reducible arrows can have more than a single barb, since the radical squared is generated by compositions of arrows. This is a special case of Proposition 2 below.

Definition. A *complete lexicographically-oriented* graph on n vertices is a graph with vertices $v_1, \ldots, v_n$ and an edge oriented from v_i to v_j for each pair i, j with $i < j$.

Proposition 2. *Let Q be a basis graph without loops. If it contains no complete, lexicographically-oriented subgraph of order k, then for any algebra A with Q as basis graph, we have*

$$\mathrm{Rad}(A)^{k-1} = 0. \tag{35}$$

Proof. Let us denote $\mathrm{Rad}(A)$ by J. Assume that we begin with a basis for A which respects the Peirce decomposition by the idempotents corresponding to the vertices of Q. J^{k-1} is generated by products $x_{k-1}x_{k-2}\cdots x_1$ of the basis elements. We will assume that some such product is nonzero and show that we get a contradiction to our assumption.

Since the total product is nonzero, the set of arrows must form a path: x_1 from v_1 to v_2, x_2 from v_2 to v_3, and so forth. Our first claim is that all the vertices $v_1, \ldots, v_k$ must be distinct. If not, there would be a subproduct $x_{j-1}\cdots x_i$ which would form a cycle, that is to say, it would lie in some Peirce factor eAe, where e is the idempotent corresponding to v_i. However, by assumption Q has no loops, which means that $(eAe) \cap J$ is zero. Since a product of radical elements must lie in the radical, the given subproduct must be zero and thus the entire product. Q thus contains k distinct vertices $v_1, \ldots, v_k$.

For each pair i, j with $i < j$, we consider the subproduct $x_{j-1}\cdots x_i$. It must be nonzero since the entire product is nonzero. Thus, there must be at least one arrow from v_i to v_j. We have thus found a complete, lexicographically oriented subgraph of Q, in contradiction to our assumption. □

The following result was obtained independently by C. Cibils using homological methods.

Corollary 2.1. *A loopless basis graph algebra whose basis graph contains no reducible arrows and no cycles consisting of two arrows is rigid.*

Proof. We are assuming that our basis graph is loopless and, thus, as mentioned above, could only deform to an algebra with the same basis graph but with different weights. Since we have excluded two-arrow cycles, we have excluded the possibility of the creation of matrix units which always come paired in two-arrow cycles. Since we have excluded reducible arrows, J^2 is zero and the only possible deformation is the basis graph algebra itself. □

We are trying to determine all possible deformations of a loopless basis graph algebra. Since one possibility is that a portion of the graph deforms into a matrix block we must first answer the following question: What are the conditions on a basis graph algebra which allow it to deform into an algebra with matrix units?

Lemma 2. *A set of n vertices in a basis graph algebra A_Q can be deformed to a matrix block if and only if they are connected, they are completely symmetric with regard to all permutations, and the number of loops at each vertex is one less than the number of arrows from each point to another.*

Proof. That these conditions are necessary is obvious since the matrix units determine K-vector space isomorphisms among the blocks e_iAe_j. The problem is to show that they are sufficient. From the elements in the radical represented by the arrows in the double complete graph which must exist among the points, we choose pairs e_{ij} and e_{ji} of elements which we want to deform into matrix units. If there are $m-1$ loops at each of the n vertices, then the set of n vertices and arrows among them is the quiver of the matrix algebra $M_n(R)$, where R is the basis graph algebra of the graph consisting of a point and $m-1$ loops. For each arrow α from the internal vertex v_1 to an external vertex w we choose $n-1$ symmetrical arrows and label them αe_{1j}. Similarly, for each arrow β from an external vertex w to the internal vertex v_1, we choose $n-1$ symmetrical arrows and label them $e_{j1}\beta$. The deformation is then completely determined by the standard deformation of matrix units to their corresponding basis graph algebra:

$$\begin{aligned} e_{ii}e_{ii} &= e_{ii}, \\ e_{ij}e_{jk} &= te_{ik}, \\ e_{ij}e_{ji} &= t^2 e_{ii}. \end{aligned} \tag{36}$$ □

Definition. If x_t is a basis element in A_{ji}, we indicate that its left idempotent is e_j by writing $l(t) = j$ and that its right idempotent is e_i by writing $r(t) = i$. A sequence of indices $(s, t, u, \ldots, v)$ will be called *admissible* if $r(s) = l(t), r(t) = l(u)$, etc. Such a sequence will be called *compatible* with an index p if it is admissible, $l(s) = l(p)$ and $r(v) = r(p)$. A product of basis elements can only be nonzero if the sequence of indices is admissible, and only compatible indices can occur in the product.

We now have the information required to prove the main result of this section.

Proposition 3. *Let Q be a loopless basis graph which has no complete, lexicographically-oriented subgraphs of order 4 and no matrix block configurations. Then there is a one-to-one correspondence between irreducible components containing the basis graph algebra A_Q of Q and the maximal weightings of Q.*

Proof. Alg_n, the structure-constant scheme, is an algebraic scheme, and thus has a finite number of irreducible components. We consider those irreducible components which contain the basis graph algebra of Q.

In [**9**, Prop. 3], we proved that if one algebra A is a deformation over a curve of another, B, then the basis graph of B is obtained from the basis graph of A by coalescing vertices or equals it. Here it is always understood that whenever n points coalesce into one, $n - 1$ loops are created, so that the total number of vertices and arrows remains fixed at the common dimension n of A and B. Since Q contained no matrix block configurations, and was loopless, every deformation of the basis graph algebra A_Q must be basic and have Q as its underlying basis graph. We take a particular irreducible component. From [**9**, Prop. 2 and **11**] the generic algebra in this component has a fixed weighted basis graph Q'. For any algebra with this weighted basis graph we can choose a basis which respects the Peirce decomposition and also gives bases for powers of the radical J.

Since there are no complete, lexicographically-oriented subgraphs of order 4, we know by Proposition 2 that for any deformation B' of the basis graph algebra A_Q, we have $J'^3 = 0$. Let us choose basis elements corresponding to the vertices and arrows in the weighted basis graph Q', numbered so that $x_1, \dots, x_r$ are idempotents, $x_{r+1}, \dots, x_s$ correspond to arrows with one barb, and the remaining basis elements $x_{s+1}, \dots, x_n$ correspond to arrows with two barbs. For each triple i, j, t for which $t > s$, $r + 1 < i$, $j \leq s$, and the sequence (i, j) is compatible with t, we choose an indeterminate a^t_{ij}. We define a generic algebra multiplication:

$$x_i x_j = \Sigma a^t_{ij} x_t, \tag{37}$$

all other products of radical basis elements being zero. We claim that this multiplication is associative. Any nonzero product of two elements is a combination of elements x_t for which $t > s$, and thus any triple product is zero. Therefore, the associativity relations

$$(x_i x_j) x_k = x_i (x_j x_k) \tag{38}$$

are trivially fulfilled. Thus, we have a monomorphism from the affine space on these indeterminates into the scheme of structure constants, with image W.

Every algebra with weighted basis graph Q' is isomorphic to one of the algebras whose set of structure constants lies in W. Since W is isomorphic to an affine space and is therefore irreducible, it must lie completely within some irreducible component of Alg_n. We claim that this is the unique irreducible component with weighted basis graph Q'. The closure of an irreducible component is a union of orbits, since each component contains an open dense set which is a union of orbits, and, thus, its closure is invariant under the group action. Thus, any orbit which intersects W must be completely contained in the given irreducible component. This includes all orbits of algebras with weighted basis graph Q'.

Conversely, suppose we have an algebra whose weighted basis graph is maximally weighted, so that by the upper semicontinuity result brought above from [**9**], it has no deformations with a basis graph of higher weight. It must be contained in some irreducible component, so this irreducible component must have the same weighted basis graph. Thus, by the previous paragraph, this is the unique irreducible component with this weighted basis graph. □

Example 3. Let Q be the following basis graph:

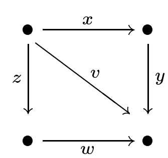

FIGURE 2

The basis graph algebra for this basis graph is not rigid. In fact, the parameter space to its versal deformation space is two-dimensional. If we let the parameters be μ_1 and μ_2, then the generic deformation is given by the following multiplication of the radical elements:

$$\begin{aligned} y \cdot x &= \mu_1 v \\ w \cdot z &= \mu_2 v. \end{aligned} \tag{39}$$

In all cases except when both parameters are zero, the weighted basis graph has two barbs on v. When both parameters are nonzero, we get a rigid algebra whose orbit is an open subvariety of Alg_9.

The two special cases when only one of the parameters is nonzero give two other algebras, making four orbits altogether represented as deformations of this basis graph algebra.

4. The computer algorithms. The final ingredient in facilitating the search for generic algebras is the use of computer algorithms to verify rigidity or to locate deformations.

In [**8**] the second author described a computer algorithm for calculating the tangent space to the deformation space of a finite dimensional algebra. This object is better known to algebraists as $H^2(A, A)$, the second Hochschild cohomology group of the algebra. If this cohomology group is zero, then the algebra has no deformations, except the trivial deformations induced by deformations of the identity automorphism. If, for a family of nonisomorphic algebras depending on continuous parameter, the dimension of $H^2(A, A)$ at a fiber is equal to the number of parameters, then the fiber is generic and the family is semirigid.

We need not actually be dependent on the accuracy of the computer program. The results in Section 3 allow us to determine the rigidity of most generic algebras on purely theoretical grounds, and the remaining cases (in dimension six there are only eight) can be checked by hand, once the computer has helped us find them. Thus, the program can be regarded as an aid for the search, not an integral part of the proof of the classification.

As we describe below, to show that the remaining algebras are not generic requires classifying them into strata and displaying a nontrivial deformation (usually an interleaving) for the generic element in each stratum. The computer produces the multiplication table of the generic deformation of each algebra, from which it is easy to choose a fairly simple deformation.

Two refinements have been made in the algorithm since the original version given in [**8**]. One, described in [**10**], calculates the dimension of the group of automorphisms of the algebra and the dimension of the orbit of the algebra which is useful information to have when trying to calculate parts of the deformation chart of Alg_n. The second, which we prove as Lemma 3 below, is simply an improvement in the efficiency of the program and the readability of the results based on the theorems in [**9**].

In [**9**, Prop. 1] it was proven that any deformation is equivalent, via some deformation of the identity automorphism, to a deformation in which the trivial sections of the basis elements preserve the good properties of these elements. In particular, idempotents remain idempotents, and factors in the Peirce decomposition determined by a complete orthogonal set of idempotents are also fixed.

Lemma 3. *In order to determine all first order deformations of A, modulo equivalence by deformations of the identity automorphism, it* *suffices to consider all deformations with fixed idempotents and Peirce decomposition, modulo all deformations of the identity automorphism which preserve the factors in the Peirce decomposition.*

Proof. Given any other first-order deformation, its equivalence class contains a deformation with the required properties, and we can replace it by the good deformation. If two good deformations are equivalent, then the automorphism connecting them converts each trivial section of a basis element to a section lying entirely within the same Peirce factor $e_j A e_k$ and leaves each idempotent fixed. The first order automorphism is of the form $I + \varepsilon M$, and thus M must map each basis element x_i to a linear combination of basis elements from the same Peirce factor, for only in that way will $(I + \varepsilon M)x_i$ be stable under left and right multiplication by the idempotents defining the Peirce factor containing x_i. □

Application of this lemma to the algorithm for computing the tangent space has considerably increased its efficiency in the case of algebras with several idempotents. An associativity equation

$$(x_i x_j)x_k = x_i(x_j x_k) \tag{40}$$

gives new information about the tangent space only when (i, j, k) is an admissible sequence, and only for a compatible index p. Furthermore, the product is known to belong to the Peirce factor defined by the left idempotent of x_i and the right idempotent of x_k. This considerably reduces the number of equations which must be checked. We also have fewer equations when dividing out by the action of the matrices $I+\varepsilon M$. Finally, if our deformed multiplication is given by

$$x_i x_j = \Sigma(a_{ij}^p + \varepsilon b_{ij}^p)x_p \tag{41}$$

then we may assume that all the variables b_{ij}^p are zero unless (i, j) is admissible and compatible with p. This is a vast reduction in the number of variables under consideration.

5. The inductive classification procedure. We now describe an inductive procedure for determining the generic algebras of a given dimension n, based on the theoretical results of the preceding sections. In order to give a clearer idea of the amount of work involved, we describe the classification in dimension six. Detailed proofs of various claims will appear in the first author's Ph.D. thesis. The classification of all six-dimensional algebras was carried out by Voghera [**12**] 80 years ago, and our results verify the accuracy of his parameterization of the continuous families.

As stated in the introduction, the explicit goal of this program is to find *all* of the generic algebras, *most* of the algebras, and *some* of the deformations between them. We have not found an exhaustive search for deformations feasible for the large number of algebras under discussion. Nor, for an algebra with a complicated internal structure, will it always be feasible to determine its precise isomorphism class. However, we try to divide up Alg_n into a finite collection of irreducible, locally closed subsets which we will call *strata*, to assign each algebra to its proper stratum, and to show that each stratum is either in the closure of another stratum or else is itself dense in some irreducible component of Alg_n.

The method we will use is a variation of Mazzola's "method of quivers," modified by the Iarrobino-Briancon method of patterns. The basic invariant of our underlying classification of algebras is the basis graph. The basis graphs can be categorized into three main groups, and in each group the nature of the work is different. The three groups are loopless, mixed, and loop-only.

Definition. A *loopless* basis graph contains no loops. A *mixed* basis graph contains both loops and nonlooped arrows. A *loop-only* basis graph has loops but no nonlooped arrows.

Every basis graph falls into exactly one of these categories, and every algebra has a unique basis graph, so we can treat each category separately. For each category we will describe the method used to find all generic algebras and give numerical results on the number of such algebras at least up to dimension six. We then summarize all the generic algebras in a single table.

5.1. *Loopless generic algebras.* The theoretical foundation for the search for generic algebras with loopless basis graph was laid in Section

3. For each loopless basis graph there is at least one generic algebra with that basis graph, and most generic algebras are of this type. The basic result is Proposition 3 above, which says that if the basis graph contains no complete lexicographically-ordered graphs on four vertices and no matrix block configurations, the generic algebras are in one-to-one correspondence with the maximal weightings of the basis graph. In dimensions less than or equal to eight there are no complete four-graphs, and, up to duality, only two matrix block configurations. Thus, the classification is purely combinatorial. There are 21 generic loopless algebras, which are drawn in Figure 3 at the end of the paper. They are listed according to the dimension of the automorphism group, starting from K^6, which has no automorphisms, down through the Kronecker algebra, for which the automorphism group has dimension 20.

5.2 *Generic algebras with mixed basis graph.* In contrast to the situation in the loopless case, in which every loopless basis graph produced at least one generic algebra, in the mixed case there are many more basis graphs than in either the loopless or loop-only cases, and hardly any of them correspond to generic algebras. Therefore, our chief concern in this section is to find theoretical results which will eliminate almost all graphs as expeditiously as possible.

In Section 3 we defined an arrow (which may or may not be a loop) to be *reducible* if it was the composition of two other arrows. We called it *reducing* if it is one of the factors in such a composition. If a loop is neither reducible nor reducing, then its product with every other basis element must be zero, and it must have weight 1, in any algebra on that basis graph. Therefore, any such algebra is obtained by a trivial interleaving (as in Proposition 1) from the product of a copy of the field with an $n-1$ dimensional algebra and is therefore not generic. This eliminates most mixed graphs immediately.

In dimension six, the small number of mixed basis graphs which remain can each be easily shown, by using Proposition 1, to be the result of an interleaving, except in two cases. The first case is the dual pair of graphs with two idempotents connected by an arrow and three loops on one idempotent. If the three loops are designated x, y, and z, with $z = xy = syx$, then we get an algebra with only infinitesimal deformations, which will be shown elsewhere to give a nonreduced component of Alg_6 [**11**]. The second case again has two idempotents, this time connected by two arrows x and y in one direction, one arrow

z in the other direction and one loop. If xz (or, dually, zx) is nonzero, then we get an algebra with no deformations. Thus, the total number of generic algebras with mixed basis graph is four.

5.3 *Generic algebras with loop-only basis graphs.* In an algebra whose basis graph has no nonlooped arrow, each idempotent corresponds to a different connected component. Assuming that we know all the generic algebras in lower dimensions, this means that we need only find the generic local algebras of dimension n.

All local algebras have the same basis graph, consisting of one point and $n-1$ loops, which represent elements of a basis of the radical J. Thus, in order to get any sort of useful classification, we must consider more delicate discrete invariants of the algebra. As our coarse invariant we will take the weighting of the basis graph, which depends on the dimensions of the vector spaces J^i/J^{i+1}. As a finer invariant we will adapt Iarrobino's concept of a pattern to our noncommutative situation.

In Section 3 for loopless algebras we used the fact that if $J^3 = 0$, all associativity relations are automatically satisfied. This is equally true in our local situation, and, therefore, we obtain a large but simply understood class of algebras.

Definition. Following Mazzola, we will call local algebras with radical-cubed zero *Scorza algebras*. If c is the dimension of J/J^2 and d is the dimension of J^2, then the multiplication is completely determined by a bilinear form mapping J/J^2 onto J^2, just as in Section 3.

Definition. The subset of Alg_n parameterizing all such algebras, for given c and d, will be denoted by $\mathrm{Scorza}(c,d)$. Its closure will be denoted by $\mathrm{Scorza}(c,d)^-$.

We claim that the family $\mathrm{Scorza}(c,d)^-$ is irreducible. Each $\mathrm{Scorza}(c,d)$ algebra structure on a vector space V determines a flag of vector spaces $V \supset J \supset J^2$, which are parameterized by points on a flag manifold. Then for each point in the c^2d-dimensional space of bilinear mappings from $J/J^2 \times J/J^2$ into J^2 we get an algebra in $\mathrm{Scorza}(c,d)^-$. A vector bundle over a flag manifold is an irreducible variety. Thus, every algebra in $\mathrm{Scorza}(c,d)$ is a specialization of the general $\mathrm{Scorza}(c,d)$ algebra, and it is therefore sufficient to check the general algebra for rigidity or semirigidity. The general family is given by fixing a radical flag and taking d arbitrary $c \times c$ matrices which determine the bilinear form. In the case of dimension six, there are three possible Scorza families corresponding to the pairs of integers (4,1), (3,2) and (2,3).

Of these, the computer output shows that the first two give generic families. For Scorza$(4,1)$, the set of orbits are parameterized by two parameters, and one can easily locate algebras for which the space of infinitesimal algebras is exactly two. A typical generic algebra is given by

$$xy = s(yx) = zw = t(wz)$$

with all other products of x, y, z, w equal to zero.

Similarly, for Scorza$(3,2)$, there is a six-parameter family of orbits, and one can easily find elements of this family which have a six-parameter family of infinitesimal deformations, indicating that the family is semirigid and gives a generic family.

The remaining Scorza family, Scorza$(2,3)$ can be given by taking x^2, xy and yx linearly independent and letting y^2 be a one-parameter combination of these elements. The computer output indicates that there are many first order idempotent-splitting deformations, and we successfully computed one of them as a function of the parameter, proving that the algebra was not rigid.

Having disposed of the Scorza algebras (which numerically form the bulk of the local algebras) we are left considering weightings in which the radical cubed is nonzero. Let s be the length of the radical, the number such that J^s is not zero, but J^{s+1} *is* zero and divide into cases according to s. For each s, there may be several different weights to consider. Then for a fixed s and a fixed weighting, we divide further according to the following:

Definition. A *pattern* of a local algebra is a basis of monomials filtered by the radical such that each basis element of degree $d > 1$ is the product of a basis element of degree $d-1$ by a basis element of degree 1.

Definition. A pattern is called *normal* if there is some ordering of the degree 1 monomials such that in each degree $d > 1$ the pattern monomials, modulo commutativity, map onto an initial segment of the sequence of commutative degree d monomials in lexicographical order.

For example, we would consider all of the patterns

$$1, x, y, x^2, xy$$
$$1, x, y, x^2, yx$$
$$1, x, y, x^2, xy, yx$$

to be normal, since in each case the degree 2 monomials, modulo commutativity, map onto the initial segment (x^2, xy) of the lexicographically ordered set (x^2, xy, y^2) of second degree commutative monomials.

On the other hand, the following pattern is not normal because the initial monomial of degree three is x^3 rather than x^2y:

$$1, x, y, x^2, xy, x^2y.$$

Since the weightings of the graphs are semicontinuous, there are a finite number of general algebras for each weighting, and the general algebras can usually be shown to have a normal pattern. (For characteristic zero and a quiver with two loops; this is a theorem of Iarrobino [**6**]).

In dimension six, each weighting of the basis graph for a local algebra has a single general algebra which has a normal pattern. The following patterns for six-dimensional local non-Scorza algebras are normal when the degree one terms are ordered alphabetically:

(a) $$1, x, x^2, x^3, x^4, x^5,$$
(b) $$1, x, y, x^2, x^3, x^4,$$
(c) $$1, x, y, z, x^2, x^3,$$
(d) $$1, x, y, x^2, xy, x^3.$$

The first, (a), is the pattern of the commutative truncated polynomial algebra and deforms to the product of six copies of the field. The second two, (b) and (c), give noncommutative families which are not semirigid but rather deform, via an interleaving, to the product of two copies of the field times the family Scorza$(2, 1)$. An interleaving for the unique family of algebras with pattern (b) is given in Example 2.3 above. The last, (d), gives an irreducible family which has infinitesimal deformations but which is not generic. Details are given in the first author's thesis.

We now summarize the enumeration of generic algebras for dimension six. As one can see, the proportion of generic algebras with loopless basis graph remains high, though no longer as high as it was in dimension five.

Enumeration of generic algebras to dimension six:

		Properties		
Dimension	Total	Loopless	Mixed	Looped
2	1	1		
3	2	2		
4	5	4		1
5	10	9		1
6	26	21	2	3

We now list the generic six-dimensional algebras, together with some of their discrete invariants: The number of idempotents, the number of parameters, the depth s, and the dimension of the automorphism group of the algebra. For the loopless algebras, numbered 1 to 21, the algebras are defined to be the unique algebras with the weighted basis graphs given in Figure 3. Except for algebra 8, the upper triangular matrices, these are all radical-squared zero algebras.

The mixed algebras both have two idempotents, labeled e_1 and e_2. They are described by giving generators and relations products. Of the loop-only algebras, all are derived from Scorza algebras.

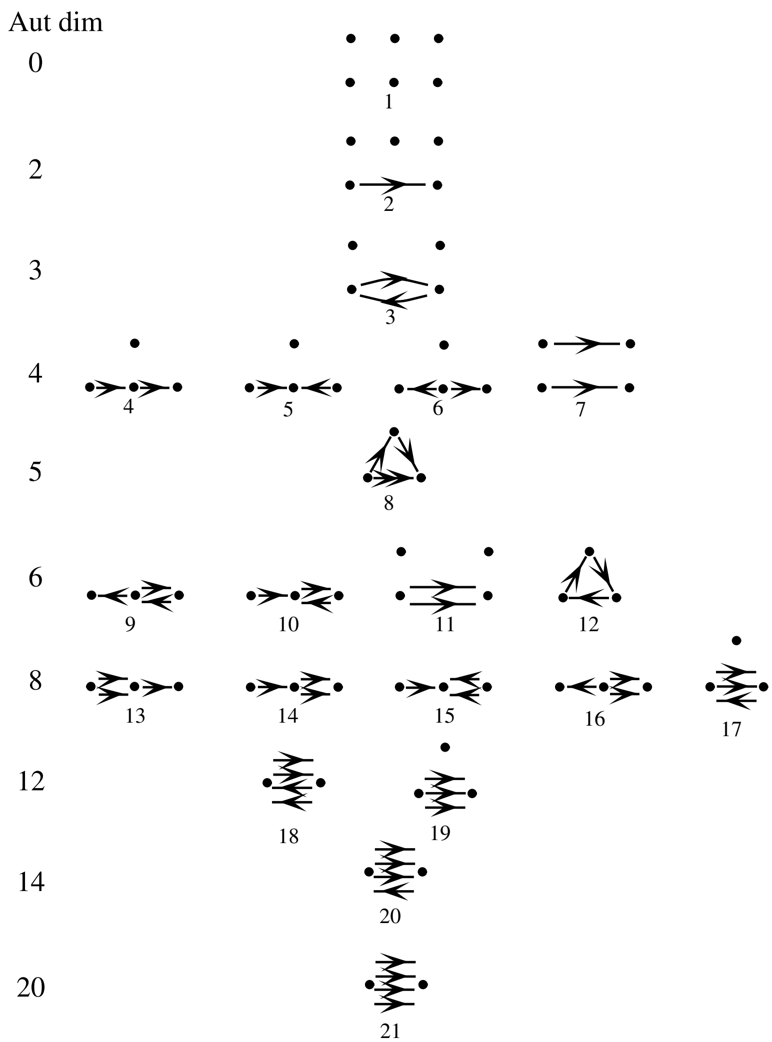
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FIGURE 3

	Algebra	No. of Idem-potents	No. of Param-eters	Depth	Dimension of the Aut. Group
	Mixed				
22.	$x, y \in e_1 A e_2)$				
	$z \in e_2 A e_1)$				
	$yz = zx = zy = 0$	2	0	2	7
23.	Dual to 22	2	0	2	7
	Loop-Only				
24.	$\text{Scorza}(2,1) \times K^2$	3	1	2	4
25.	$\text{Scorza}(4,1)$	1	2	2	7
26.	$\text{Scorza}(3,2)$	1	6	2	7

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