ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 1, Winter 1992

ON CONNECTED GROUPS AND RELATED TOPICS

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ABSTRACT. The study of pro-Lie groups and residual Lie groups [**2, 3**] led to the present work on connected groups and related topics. During the course of our study, we found that some theorems (Theorem 8, for example) of V.M. Gluskov as appeared in [**5**] may not have been accurately stated. In a related context we show that if G is a connected, locally connected, locally compact group with its center metrizable, then there exists a neighborhood of the identity element each of whose points lies in a one parameter subgroup. In terms of the characteristic index of a connected locally compact group defined by K. Iwasawa [**8**], we have introduced the concept of total index for a closed normal subgroup of a connected locally compact group. A counterexample to a conjecture made in [**6**] is given at the end of the paper.

1. Introduction. This paper constitutes part of our recent efforts in studying the structure of locally compact groups. Our study of pro-Lie groups and residual Lie groups [**2, 3**] led to the present work.

In Section 2 we study some properties of connected locally compact groups and the question of lifting one-parameter subgroups. We show that the arc component of the identity element of a connected locally compact group is generated by all one parameter subgroups. During this course of our study, we found that some theorems (Theorem 8, for example) of V.M. Gluskov as appeared in [**5**] may not have been accurately stated. In a related context we show that if G is a connected, locally connected, locally compact group with its center metrizable, then there exists a neighborhood of the identity element each of whose points lies in a one-parameter subgroup. We also study local connectedness of homogeneous spaces at the end of the section.

If G is a connected locally compact group, then there exists a maximal compact connected subgroup K such that $G = KE$, where the space E is homeomorphic to γ-dimensional Euclidean space. The integer γ

1980 *Mathematics Subject Classification.* Primary 22D05.
Key words and phrases. Locally compact group, connected group, one parameter group, inverse limit, characteristic index, characteristic subgroup.
Received by the editors on March 2, 1987.

is called the characteristic index of the connected group G, [**8**, p. 549]. We introduce the concept of total index for a closed normal subgroup of a connected locally compact group in Section 3. We show that if N is a closed normal subgroup of a connected locally compact group G, then G/N is compact if and only if the total index of N equals the characteristic index of G.

In Section 4 we consider various locally compact groups involving connectedness. Results obtained in this section are closely related to some of those in [**6**]. A counterexample to a conjecture made in [**6**] is given at the end of this section.

Throughout this paper, we shall use G^α to denote the arc component of the identity element of the locally compact group G with G_0 stands for the identity component of G. The group of automorphisms of G will be denoted by $\mathrm{Aut}\,(G)$. $\mathrm{Aut}\,(G)$ is assumed to have the Birkhoff topology [**7**] when appropriate. For a subgroup S of G, the centralizer of S in G is denoted by $C_G(S)$, or simply $C(S)$. We use the notation $G \cong H$ to denote that the topological groups G and H are isomorphic topologically and use the notation $G \simeq H$ to denote that G and H are homeomorphic.

2. Connected groups and Gluskov's Theorems. In this section we discuss arcwise connected and connected, locally compact groups. In the course of our study, we find that some theorems of Gluskov's in [**5**] may not have been accurately stated.

For a connected, locally compact group G, we shall say that every finite-dimensional factor is a Lie group if for every compact normal subgroup K of G, G/K is a Lie group whenever G/K is finite-dimensional.

Proposition 2.1. *Let G be a connected, locally compact group. Suppose that*

(1) *The maximal compact normal subgroup has only finitely many components, and*

(2) *If A is the maximal compact connected central subgroup of G, then every finite-dimensional factor of A is a torus.*

Then every finite-dimensional factor of G is a Lie group.

Proof. Let K be a compact normal subgroup of G such that G/K is finite-dimensional. Let B be the maximal compact connected normal subgroup of G, and let B' be the maximal compact normal subgroup. Then G/B' is a Lie group. Hence, G/B is also a Lie group. Note that B'/B is finite. Consequently, we may assume that $K \subset B$. Since G/B is Lie and $K \subset B$, B/K is finite-dimensional. Now $B = AS$, where A is a compact connected abelian group and S is a product of compact simple Lie groups, i.e., $S = \pi S_\lambda$, each S_λ is a compact simple Lie group. We claim that G/K is Lie if and only if $A/A \cap K$ is Lie. Suppose that G/K is a Lie group. Then certainly B/K is a Lie group. Since $B = AS$, AK/K is a Lie group, hence $A/A \cap K$ is a Lie group. Conversely, if $A/A \cap K$ is a Lie group, then AK/K is Lie. Since $AS/AK \cong S/AK \cap S$ is Lie, AS/K is a Lie group. This proves the claim. The proposition now follows from hypothesis (2). □

If G is a connected locally compact metrizable group, and if K is a normal subgroup of G such that G/K is finite-dimensional and $G = G^\alpha K$, where G^α denotes the arc component of the identity in G, then it is clear that G/K is arcwise connected. Since G/K is finite-dimensional, this implies that G/K is locally connected and, hence, is a Lie group.

We also note that if G is not metrizable and if G/H is locally connected, $G \neq G^\alpha H$ in general, for a subgroup H.

Proposition 2.2. *Let G be a connected locally compact metrizable group, and let K be a compact normal subgroup such that G/K is locally connected. If A is the maximal compact connected central subgroup of K and if every arc in G/A can be lifted to G, then $G^\alpha K = G$.*

Proof. Since the mapping $G/K_0 \to G/K$ has compact and totally disconnected fibers every arc in G/K can be lifted to G/K_0 (cf. [**10**, p. 236]).

Let $K_0 \cong A(\pi S_\lambda)$, where A is a compact connected abelian group and each S_λ is a compact simple Lie group. The natural mappings $G \to G/K_0$ and $G \to G/A$ induce a homomorphism $\phi : G/A \to G/K_0$ with kernel $\phi \cong \pi S_\lambda$. Write $G_1 = G/A$ and $G_2 = G/K_0$. Then $G_2 \cong G_1/\pi S_\lambda$. Since each S_λ is a compact simple Lie group, we may assume $G_1 = \varprojlim G_\lambda$, where the kernel of the homomorphism $f_{\lambda+1} : G_{\lambda+1} \to G_\lambda$ is a compact simple Lie group. Hence, we can lift arcs from G_λ to $G_{\lambda+1}$. After taking the inverse limit, we see that every arc in G_2 can be lifted to G_1. Consequently, if G/K is locally connected, then every arc can be lifted to G/A. This implies that $(G/A)^\alpha(K/A) = G/A$. Hence, if every arc in G/A can be lifted to G, then we may reach the conclusion that $G^\alpha K = G$. □

Remark 2.1. If G is a connected locally compact metrizable group, if K is a compact normal subgroup such that G/K is locally connected, and if A is a maximal compact connected central subgroup of K which has a totally disconnected subgroup D such that A/D is a torus (which may be infinite-dimensional), it follows from the above considerations that $G^\alpha K = G$.

Using the concept of Lie algebra in the sense of Lashof [**9**], we in fact can improve Proposition 2.2 to

Proposition 2.3. *Let G be a connected locally compact metrizable group, and let H be a closed normal subgroup such that G/H is locally connected. Then $G^\alpha H = G$.*

Proof. For a locally compact group X, let $L(X)$ be the Lie algebra of X in the sense of Lashof. Consider the following diagram:

$$\begin{array}{ccc} L(G) & \xrightarrow{d\phi} & L(G/H) \\ \Big\downarrow{\scriptstyle \exp} & & \Big\downarrow{\scriptstyle \exp} \\ G & \xrightarrow[\phi]{} & G/H \end{array}$$

Since G is metrizable, G/H is arcwise connected, hence $\exp L(G/H) = G/H$. Using the commutativity of the diagram in the proof of [**7**, Lemma 3.4] and the fact that $d\phi$ is onto and $\exp L(G) \subseteq G^\alpha$, we see that $\phi(G^\alpha) = G/H$. Hence, $G^\alpha H = G$. □

Theorem 2.4. *Let G be a connected locally compact group, and let G^α be the arc component of G at the identity. Then G^α is generated by all one-parameter subgroups of G. Furthermore, given any local* *decomposition of a neighborhood V of e, $V = K \times L$, where K is a compact normal subgroup and L a local Lie group, $K^\alpha \times L$ generates G^α and each $x \in K^\alpha \times L$ is in a one-parameter group.*

Proof. Let V be a neighborhood of e, and let $V = K \times L$ be a local decomposition of V, where K is a compact normal subgroup and L a local Lie group. If $x \in G^\alpha \cap (K \times L)$, there exists an arc connecting x and e, this arc must pass through $K_0 \times L$. We consider $(K_0 \times L)^\alpha = K_0^\alpha \times L$. Let $K_0 = AS$, where A is a compact connected abelian group and S a product of compact simple Lie groups. Then $K_0^\alpha = A^\alpha S$, and hence $(K_0 \times L)^\alpha = A^\alpha SL$. Now $(A^\alpha SL)^n = (A^\alpha S)L^n$, hence $A^\alpha SL \subset G^\alpha$, where $L = \cup_{n=1}^\infty L^n$.

An arc connecting e and $a \in G^\alpha$ is covered by a chain of translates of V. Hence, there is $a_1 \neq e$ such that $a_1 V \cap V \neq \varnothing$ and there exists $b_1 \in K_0^\alpha L$ such that $a_1 \in b_1 K_0^\alpha L$. Likewise, we choose a_2 and b_2 such that $b_2 \in a_1 K_0^\alpha L$ and $a_2 \in b_2 K_0^\alpha L$. Continuing in this way, we see that $G^\alpha \subset \cup_{n=1}^\infty (A^\alpha SL^n) \subset A^\alpha SL$. Hence, $G^\alpha = A^\alpha SL$. Note that for any other choice of neighborhood V' and local decomposition $V' = K' \times L'$, $K_0'^\alpha \times L'$ again generates the arc components G^α. In fact, $K_0'^\alpha \times L'$ generates the same set of one-parameter subgroups.

The second assertion is now clear. □

Theorem 2.5. *Let G and H be locally compact groups and let $\phi : G \to H$ be an onto homomorphism with compact kernel. If G is metrizable, every one-parameter subgroup in H can be lifted to G.*

Proof. Let K be the compact kernel of ϕ, and let $\{\alpha(t) \mid t \in \mathbf{R}\}$ be a one parameter subgroup in H. Let $A = \overline{\{\alpha(t) \mid t \in \mathbf{R}\}}$, let $F = \phi^{-1}(A)$, and let $M = (Z_F(K))_0$, the identity component of the centralizer of K in F. Since F/K is connected, $F = MK$. Thus, $M/M \cap K \cong MK/K \cong F/K = A$. We consider the following two cases:

Case 1. $A \cong \mathbf{R}$ or A is a torus. In this case, since $M \cap K$ is central in M, there exists a one parameter group $\beta(t)$ in M which covers $\alpha(t)$.

Case 2. A is compact but is not a torus. Then $M/M \cap K \cong A$ implies that M is compact, connected and nilpotent since $M \cap K$ is central in

M and $M/M \cap K$ is abelian. Since every compact connected nilpotent group is an abelian group, M is abelian. Hence, if $Q = M \cap K$, we have the exact sequence

$$1 \longrightarrow Q \longrightarrow M \xrightarrow{\pi} A \longrightarrow 1.$$

Now M^α and A^α, respectively, are the union of one-parameter groups in M and A, [**1**, Theorem 8.19, p. 110]. Hence, $\pi(M^\alpha) = A^\alpha \supset \{\alpha(t) \mid t \in \mathbf{R}\}$. Note that, if $M' = M/Q_0$ and $Q' = Q/Q_0$, $M'/Q' \cong A$. Hence, the kernel of the natural map $\pi' : M' \to A$ is totally disconnected and there exists a unique $\beta'(t)$ in M' that covers $\alpha(t)$. Hence, we may assume that Q is connected in the exact sequence above.

Since G is metrizable, Q is metrizable and, hence, there exists a sequence of closed normal subgroups N_i such that $Q = \varprojlim Q/N_i = \varprojlim T_i$, where each T_i is a finite-dimensional torus and that $M = \varprojlim M/N_i$. For each i let $M_i = M/N_i$, then the homomorphism $f_{i+1,i} : M_{i+1} \to M_i$ has fiber T_i which is a finite-dimensional torus. Now we lift $\alpha([0,1])$ to M_i as $\beta_i([0,1])$ such that the mapping $\beta_{i+1}([0,1]) \to \beta_i([0,1])$ forms a local one parameter group. Finally, taking $\varprojlim \beta_i$ to be β on $[0,1]$, we may lift $\alpha(t)$ to G. □

In [**5**], V.M. Gluskov has stated the following theorem: In every locally connected locally compact group, there exists a neighborhood of the identity U every element of which lies in a real one-parameter subgroup. This means that any two elements in U can be joined by a path (in G), hence any two elements in U^m, for any m, can be joined by a path (in G). Hence, in particular, if G is connected, G is generated by U, which would imply that G is arcwise connected. However, J. Dixmier [**4**] has an example showing that connectedness and local connectedness do not imply arcwise connectedness. Therefore, it does seem that the above Gluskov's theorem is not accurately stated; we shall focus on this problem in the next several results.

Proposition 2.6. *Let G be a connected, locally connected, locally compact group. Then the maximal compact connected normal subgroup and the maximal compact normal subgroup of G are locally connected.*

Proof. Let B be the maximal compact connected normal subgroup of G. We shall prove that B is locally connected. As we noted in the proof of Proposition 2.1, B has finite index in the maximal compact normal subgroup. Let V be a connected neighborhood of e. By Theorem 11, [**12**], we can choose $V \cong K \times L$, where K is a compact normal group and L a local Lie group. Since V is connected, K is connected. Consequently, BK is connected. Since BK is compact and normal, we have $BK \subset B$. This implies that B is locally connected. □

The following is a version of Gluskov's theorem stated above.

Theorem 2.7. (Gluskov). *If G is a connected, locally connected, locally compact group with its center metrizable, then there exists a neighborhood V of e, every point of which is in a one-parameter group.*

Proof. Let B be the maximal compact connected normal subgroup of G. As in the proof of Proposition 2.6 we may choose a connected neighborhood V of e, with $V \cong K \times L$, where K is a compact normal subgroup and L a local lie group. Then $BK \subset B$. Let $B = AS$, where A is a compact connected abelian group and S is a product of compact simple Lie groups. Now B is metrizable, hence B is arcwise connected. By Corollary 1, p. 223, of [**11**], every point $x \in B$ is in a one parameter subgroup. Since $K \subset B$ and $V \cong K \times L$, it follows that every element of V is in a one-parameter group as desired. □

We carry the result of Theorem 2.7 further. Consider the decomposition $V \cong K \times L$ in the proof of that theorem. We let $K = AS$, where A is a compact connected metrizable abelian group and S is a direct product of compact simple Lie groups. Then $A \cong T^n$, a torus with n possibly infinite [**1**, p. 113]. We now show that $\mathcal{L} = \cup L^k$, the subgroup generated by L is a normal subgroup of G. Let $g \in G$ and choose a neighborhood $V_1 \subset V$ of e such that $gV_1g^{-1} \subset V$. Now V_1 contains a local Lie group L_1 (open in L) and $gL_1g^{-1} \subset L$. Since $\mathcal{L}$ is connected, $\cup L_1^k = \mathcal{L}$. It follows that $g\mathcal{L}g^{-1} = g(\cup L_1^k)g^{-1} = \cup L_1^k = \mathcal{L}$ as desired.

We can provide $\mathcal{L}$ with a unique Lie group topology by taking L to be an open subset and let the resulting Lie group be denoted by $\mathcal{L}^*$. Then the canonical mapping θ' of $\mathcal{L}^*$ onto $\mathcal{L}$ is a continuous isomorphism, and we have the mapping $\theta : K \times \mathcal{L}^* \to G$ defined by $\theta(k, x^*) = k\theta'(x^*) = kx$ for $(k, x^*) \in K \times \mathcal{L}^*$. Consequently, we have

Theorem 2.8. [**5**, Theorem 7]. *Under the hypotheses of Theorem* 2.7, *there exists a homomorphism of* $\mathbf{R}^n \times \pi S_\lambda \times \mathcal{L}^*$ *onto* G *whose kernel is a totally disconnected central subgroup, where* n *is possibly infinite and each* S_λ *is a compact simple Lie group.*

For the rest of this section we consider local connectedness of coset spaces.

Following [**12**] we say a closed subgroup H of a locally compact group G contains all small subgroups of G if there exists a neighborhood U of the identity of the group G such that all subgroups of G contained in U are also contained in H. If G is a locally compact group and H is a closed subgroup of G, then H contains all small subgroups of G if and only if G/H is a manifold [**2**, Theorem 4].

Lemma 2.9. *Let* G *be a connected locally compact group,* H *a closed subgroup of* G *such that* G/H *is locally connected. If* C *is the maximal compact normal subgroup of* G *and if* $F = C \cap H$*, then for every compact normal subgroup* K *of* C *such that* C/K *is finite-dimensional,* KF *contains all small subgroups of* C.

Proof. Since G is connected, $CC_G(C) = G$. Hence, every compact normal subgroup K of C is also normal in G. Now let K be a compact normal subgroup of C such that C/K is finite-dimensional. Clearly, G/K is finite-dimensional. Thus, G/KH is locally connected and finite-dimensional, consequently a manifold. Hence, KH contains all small subgroups of G [**12**, Theorem 4]. In particular, KH contains all small subgroups of CH. Hence, CH/KH is a manifold also [**12**, Theorem 4]. Now the homomorphism $C \to CH/KH$ is onto, and the canonical map $C/C \cap KH \to CH/KH$ is a homeomorphism. It follows that $C/C \cap KH$ is a manifold. Since $K \subset C$, $C \cap KH = K(C \cap H) = KF$ and C/KF is a manifold. Thus, KF contains all small subgroups of C. □

Corollary. *Under the hypotheses of Lemma* 2.9, *if C is the maximal compact normal subgroup of G, if $F = C \cap H$, and if K is a compact normal subgroup C such that C/K is finite-dimensional, then K_0F has finite index in KF.*

Proof. Since C/K is finite-dimensional, so is C/K_0. Hence, K_0F contains all small subgroups by Lemma 2.9, and hence $K = K_0D$, where D is totally disconnected. Therefore, K_0F has finite index in KF. □

Proposition 2.10. *Let G be a connected locally compact group, H a closed subgroup such that G/H is locally connected. If C is the maximal compact normal subgroup of G, then $C/C \cap H$ is locally connected.*

Proof. Let $F = C \cap H$ and V be a neighborhood of e in C. Then there exists a compact connected normal subgroup $K \subset V$ of C such that C/KF is locally connected. Let V' be a neighborhood of e in C such that $V'K \subset V$. We can choose a neighborhood V'' of e in C such that $KV'' \subset V'K$. Now C/KF is locally connected and $V''KF/F$ is a neighborhood of the identity in C/KF. There exists a connected neighborhood $W \subset V''KF/F$. If we let $\phi : C/F \to C/KF$ denote the natural map, $\phi^{-1}(W) \subset KV''KF/F \subset V'KF/F \subset \pi(V)$, where $\pi : C \to C/F$ is the natural map. Now $\phi^{-1}(W)$ is connected since the inverse image of each $w \in W$ has connected fiber wKF/F and is contained in $\pi(V)$. Hence, $C/C \cap H$ is locally connected. □

Theorem 2.11. *Let G be a connected locally compact group, and let C be a maximal compact normal subgroup of G. If H is a closed subgroup of G such that G/CH is a manifold, then G/H is locally connected if and only if CH/H is locally connected.*

Proof. It is clear that if CH/H is locally connected, then G/H is locally connected.

Now assume that G/H is locally connected. Since G/CH is finite-dimensional, the fibration of G/H over G/CH is locally trivial [**12**, Theorem 13′]. This implies that, locally, it is the direct product of base space and fiber. Now, since G/CH is a manifold and G/H is locally connected, the fiber has to be locally connected. Hence, CH/H is locally connected. □

3. Total index of a closed normal subgroup. Let G be a connected locally compact group. Then there exists a maximal compact connected subgroup K of G such that $G = KE$, where the space of E is homeomorphic to r-dimensional Euclidean space. The integer r is called the characteristic index of G [**8**, p. 549], abbreviated as $\mathrm{ci}(G)$. In this section we shall define the total index of a closed normal subgroup of a connected, locally compact group and discrete index, denoted by $\mathrm{ti}(G)$ and $\mathrm{di}(G)$, respectively.

The following remark is clear.

Remark 3.1. (1) A connected locally compact group G is compact if and only if $\mathrm{ci}(G) = 0$.

(2) If G is a connected locally compact group and if N is a connected closed subgroup of G, then G/N is compact if and only if $\mathrm{ci}(N) = \mathrm{ci}(G)$.

Now let G be a connected locally compact group and let N be a closed normal subgroup of G. If $G' = G/N_0$ and $N' = N/N_0$, then N' is a totally disconnected normal subgroup of the connected group G' and hence is central. Because N' is compactly generated, $N' \cong Z^a \times F$, where F is a compact group and a is an integer. We shall call the integer a the discrete index of N/N_0 and define the total index of N by

$$\mathrm{ti}(N) = a + \mathrm{ci}(N_0).$$

Note that if N is connected, $\mathrm{ti}(N) = \mathrm{ci}(N)$, and that, if N/N_0 is compact, $\mathrm{ti}(N) = \mathrm{ci}(N_0)$.

Theorem 3.1. *Let G be a connected locally compact group and N a closed normal subgroup of G. Then G/N is a compact group if and only if the total index of N equals the characteristic index of G.*

Proof. Suppose first that $\mathrm{ti}(N) = \mathrm{ci}(G)$. Since $\mathrm{ci}(G) = \mathrm{ci}(N_0) + \mathrm{ci}(G/N_0)$, we see that $\mathrm{ti}(N) = \mathrm{ci}(N_0) + \mathrm{ci}(G/N_0) = \mathrm{ci}(N_0) + \mathrm{di}(N/N_0)$. Hence, $\mathrm{ci}(G/N_0) = \mathrm{di}(N/N_0)$, and, therefore, $G' = G/N_0 \supset N/N_0 = Z^a \times Q$, where Q is a compact group and a is the discrete index of N. Let

K be a maximal compact connected subgroup of G'. Then $G' = KE$ with $\dim E = a$. Since Z^a is central in G', and $Z^aK/K \cong Z^a$, G'/N' is compact, where $N' = N/N_0$. Consequently, G/N is compact.

Conversely, assume that G/N is compact. Then G'/N' is compact. Now let $N' \cong Z^a \times Q$, where Q is a compact group. Then $(G'/Q)/((Z^a \times Q)/Q)$ is compact which implies that $\mathrm{ci}(G') = a$. Hence, $\mathrm{ti}(N) = \mathrm{ci}(G)$. □

Theorem 3.2. *Let G be a connected locally compact group. Let F and H be normal subgroups of G with $F \supset H$ and G/H a Lie group. Then F/H is compact if and only if F and H have the same total index.*

Proof. Note first that, since G/H is a Lie group, F/H is a Lie group and, hence, F_0H is open in F [**4**, Theorem 1.3].

Assume now that F/H is compact. Then F_0H/H is compact, hence $F_0/F_0 \cap H$ is compact since F_0H is open. Thus, $\mathrm{ti}(F_0 \cap H) = \mathrm{ci}(F_0)$. Let $H_1 = F_0 \cap H$. Since $F/F_0 \supset HF_0/F_0$ and $F/F_0/HF_0/F_0 \cong F/HF_0$ is compact, we have $\mathrm{di}(F/F_0) = \mathrm{di}(HF_0/F_0)$. Hence,

$$\begin{aligned}
\mathrm{ti}(F) &= \mathrm{ci}(F_0) + \mathrm{di}(F/F_0)\\
&= \mathrm{ci}(F_0) + \mathrm{di}(HF_0/F_0)\\
&= \mathrm{ti}(H_1) + \mathrm{di}(HF_0/F_0)\\
&= \mathrm{ti}(H_1) + \mathrm{di}(H/H \cap F_0)\\
&= \mathrm{ci}((H_1)_0) + \mathrm{di}(H_1/(H_1)_0)\\
&\quad + \mathrm{di}(H/H_1)\\
&= \mathrm{ci}(H_0) + \mathrm{di}(H_1/(H_1)_0)\\
&\quad + \mathrm{di}(H/H_1)\\
&= \mathrm{ci}(H_0) + \mathrm{di}(H/H_0)\\
&= \mathrm{ti}(H).
\end{aligned}$$

Conversely, we now assume that $\mathrm{ti}(H) = \mathrm{ti}(F)$. Since

$$\begin{aligned}
\mathrm{ti}(F) &= \mathrm{ci}(F_0) + \mathrm{di}(F/F_0)\\
&= \mathrm{ci}(F_0/H_0) + \mathrm{ci}(H_0) + \mathrm{di}(F/F_0)\\
&= \mathrm{ci}(H_0) + \mathrm{ci}(F_0/H_0) + \mathrm{di}(F/F_0),
\end{aligned}$$

and $\mathrm{ti}(H) = \mathrm{ci}(H_0) + \mathrm{di}(H/H_0)$, then $\mathrm{ci}(F_0/H_0) + \mathrm{di}(F/F_0) = \mathrm{di}(H/H_0)$. Therefore,

$$\begin{aligned}\mathrm{ti}(F/H_0) &= \mathrm{ci}((F/H_0)_0) + \mathrm{di}(F/H_0/(F/H_0)_0)\\ &= \mathrm{ci}(F_0/H_0) + \mathrm{di}(F/F_0)\\ &= \mathrm{di}(H/H_0)\\ &= \mathrm{di}(H \cap F_0/H_0) + \mathrm{di}(HF_0/F_0).\end{aligned}$$

Since $F_0/H_0 \supset H \cap F_0/H_0$, $\mathrm{ci}(F_0/H_0) \geq \mathrm{di}(H \cap F_0/H_0)$. If $\mathrm{ci}(F_0/H_0) > \mathrm{di}(H\cap F_0/H_0)$, then $\mathrm{di}(F/F_0) < \mathrm{di}(HF_0/F_0) = \mathrm{di}(H/H \cap F_0)$. This contradiction implies that $\mathrm{ci}(F_0/H_0) = \mathrm{di}(H \cap F_0/H_0)$. Hence, F/H is compact. □

4. Related results. We consider various locally compact groups involving connectedness in this section. The results are closely related to those in [**6**]. The reader is referred to [**6**] for definitions of classes of locally compact groups not defined here. A counter-example to a conjecture made in [**6**] is given at the end of this section.

Let G be a locally compact group and let $\mathcal{B}$ be a group of automorphisms of G, where $\mathcal{B} \supset \mathrm{In}(G)$, the group of inner automorphisms of G. Suppose K is a compact normal subgroup of G which is $\mathcal{B}$-invariant. Then we have the exact sequence

$$K \longrightarrow G \longrightarrow H = G/K.$$

If H is connected, we have $G = (C_G(K))_0 K$. It is easy to see that $(C_G(K))_0$ is $\mathcal{B}$-invariant. Hence, $K \cap (C_G(K))_0$ is $\mathcal{B}$-invariant, and we have the sequence

$$K_1 \longrightarrow (C_G(K))_0 \longrightarrow H,$$

where

$$K_1 = K \cap (C_G(K))_0.$$

Theorem 4.1. *Let G be a locally compact group and K be a compact normal subgroup of G. Let $\mathcal{B}$ be a group of automorphisms of G such that*

(1) $\mathcal{B} \supset \mathrm{In}(G)$

(2) *$\mathcal{B}$ is relatively compact in the Birkhoff topology, and*

(3) *K is $\mathcal{B}$-invariant.*

Suppose that G/K is connected and pointwise fixed by the induced group of automorphisms $\tilde{\mathcal{B}}$ on G/K. Then $G = KL$, where

$$L = \{x \in G : \theta(x) = x \text{ for } \theta \in \mathcal{B}\}.$$

Proof. Note first that $\bar{\mathcal{B}}$ also satisfies (1), (2), and (3); hence, we may, in fact, assume that $\mathcal{B}$ is compact. Now $G \in [\mathrm{SIN}]_{\mathcal{B}}$. Since G/K is connected and $\tilde{\mathcal{B}}$ acts trivially on G/K, we claim that $G = K(C_G(K))_0$. To see this, let $\rho : G \to \mathrm{Aut}\,(K)$ be the canonical map. Then $\rho(K) = \mathrm{In}(K)$. Since $\mathrm{Aut}\,(K)/\mathrm{In}(K)$ is totally disconnected and G/K is connected, we see that $\rho(G) \subset \rho(K) = \mathrm{In}(K)$. Hence, $G = KC_G(K)$. Now G is σ-compact since G/K is connected; hence, $C(K)$ is σ-compact, and the canonical map $C_G(K)/C_G(K) \cap K \to G/K$ is a homeomorphism. Since $C_G(K) \cap K \subset (C_G(K) \cap K)(C_G(K))_0 \subset C_G(K)$, $C_G(K)/(C_G(K) \cap K)(C_G(K))_0$ is totally disconnected, and G/K is connected, we have $(C_G(K) \cap K)(C_G(K))_0 = C_G(K)$, i.e., $G = K(C_G(K))_0$ as claimed.

Since K is $\mathcal{B}$-invariant, $C(K)$ is $\mathcal{B}$-invariant, and so is $C(K)_0$. Hence, we may assume from now on that $G = C(K)_0$, K is central in G, and G/K is connected and abelian.

Since $G \in [\mathrm{SIN}]_{\mathcal{B}}$, let M be any small $\mathcal{B}$-invariant compact normal subgroup of G such that $G/M = G'$ is a Lie group. If $K' = MK/M$, we have the exact sequence

$$(K')_0 \longrightarrow K' \longrightarrow G' \longrightarrow G'/K'.$$

For each $\theta \in \mathcal{B}$, $d\theta$ acts on the Lie algebra $L(G')$ of G', hence $\mathcal{B}$ acts on $L(G')$ by the differentials $d\theta$, $\theta \in \mathcal{B}$. Since $\mathcal{B}$ is compact there is a nondegenerate positive definite quadratic form β on $L(G')$ which is $\mathcal{B}$-invariant. Let $\mathcal{S}$ be the orthogonal complement of $L(K')$ in $L(G')$. If $\pi : G' \to G'/K'$ denotes the natural map, $d\pi : L(G') \to L(G'/K')$ induces $d\pi : \mathcal{S} \to L(G'/K')$ which is one-to-one and onto.

Since $\mathcal{B}$ acts on $L(G'/K')$ trivially, $d\pi$ acts on $\mathcal{S}$ trivially. If $S = \exp \mathcal{S}$, then $K'S$ is open in G', and, since G'/K' is connected, $K'S = G'$.

This shows that for each small $\mathcal{B}$-invariant compact normal subgroup M of G such that G/M is a Lie group there is a normal subgroup S_M of $G'_M = G/M$ such that $G'_M = K'_M S_M$, where $K'_M = MK/M$. Now $G = \varprojlim G'_M = \varprojlim K'_M S_M$, $K = \varprojlim K'_M$, and $G = KL$, where $L = \varprojlim S_M$. It is clear that $L = \{x \in G : \theta(x) = x \text{ for } \theta \in \mathcal{B}\}$. This completes the proof. □

Definition 4.1. Let G be a locally compact group. Then G is said to have property C or $G \in [C]$ if for every nontrivial characteristic subgroup H of G, $C_G(H)$ is compact.

Lemma 4.2. *If $G \in [C]$, then G_0 is compact.*

Proof. Let Q be a maximal compact normal subgroup of G_0. Then Q is a characteristic subgroup of G and $G_0 = QC_{G_0}(Q)$. Now $C_{G_0}(Q) \subset C_G(Q)$ and $C_G(Q)$ is compact. It follows that G_0 is compact. □

The following theorem resembles Theorem 4.2 of [**4**].

Theorem 4.3. *Let G be a locally compact group which has the property that for any compact nontrivial subgroup M, $C_G(M)$ is compact. If $G \in [C]$ and $G_0 \in [\mathrm{SIN}]_G$, then G is either compact or totally disconnected.*

Proof. First we note that $G_0 \in [\mathrm{SIN}]_G$ if and only if the canonical map $\rho : G \to \mathrm{Aut}\,(G_0)$ has relatively compact image. We shall call G_0 stable if $G_0 \in [\mathrm{SIN}]_G$.

Since $G \in [C]$, G_0 is compact by Lemma 4.2, so we may write $G_0 = AS$, where A is a compact connected abelian group and S is a product of compact simple Lie groups. Both A and S are characteristic subgroups of G. Since G_0 is stable, both A and S are stable in G. Note also that either A or S cannot be trivial unless G_0 is trivial. Suppose now that G is not totally disconnected. Then G_0 is not trivial. We consider two cases.

Suppose that A is not trivial. There exists a compact normal subgroup N of G such that $A/N \cong T$, an n-dimensional torus. Then $\mathrm{Aut}\,(A/N)$ is discrete. Since the group of automorphisms $\tilde{\mathcal{B}}$ induced on A/N by $\mathrm{Int}\,(G)$ is relatively compact, $\tilde{\mathcal{B}}$ is finite. Hence, there exists a closed normal subgroup G_1 of G such that G/G_1 is finite, and $\tilde{\mathcal{B}}$ leaves every point in A/N fixed. Hence, by Theorem 4.1, $A = NL$, where

$$L = \{x \in A : gxg^{-1} = x \text{ for each } g \in G_1\}.$$

Suppose now that S is not trivial. By the stability of S, there exists a compact normal subgroup N of S such that $S/N = J$ is a compact semisimple group. Hence $\mathrm{Aut}\,(J)$ is compact, and there exists a closed normal subgroup G_1 of G such that G/G_1 is compact, and, again, we have $S = NL$ where $L = \{x \in S : gxg^{-1} = x \text{ for each } g \in G_1\}$.

Note that, in either case, L is compact. Hence, $C_G(L)$ is compact by assumption. Since $C_G(L) \supset G_1$, G_1 is compact, and, consequently, G is compact.

The following conjecture appears in [**6**, p. 122]:

If $G \in [\mathrm{SIN}] \cap [AF]^- \cap [LF]^-$, then G is either compact or totally disconnected. The following example seems to provide a counterexample to this conjecture.

Example 4.1. Let a_i, $c_{j,k}$, where $j \neq k$, $i, j, k = 1, 2, 3, \ldots$, be distinct elements subject to the following conditions: $a_i^2 = c_{i,j}^2 = e$, $a_i c_{j,k} = c_{j,k} a_i$, $c_{i,j} = c_{j,i}$, and $a_i a_j = a_j a_i c_{i,j}$ for each i, j, k. We shall embed each $c_{i,j}$ in the circle group $T_{i,j}$. Let $A_k = \langle a_k \rangle$ be the discrete group generated by a_k, and let $C = \prod_{i \neq j} T_{i,j}$ be the product group with the product topology.

Let $G = \{c a_{i_1} a_{i_2} \cdots a_{i_m} : i_1 < i_2 < \cdots < i_m,\ m \text{ a positive integer, and } c \in C\}$, and let the group operation in G be induced by the product

$$(c_1 a_i)(c_2 a_j) = \begin{cases} c_1 c_2 a_i a_j, & i = j \\ c_1 c_2 c_{i,j} a_j a_i, & i \neq j \end{cases}.$$

Denote by ΣA_k the direct sum with the discrete topology and, with obvious identification, endow G with the relative topology of the product space $C \times \Sigma A_k$. Then G is a locally compact group in which C is open and central in G.

For any two elements $x = a_{i_1}a_{i_2}\cdots a_{i_m}$, $i_1 < i_2 < \cdots < i_m$, and $y = a_{j_1}a_{j_2}\cdots a_{j_n}$, $j_1 < j_2 < \cdots < j_n$, in G, we may write

$$(*) \qquad \begin{aligned} xy &= (a_{j_1}a_{i_2}\cdots a_{i_m})(a_{j_1}a_{j_2}\cdots a_{j_n}) \\ &= a_{l_1}a_{l_2}\cdots a_{l_{m+n}}c_1 \end{aligned}$$

by shifting the $a_{j_1}a_{j_2}\cdots$ into the position where $l_1 < l_2 < \cdots < l_{m+n}$ by using the property that $a_\alpha a_\beta = a_\beta a_\alpha c_{\alpha,\beta}$ for $\beta < \alpha$. Similarly, we may write

$$(**) \qquad \begin{aligned} yx &= (a_{j_1}a_{j_2}\cdots a_{j_n})(a_{i_1}a_{i_2}\cdots a_{i_m}) \\ &= a_{l_1}a_{l_2}\cdots a_{l_{m+n}}c_2. \end{aligned}$$

In $(*)$, c_1 is the product of those $c_{i_\alpha}c_{j_\beta}$ where $i_\alpha > j_\beta$, and, in $(**)$, c_2 is the product of those $c_{j_\beta}c_{i_\alpha}$ where $j_\beta > i_\alpha$. For example, $(a_1a_3)(a_2a_5) = a_1a_2a_3a_5c_{3,2}$, while $(a_2a_5)(a_1a_3) = a_1a_2a_3a_5c_{2,1}c_{5,1}c_{5,3}$. Hence, if H is an abelian subgroup of G, every element of H is of the form ca_i for some $c \in C$. Thus, every closed abelian subgroup of G is a closed subset of $C \times (\pi A_k)$, where πA_k has the product topology, and, hence, is compact. Hence, $G \in [\mathrm{AF}]^-$.

Since G contains C as an open central subgroup, $G \in [\mathrm{SIN}]$. It is clear that $G \in [\mathrm{LF}]^-$. Hence, $G \in [\mathrm{SIN}] \cap [\mathrm{AF}]^- \cap [\mathrm{LF}]^-$, but G is neither compact nor totally disconnected. Note that C is a compact normal subgroup, and $G/C \cong \Sigma Z_2$ is an infinite discrete abelian group. Hence, $G/C \notin [\mathrm{AF}]^-$.

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