ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 1, Winter 1992

ON THE GROTHENDIECK AND NIKODYM PROPERTIES OF BOOLEAN ALGEBRAS

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ABSTRACT. In this paper we investigate the Grothendieck and Nikodym properties on a Boolean algebra. We obtain a sufficient condition for the Nikodym property that is not sufficient for the Grothendieck property.

1. Introduction. Schachermayer [**9**] proved that the family J of Jordan-measurable sets in [0,1] has the Nikodym property (NP) but lacks the Grothendieck property (GP). This is the first algebra known with these characteristics and it has been of major importance in the study of the relations between the Grothendieck and Nikodym properties (cf. [**3**]). Graves and Wheeler [**5**] made another important contribution to this subject, the main focus of which is the study of the properties (NP) and (GP) for certain Jordan-type algebras of Baire, Borel and universally measurable sets.

Haydon [**6**] proved that a subsequentially complete Boolean algebra (SC) has the Grothendieck property. He also gives an example of a Boolean algebra with the property (SC) that does not have Rosenthal's property. Hence, he obtains a sufficient condition for Grothendieck's property that is not sufficient for Rosenthal's property.

Dashiell [**2**] proved that an algebra that is up-down-semi-complete (udsc) and that has an additional property, has Rosenthal's and Nikodym's properties. Hence, such algebra has Grothendieck's property. An algebra with that additional property will be called (aD).

Dashiell [**2**] also proved that the Boolean algebra D of the simultaneously G_δ and F_σ sets in [0,1] has the properties (udsc) and (aD). We observe that the algebra J is (udsc) and, hence, the property (aD) gives to D properties that J does not have.

Talagrand [**10**] proves that, assuming the continuum hypothesis, there exists a Boolean algebra with the Grothendieck property that lacks the Nikodym property.

Received by the editors on November 5, 1988 and in revised form on August 14, 1989.

In the relations between the Grothendieck and Nikodym properties, we believe that there are some interesting problems.

To have a condition that is sufficient for (NP) (respectively, (GP)) but not for (GP) (respectively, (NP)), (rather than just an example to show that the one does not imply the other).

To decide if the assertion, "the Grothendieck property does not imply the Nikodym property" holds in all models of set theory, or whether Martin's axiom yields (GP) $\Rightarrow$ (NP).

In this paper we obtain a sufficient condition for (NP) that is not sufficient for (GP), and we investigate the fundamental differences between the algebras J and D.

2. Notations. In this paper F is a Boolean algebra with the operations of sup, inf and c. F has a unique representation as the field of clopen sets of the Stone representation space S of F.

A bounded function $\mu : F \to \mathbf{R}$ is called a measure if it is additive, i.e., $\mu(A_1 \vee A_2) = \mu(A_1) + \mu(A_2)$ whenever A_1, A_2 in F are disjoint.

Kuratowski [**7**] proves that a topological space X can be written as $X = P \cup D$ where P is a perfect set and D is dispersed. This decomposition will be called the perfect-dispersed decomposition of X. P (respectively, D) will be called the perfect-kernel (respectively, dispersed-kernel) of X. It is well known that if S is the Stone-space of a Boolean algebra with the property of Grothendieck, then in S there do not exist nontrivial convergent sequences. This is also valid if S is the Stone-space of a Boolean algebra with the property of Nikodym (cf. [**4,1**]). For these reasons the Boolean algebras that will be considered here are algebras with an infinite number of elements and whose Stone-spaces do not have nontrivial convergent sequences. If $S = P \cup D$ is the Stone space of an algebra with these characteristics, then $P \neq \varnothing$, every point in D is an isolated point and every infinite closed set $T \subset S$ is uncountable.

A point x of a space X is called a P-point if the intersection of countably many neighborhoods of x is again a neighborhood of x. In this paper the set of the P-points of S will be denoted by P_1; it is clear that $P_1 \supset D$.

A point $x \in X$ is not a P-point if and only if x is in the frontier of a cozero set in X. In this paper the set of the non P-points of S will be denoted by P_2; it is clear that $P_2 \subset P$.

3. Main results.

Lemma 3.1. *Let* $(\mu_n)_{n\in\omega}$ *be a sequence in* $C(S)^*$ *that is pointwise bounded* (pb) *on* F.

a) $(\mu_n)_{n\in\omega}$ *is uniformly bounded* (ub) *on* F *if and only if, for every* $x \in P$, *there exists an* $A \in F$ *with* $x \in A$ *such that the sequence* $(|\mu_n|(A))_{n\in\omega}$ *is bounded.*

b) $(\mu_n)_{n\in\omega}$ *is uniformly strongly additive* (usa) *on* F *if and only if, for every* $x \in P$, *there exists an* $A \in F$ *with* $x \in A$ *such that* $(\mu_n)_{n\in\omega}$ *is* usa *in* F_A.

Proof. a) The necessity is evident. For every $x \in P$, let $A_x \in F$ be such that the sequence $(|\mu_n|(A_x))_{n\in\omega}$ is bounded. Since

$$S = \left(\bigcup_{x\in P} A_x\right) \cup \left(\bigcup_{x\in D} \{x\}\right),$$

we can write $S = B_1 \cup \ldots \cup B_n$ where every B_i is either a set A_x or a unitary clopen; hence, $(|\mu_n|(B_i))_{n\in\omega}$ is a bounded sequence, for every $i = 1, 2, \ldots, n$. Since, for $i \in \omega$, $||\mu_i|| = |\mu_i|(s) \leq |\mu_i|(B_1) + \cdots + |\mu_i|(B_n)$, it is clear that $(||\mu_i||)_{i\in\omega}$ is a bounded sequence.

b) The necessity is also evident. We can write, as in a), $S = B_1 \cup \ldots \cup B_n$, where, for $i \in \{1, 2, \ldots, n\}$, $(\mu_k)_{k\in\omega}$ is usa in F_{B_i}. We can write $S = C_1 \cup \ldots \cup C_n$ where the C_i are mutually disjoint and $(\mu_k)_{k\in\omega}$ is usa in F_{C_i}.

Let $(A_i)_{i\in\omega}$ be a disjoint sequence in F. For $j = 1, 2, \ldots, n$ and $i \in \omega$, we denote $A_i^j = A_i \cap C_j$. We have that, for every $\varepsilon > 0$ and $j = 1, 2, \ldots, n$, there exists an $i_j \in \omega$ such that if $i > i_j$ then $|\mu_m(A_i^j)| < \varepsilon/n$, for every $m \in \omega$.

If we denote $i_0 = \max(i_1 \ldots i_n)$, we have $|\mu_m(A_i)| \leq |\mu_m(A_i^1)| + \cdots + |\mu_m(A_i^n)| < \varepsilon$, for every $m \in \omega$ and $i \geq i_j$. Hence, $\lim_{i\to\infty} |\mu_m(A_i)| = 0$ uniformly in $m \in \omega$. □

Remark 3.2. It can be proved easily that there exist in S an uncountable number of points that are not P-points.

If $T \subset S$ is an infinite closed set and $x \in T$ is not a P-point, for the relative topology in T, then x is not a P-point for the topology of S. The converse is false. In the Stone space $\beta\omega$ of $P(\omega)$, we have that every point of $\omega^* = \beta\omega/\omega$ (the perfect-kernel of $\beta\omega$) is not a P-point in the topology of $\beta\omega$. Nevertheless, assuming CH, in ω^*, with its relative topology, there exist points that are P-points (cf. [**8**, Corollary 1.7.2]).

A closed set $T \subset S$ whose points are P-points is necessarily finite.

If $\mu \in C(S)^*$ and if x is a P-point in S such that $x \in \operatorname{car}\mu$, then it is also a P-point in $\operatorname{car}\mu$ (endowed with its relative topology). Since $\operatorname{car}\mu$ is a zero-dimensional Hausdorff compact space with the property (ccc), there are no P-points in its perfect kernel. Hence, $\{x\}$ is an isolated point in $\operatorname{car}\mu$ and there exists an $A \in F$ such that $x \in A$ and $A \cap \operatorname{car}\mu = \{x\}$. This proves that $|\mu|(A\backslash\{x\}) = 0$.

Let us suppose that $(\mu_n)_{n\in\omega}$ is a pointwise bounded sequence of measures in F and that x is a P-point in S. For $A \in F$, let $\mu(A) = \sum_1^\infty (1/2^n)(|\mu|(A)/||\mu_n||)$. There exists an $A \in F$ such that $x \in A$ and $|\mu_n|(A\backslash\{x\}) = 0$ for every $n \in \omega$. Hence, in $F_{\mathbf{A}}$, we have either μ_n is a multiple of δ_x or $\mu_n = 0$, for every $n \in \omega$, and $(\mu_n)_{n\in\omega}$ is uB and uSA in $F_{\mathbf{A}}$. This proves that Lemma 3.1 can be improved in the following form.

Corollary 3.3. *Let $(\mu_n)_{n\in\omega}$ be a pointwise bounded sequence of measures in $C(S)^*$.*

a) *$(\mu_n)_{n\in\omega}$ is* ub *in F if and only if for every $x \in P_2$ there exists an $A \in F$ with $x \in A$ and such that $(|\mu_n|(A))_{n\in\omega}$ is a bounded sequence.*

b) *$(\mu_n)_{n\in\omega}$ is* usa *in F if and only if for every $x \in P_2$ there exists an $A \in F$ with $x \in A$ and such that $(\mu_n)_{n\in\omega}$ is* usa *in $F_{\mathbf{A}}$.*

The following result, whose proof is an immediate consequence of Corollary 3.3, characterizes the Nikodym, Grothendieck and Vitali-Hahn-Saks properties through the points that are not P-points of the perfect kernel of S.

Corollary 3.4. *A Boolean algebra F does not have the Vitali-Hahn-Saks (respectively, Grothendieck's, respectively Nikodym) property if and only if there exists a sequence $(\mu_n)_{n\in\omega}$ in $C(S)^*$ pointwise convergent to zero in F (respectively, bounded and pointwise convergent to zero; respectively, pointwise bounded in F) and there exists an $x \in P_2$ such that for every neighborhood $A \in F$ of x we have that $(\mu_n)_{n\in\omega}$ is not* usa *in $F_{\mathbf{A}}$ (respectively, $(\mu_n)_{n\in\omega}$ is not* usa *in $F_{\mathbf{A}}$; respectively, $(|\mu_n|(A))_{n\in\omega}$ is not bounded).*

In order to obtain a sufficient condition for (NP) that is not sufficient for (GP), we need some definitions and results.

Definition 3.5. We shall say that a Boolean algebra F has the additional property of Dashiell (aD) if and only if every positive measure $\mu \in C(S)^*$ whose carrier $\operatorname{car}(\mu)$ is contained in the frontier of a cozero set of S is not σ-additive.

The property (aD) will now be decomposed into two weaker properties.

Definition 3.6. A Boolean algebra F is said to be (aD_1) (respectively, (aD_2)) if and only if for every positive measure $\mu \in C(S)^*$ that is not atomless (respectively, atomless) and whose carrier is contained in the frontier of cozero set in S is not σ-additive.

Proposition 3.7. *The following conditions on a Boolean algebra F are equivalent:*

a) *F has the property (aD_1).*

b) *For every $x \in P_2$ there exists a decreasing sequence $(B_n)_{n\in\omega}$ in F such that $x \in B_i$ for $i \in \omega$ and $\wedge_{i\in\omega} B_i = \overline{\overset{\circ}{\cap B_i}} = \varnothing$.*

c) *For every $x \in P_2$ there exists a disjoint sequence $(A_i)_{i\in\omega}$ in F such that $\overline{\cup_{i\in\omega} A_i} = A \in F$, $x \notin A_i$ for $i \in \omega$ and $x \in A$.*

d) *$x \in P_2$ if and only if δx is not σ-additive.*

Proof. a) $\Rightarrow$ b). If $x \in P_2$, the measure δ_x is not atomless and its carrier $\{x\}$ is contained in the frontier of a cozero set in S. Hence, δ_x

is not σ-additive and there exists a decreasing sequence $(B_n)_{n\in\omega}$ such that $\wedge_{i\in\omega} B_i = \varnothing$ and $\lim_{i\to\infty} \delta_x(B_i) \neq 0$; so $x \in B_i$ for every $i \in \omega$.

The proof of b) $\Rightarrow$ c) is evident by taking complements.

c) $\Rightarrow$ a). Let $\mu \in C(S)^*$ be a nonatomless positive measure whose carrier is contained in the frontier of a cozero set in S. There exists an $x \in \mathrm{car}\,(\mu)$ such that $\mu(x) > 0$; this point x is not a P-point. Hence, there exists a disjoint sequence $(A_i)_{i\in\omega}$ in F such that $x \notin A_i$, for $i \in \omega$, and $x \in \vee_{i\in\omega} A_i = A \in F$. Since $\mu(A \setminus \cup_{i\in\omega} A_i) \geq \mu(x) > 0$ the measure μ is not σ-additive. The equivalence of c) and d) is evident. □

Theorem 3.8. *If a Boolean algebra F has the properties* (udsc) *and* (aD$_1$)*, then F has the Nikodym property.*

Proof. If F does not have the property (NP), there exists a sequence $(\mu_n)_{n\in\omega}$ in $C(S)^*$ that is pb in F but is not ub in F. Corollary 3.3 implies that there exists an $x \in P_2$ such that for every neighborhood $A \in F$ of x, the sequence $(|\mu_n|(A))_{n\in\omega}$ is not bounded. Since $x \in P_2$, there exists a decreasing sequence $(B_i)_{i\in\omega}$ such that $x \in B_i$ for every $i \in \omega$ and $\overset{\circ}{\overline{\cap B_i}} = \varnothing$. It is well known that when a sequence of measures $(\mu_i)_{i\in\omega}$ is pointwise bounded in a Boolean algebra but is not uniformly bounded, there exists, for every $p > 0$, a partition (E_p, F_p) of the unitary element of the algebra and an $n_p \in \omega$ such that

$$|\mu_{n_p}(E_p)| > p \text{ and } |\mu_{n_p}(F_p)| > p.$$

For $p = 1$, the algebra F_{B_1} and the sequence $(\mu_n)_{n\in\omega}$, there exists a partition (E_1, F_1) of B_1 and an $n_1 \in \omega$ such that $|\mu_{n_1}(E_1)| > 1$ and $|\mu_{n_1}(F_1)| > 1$.

Let us suppose that $x \in E_1$. If we write $A_1 = F_1$, we have that $x \in E_1 \cap B_2$.

For $p = 2$, the algebra $F_{E_1 \cap E_2}$ and the sequence $\{\mu_n, n > n_1\}$, there exists a partition (E_2, F_2) of $E_1 \cap B_2$ and $n_2 > n_1$ such that $|\mu_{n_2}(E_2)| > 2$ and $|\mu_{n_2}(F_2)| > 2$.

It can be obtained inductively that a disjoint sequence $(A_i)_{i\in\omega}$ and a subsequence $(\mu_{n_i})_{i\in\omega}$ of $(\mu_n)_{n\in\omega}$, that will be denoted as $(\mu_n)_{n\in\omega}$, such that, for every $i \in \omega$:

$$(1) \qquad A_i \subset B_i \quad \text{and} \quad |\mu_i(A_i)| > i.$$

Since $\overline{\overset{\circ}{\cap B_i}} = \varnothing$ and F is (udsc), then $\overline{\cup_{i\in\omega} A_i} \in F$ (cf. [**2**, Theorem 1.8]) and for every $N \subset \omega$ is also $\overline{\cup_{i\in N} A_i} \in F$. Hence, the σ-algebra Σ generated by $(A_i)_{i\in\omega}$ is contained in F and the sequence $(\mu_n)_{n\in\omega}$ is pointwise bounded in Σ. Since the σ-algebras have the Nikodym property, $(\mu_n)_{n\in\omega}$ is uniformly bounded in Σ. This contradicts (1). □

Theorem 3.9. *The algebra J of the Jordan-measurable sets of* $[0,1]$ *has the property* (aD_1).

Proof. Let $x \in P$, where P is the perfect kernel. We have $[0,1] = [0,1/2] \cup [1/2,1]$.

If $[0,1/2] \in x$, we put $I_1 = [0,1/2]$; if $[1/2,1] \in x$, we put $I_1 = [1/2,1]$. We have $I_1 = I_1^1 \cup I_1^2$ where I_1^1 and I_1^2 are closed intervals of length $1/2^2$. We put $I_2 = I_1^1$ if $I_1^1 \in x$ and $I_2 = I_1^2$ if $I_1^2 \in x$. We obtain inductively a decreasing sequence $(I_k)_{k\in\omega}$ of closed intervals such that $I_n \in x$ for every $n \in \omega$ and $\operatorname{diam}(I_n) = (1/2^n)$. Hence, there exists a point $t \in [0,1]$ such that $\{t\} = \cap_{n\in\omega} I_n$. Nevertheless, $\{t\} \notin x$ because x is not a principal filter. □

As a consequence of the proof of Theorem 3.9, we have that in the perfect-kernel of the Stone Space of J, there are no P-points.

Remark 3.10. Schachermayer [**9**] proved that J has the property (udsc) but lacks the Grothendieck property. Since J has (aD_1), we have that the property $(\mathrm{udsc}) + (\mathrm{aD}_1)$ is sufficient for the Nikodym property but is not sufficient for the Grothendieck property. Finally, we observe that J does not have the property (aD_2) because, to the contrary, J would have the property of Grothendieck. Hence, the fundamental difference between the algebra J and the algebra D of the simultaneous G_δ and F_σ subsets of $[0,1]$ is that J does not have (aD_2), while D has (aD_2).

We saw in Section 3 that several properties on a Boolean algebra F can be characterized by means of the perfect kernel P endowed with the topology of S, but not with the relative topology. If we consider P with its relative topology, we cannot obtain the same results. This will be clear through the following examples.

Example 3.11. Let F be the algebra of subsets of ω consisting of the sets $A \subset \omega$ such that for all but finitely many k, the pair $\{2k-1, 2k\}$ is either in A or in the complement (cf. [**9**, ex. 4.10]). F lacks the properties of Grothendieck, Nikodym, Vitali-Hahn-Saks and Rosenthal. Nevertheless, the perfect kernel P of F is homeomorphic to $\omega^* = \beta\omega/\omega$, which is the Stone space of a Boolean algebra with those four properties.

Example 3.12. Let $\Omega = \{(n,m) \in \omega \times \omega, n \geq m\}$. For every $i \in \omega$, let $A_i = \{(i,n) \in \omega \times \omega, n \leq i\}$. For every $A \subset \Omega$, we write $d(A) = \lim_{n\to\infty}(1/n)|A \cap A_n|$ if that limit exists there. On the contrary, we write $d(A) = \infty$. $d(A)$ will be called the density of A. Let $F = \{A \in \Omega : d(A) = 0 \text{ or } d(A) = 1\}$. Freniche [**4**] proved that F is a Boolean algebra that lacks the Grothendieck property. For every $A \in F$ such that $d(A) = 0$, $F_{\mathbf{A}} = 2^{\mathbf{A}}$ is complete and that the Stone space of F has no nontrivial convergent sequences, let $z = \{A \in \Omega, d(A) = 1\}$. It is clear that z is a nonprincipal ultrafilter in F. Hence, z is in the perfect kernel P of F and $d(A) = \delta_z(A)$ for $A \in F$. Let $x \neq z$ be a maximal filter on F. There exists an $A \in z$ such that $A \notin x$. Since $d(A) = 1$, $d(A^c) = 0$ and $A^c \in x$. Hence, for every $x \in P$, $x \neq z$, there exists a clopen neighborhood A of x such that $F_{\mathbf{A}}$ is a complete algebra. If the point z had the same property, the results of section 3 would tell us that F has the Grothendieck property. Hence, for every clopen neighborhood A of z, $F_{\mathbf{A}}$ does not have Grothendieck's property.

The next theorem tells us that every Boolean algebra with similar characteristics to the one showed on Example 4.2 and that has a weak separation property on the sequences of disjoint sets has the Vitali-Hahn-Saks property.

Theorem 3.13. *Let F be a Boolean algebra such that for every disjoint sequence $(A_i)_{i\in\omega}$ in F there exist two disjoint infinite sets*

$N_1, N_2 \subset \omega$ such that, if $i \in N_1$, then $A_i \subset A$ and if $i \in N_2$, then $A_i \cap A = \varnothing$, for some $A \in \mathcal{F}$. If for every $x \in P$, except a discrete set $M \subset P$, there exists a neighborhood $A \in F$ of x such that $F_{\mathbf{A}}$ has the property of Vitali-Hahn-Saks (respectively, Nikodym, respectively, Grothendieck) then F has the property of Vitali-Hahn-Saks (respectively, Nikodym, respectively, Grothendieck).

Proof. If F does not have the property of Vitali-Hahn-Saks, then there exists a sequence $(\mu_n)_{n\in\omega}$ in $C(S)^*$ pointwise convergent to zero, an $\varepsilon > 0$ and a disjoint sequence $(A_i)_{i\in\omega}$ in F such that $|\mu_i(A_i)| > \varepsilon$ for every $i \in \omega$. Let $A \in F$ and two disjoint infinite sets $N_1, N_2 \subset \omega$ be such that $A_i \subset A$ if $i \in N_1$ and $A_i \cap A = \varnothing$ if $i \in N_2$. We consider two cases.

a) Let $M = \{z\}$. If $z \in A$ we write $B = A^c$ and $N = N_2$. If $z \in A^c$ we write $B = A$ and $N = N_1$. It is clear that $(\mu_i)_{i\in N}$ is a sequence that is pointwise convergent to zero in $F_{\mathbf{B}}$. $(A_i)_{i\in N}$ is a disjoint sequence in $F_{\mathbf{B}}$ and

$$|\mu_i(A_i)| > \varepsilon \tag{2}$$

for every $i \in N$. For every $x \in B \cap P$, there exists a clopen neighborhood $c \in F$, $c \subset B$, such that F_c has the Vitali-Hahn-Saks property. Hence, $F_{\mathbf{B}}$ is Vitali-Hahn-Saks, which is in contradiction to (2).

b) If M is a discrete set, with an arbitrary cardinal number. we have that for every $x \in P$ there exists a neighborhood $A \in F$ of x such that $A \cap M$ is either empty or unitary. Hence, a) shows that $F_{\mathbf{A}}$ and F have the Vitali-Hahn-Saks property.

In a similar way, the theorem can be proved for the properties of Nikodym and Grothendieck. □

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