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UNBOUNDED MULTIVALUED NEMYTSKII OPERATORS IN SOBOLEV SPACES AND THEIR APPLICATIONS TO DISCONTINUOUS NONLINEARITY

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Introduction. The aim of this paper is to provide tools for solving boundary value problems for partial differential inclusions of the type

$$\mathrm{L}u \in F(x, u, Du, \dots), \qquad x \in \Omega,$$

where the order of a linear operator L exceeds the order of a Carathéodory multifunction F with convex values in R^M. Such inclusions may be used for dealing with partial differential equations with discontinuous nonlinearities, as we illustrate by an example at the end of this paper.

In all publications on that topic known to the author, F was assumed as either bounded or growing at most linearly in function variables (cf. [**4, 5, 13, 15**]). However, if $\dim \Omega = 1$, such assumptions can be relaxed, as shown, e.g., in [**9**] or [**11**]. We establish here results (Theorem 1 and Corollary 1) showing that the linear growth need not be assumed with respect to those partial derivatives of u which, due to the Sobolev imbedding theorem, imbed to C.

The next presented results (Lemma 2 and Theorem 3) show how solutions of differential inclusions can be approximated by solutions of equations

$$\mathrm{L}u = f_\varepsilon(x, u, Du, \dots), \qquad x \in \Omega,$$

where f_ε is a Carathéodory function whose graph approximates the graph of F. The consequences of Theorem 3 are twofold; on one hand, it provides a theoretical interpretation of a common practice of ignoring continuity assumptions while using numerical methods in differential equations. On the other hand, we hope to use Theorem 3 for directly deriving existence results for partial differential inclusions from the corresponding results for partial differential equations without repeating the same arguments for multivalued operators.

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As an example, we generalize a result due to Brézis and Turner [**2**] concerning a nonlinear elliptic Dirichlet boundary value problem, to the case of an equation with a discontinuous nonlinear right-hand side.

Main results. The class of all nonempty closed convex subsets of a normed space X is denoted by CX. Given a bounded subset $B \subset X$, we use the notation $||B|| := \sup\{||u|| \, |u \in B\}$. If $X = R^N$, with the Euclidean norm, we shall use the modulus notation $|B|$, $|u|$, instead. By R^+ we mean the interval $[0, \infty)$.

Let us recall that a multifunction $F : \Omega \times R^K \to CR^M$, where Ω is a domain in R^N, is called a *Carathéodory multifunction* if it satisfies the following two conditions:

C1. For each $u \in R^K$, the multifunction $F(\cdot, u)$ is measurable;

C2. For a.e. $x \in \Omega$, the map $F(x, \cdot)$ is upper semicontinuous.

By measurable functions we mean those Lebesgue measurable (for the definition and properties of measurable multifunctions, see [**3**]). It is known (cf. [**14**]) that C1 and C2 together imply the following

C1′. For each measurable $u : \Omega \to R^K$, the map $F(\cdot, u(\cdot))$ has measurable single-valued selections.

In fact, it is C1′ with C2 which is used in applications. To any Carathéodory multifunction F we assign the *multivalued Nemytskii operator* $F^* : X \to CY$ defined by

$$F^*u(x) := \{v \in Y | v(x) \in F(x, u(x)), \text{ a.e. } x\}, \qquad u \in X,$$

where X and Y are normed function spaces, to be decided, containing functions $u : \Omega \to R^K$, $v : \Omega \to R^M$, respectively.

A mapping $F : X \to CY$ is called *weakly compact* if it sends bounded sets to relatively weakly sequentially compacts sets. F is called *demi-continuous* if, for each pair of sequences $u_n \to u \in X$ (norm convergence) and $v_n \rightharpoonup v \in Y$ (weak convergence) with $v_n \in Fu_n$, $n = 1, 2, \dots$, we have $v \in Fu$.

Lemma 1. *If $F : X \to CY$ is a weakly compact demi-continuous map and $T : Y \to X$ is a compact linear operator, then the composition $TF : X \to CX$ is upper semicontinuous and compact.*

Proof. Since TF sends bounded sets to relatively compact sets, we must only show that the graph of TF is closed in $X \times Y$. For, let $u_n \to u$ and $Tv_n \to w$, where $v_n \in Fu_n$, $n = 1, 2, \dots$. We need to show that $w \in TFu$. Since $\{v_n\} \subset F(\{u_n\})$ is relatively weakly compact, there is a subsequence $v_{n_k} \rightharpoonup v$. Hence, $v \in Fu$. Any continuous linear operator is weakly continuous, so $Tv_{n_k} \to Tv$. Consequently, $w = Tv \in TFu$. $\square$

Theorem 1. *Let Ω be a bounded domain in R^N, let $p \geq 1$, $q > 1$, and let $F : \Omega \times R^{K+L} \to CR^M$ be a Carathéodory multifunction of variables $x \in \Omega$ and $u = (y, z) \in R^K \times R^L$ satisfying the following condition. For any bounded set $B \subset R^K$, there is a function $\varphi_B \in L^q(\Omega, R^+)$ and a constant $c > 0$ such that*

$$|F(x, y, z)| \leq \varphi_B(x) + c|z|^{\frac{p}{q}} \tag{1}$$

for all $y \in B$, $z \in R^L$, and a.e. $x \in \Omega$. Then the corresponding Nemytskii operator $F^* : C(\overline{\Omega}, R^K) \times L^p(\Omega, R^L) \to CL^q(\Omega, R^M)$ is well defined, weakly compact and demi-continuous. Moreover, we may allow $q = 1$ if we assume $c = 0$.

Proof. It is clear from (1) that F^* is well defined with nonempty closed convex values and that it sends bounded sets to bounded sets. Consequently, F^* is weakly compact; indeed, this is clear if $q > 1$ since then the space L^q is reflexive. In the case $q = 1$, $c = 0$, the conclusion follows from the Dunford-Pettis theorem [**7**, Corollary IV.8.11] and from (1). We now show that F^* is demi-continuous. Let u_n and u be functions from Ω to R^{K+L}, v_n and v from Ω to R^M, $||u_n - u||_{C \times L^p} \to 0$, $v_n \rightharpoonup v$ (weakly in L^q) and $v_n \in F^* u_n$ for $n = 1, 2, \dots$. We need to show that $v \in F^* u$. By the Mazur theorem [**8**], for any $k = 1, 2, \dots$, there is a sequence $\{w_n^k\} \subset \omega\{v_k, v_{k+1}, \dots\}$ which is norm-convergent to v, as $n \to \infty$. The convergence in C implies pointwise convergence and, since Ω is bounded, the convergence in L^p or L^q norm implies the convergence almost everywhere of a subsequence. By passing to subsequences, we may assume that $u_n(x) \to u(x)$ and $w_n^k(x) \to v(x)$ for all k and a.e. x.

Let A be the set of those x for which the above sequences converge, $F(x, \cdot)$ is u.s.c., and $v_n(x) \in F(x, u_n(x))$ for all n. It follows that the complement of A in Ω has measure zero. Now let $x \in A$. Then

$w_n^k(x) \to v(x)$, for all k, and $u_n(x) \to u(x)$. Since $F(x, \cdot)$ is u.s.c., therefore, for each $\varepsilon > 0$, there is m with

$$v_n(x) \in C_\varepsilon := F(x, u(x)) + \overline{B}_\varepsilon$$

for all $n \geq m$, where $\overline{B}_\varepsilon$ is the closed ε-ball about the origin. Since the set C_ε is closed and convex, $w_n^m(x)$ belongs to it for all n and, consequently, $v(x) \in C_\varepsilon$. That holds for all ε and all $x \in A$; therefore, $v \in F^* u$, and the proof is complete. $\square$

Now let Ω be a bounded domain in R^N with a smooth boundary, let $p \geq 1$, and let integers k, n, N be related by

$$k := \left[n - \frac{N}{p} \right] \geq 0,$$

where $[\cdot]$ stands for an integer part. Then, by the imbedding theorem [**12**, Theorem 7.26], the Sobolev space $W^{n,p}(\Omega, R^M)$ is continuously imbedded into $C^k(\overline{\Omega}, R^M)$. We let

$$K := \binom{N+k}{k} M, \qquad L := \binom{N+n}{n} M - K, \tag{2}$$

so that K is the dimension of a vector $(D^\alpha u(x))_{0 \leq |\alpha| \leq k}$, and L is the dimension of $(D^\alpha u(x))_{0 \leq \alpha \leq k}$, where $u \in W^{n,p}(\Omega, R^M)$ and $x \in \Omega$ (under the identification of mixed partial derivatives). Since $W^{n,p}(\Omega, R^M)$ is isometric to the product of $K + L$ copies of $L^p(\Omega, R)$, we obtain the continuous imbedding

$$W^{n,p}(\Omega, R^M) \underset{\to}{\subseteq} C(\overline{\Omega}, R^K) \times L^P(\Omega, R^L). \tag{3}$$

As a consequence of (3) and of Theorem 1 we have the following

Corollary 1. *Let Ω be a bounded domain in R^N with a smooth boundary, let K and L be defined by* (2), *and suppose that a multifunction $F : \Omega \times R^{K+L} \to CR^M$ satisfies the hypothesis of Theorem 1. Then the multivalued Nemytskii operator $F^* : W^{n,p}(\Omega, R^M) \to CL^q(\Omega, R^M)$ is well-defined, weakly compact and demi-continuous.*

For the reader's convenience, we recall the following well-known result [**1**, approximate selection theorem]:

Theorem 2. *Let X be a subset of a normed space, Y, a normed space, and let $F : X \to CY$ be a u.s.c. map. Then, for any $\varepsilon > 0$, there is a continuous map $f_\varepsilon : X \to \operatorname{co} F(X)$ with the property that for each $u \in X$ there exists $\hat{u} \in X$ and $\hat{v} \in F\hat{u}$ such that $||(u, f_\varepsilon u) - (\hat{u}, \hat{v})|| \le \varepsilon$, where $|| \cdot ||$ is the product norm on $X \times Y$.*

If $F(x, u)$ is a Carathéodory multifunction, then, by $f_\varepsilon(x, u)$, we mean the ε-approximation of the u.s.c. map $F(x, \cdot)$ in the sense of Theorem 2. Consequently, $f_\varepsilon(x, \cdot)$ is continuous for a.e. x. We will assume throughout that one can find $f_\varepsilon(x, u)$ for every ε, which is measurable in x for all u and that, given a measurable $u(x)$, the functions $\hat{u}(x)$ and $\hat{v}(x)$ of a parameter x, as they appear in Theorem 4, are measurable. Since such approximations f_ε are practically constructed for a given F, there is not much loss of generality in making that assumption. Thus, ε-approximations of Carathéodory multifunctions, in the above sense, are single-valued Carathéodory functions.

Lemma 2. *Let Ω be a bounded domain and suppose that a Carathéodory multifunction $F : \Omega \times R^K \to CR^M$ yields a well-defined multivalued Nemytskii operator $F^* : L^p(\Omega, R^K) \to CL^q(\Omega, R^M)$, where $p, q \ge 1$. Then there is a constant c depending only on the measure of Ω such that the Nemytskii operator f_ε^* for any ε-approximation f_ε of F is an εc-approximation of F^*, in the sense of Theorem 2.*

Proof. For any $u \in L^p(\Omega, R^K)$, there exist measurable $\hat{u}(x)$ and $\hat{v}(x) \in F(x, \hat{u}(x))$ such that $|u(x) - \hat{u}(x)| < \varepsilon$ and $|f_\varepsilon(x, u(x)) - \hat{v}(x)| < \varepsilon$ for a.e. $x \in \Omega$. By integrating both sides of those inequalities with exponents p and q, respectively, we get

$$||u - \hat{u}||_{L^p} < \varepsilon[m(\Omega)]^{\frac{1}{p}} \quad \text{and} \quad ||f_\varepsilon^* u - \hat{v}||_{L^q} < \varepsilon[m(\Omega)]^{\frac{1}{q}}$$

where $m(\Omega)$ is the Lebesgue measure of Ω, and the conclusion follows. □

Theorem 3. *Suppose that $F : \Omega \times R^{K+L} \to CR^M$ satisfies the* *hypothesis of Corollary 1, that $T : L^q(\Omega, R^M) \to W^{n,p}(\Omega, R^M)$ is a linear compact operator f_{ε_i}, an ε_i-approximation of F, and $\{u_i\}$ is a sequence with $u_i = Tf^*_{\varepsilon_i} u_i$, where $\varepsilon_i \to 0$. Then $\{u_i\}$ has a convergent subsequence and a limit of any such subsequence is a fixed point of TF^*, i.e., a point $\bar{u} \in TF^*\bar{u}$.*

Proof. By Lemma 1, TF^* is a u.s.c. compact map. For any u_i, there are $\hat{u}_i$ and $\hat{v}_i \in TF^*\hat{u}_i$ with $||u_i - \hat{u}_i|| < c\varepsilon_i$ and $||u_i - \hat{v}_i|| \leq c\varepsilon_i ||T||$, where c is the constant from Lemma 2. Since $\{u_i\}$ is bounded and $TF^*(\{u_i\})$ is relatively compact, therefore, $\{\hat{v}_i\}$ has a subsequence $\{\hat{v}_{ik}\}$ convergent to some limit $\bar{u}$. But $\varepsilon_{i_k} \to 0$, so $\{u_{i_k}\}$ and $\{\hat{v}_{i_k}\}$ converge to the same limit $\bar{u}$. Since TF^*, as a u.s.c. map, has a closed graph, we obtain $\bar{u} \in TF^*\bar{u}$. $\square$

Remark 1. For simplicity of presentation, we assumed in Lemma 1 and Theorem 3 that T is compact. However, it is enough to assume, instead, that T sends weakly sequentially compact sets to compact sets.

Remark 2. The smoothness of $\delta\Omega$, in Corollary 1, can be relaxed to a Lipschitz condition as in Theorem 7.26 in [**12**], whereas it can be dropped if we consider the space $W_0^{n,p}$ instead of $W^{n,p}$.

Remark 3. The conclusion of Lemma 2 remains true if the space $L^p(\Omega, R^K)$ is replaced by $C(\overline{\Omega}, R^K)$, since the second space is continuously imbedded into the first. Analogously, in Theorem 3 we may replace $W^{n,p}(\Omega, R^M)$ by $C^n(\overline{\Omega}, R^M)$.

Example of application. We are concerned with the boundary value problem

$$(4) \qquad \mathrm{L}u = g(u, Du), \qquad \text{a.e. } x \in \Omega; \qquad u = 0, \qquad x \in \delta\Omega$$

where Ω is a smooth domain in R^N, L is a linear uniformly elliptic operator satisfying the strong maximum principle [**12**, Theorem 3.5] and $g : R^+ \times R^N \to R^+$ is a nonnegative function. Let λ_1 be the lowest eigenvalue of the adjoint operator L$'$, $\lambda_1 > 0$, and let $\beta := (N+1)/(N-1)$. We assume that g satisfies the following conditions, uniformly for $p \in R^N$:

$$\underline{\lim}_{u\to\infty} u^{-1} g(u,p) > \lambda_1 \tag{5}$$

$$\lim_{u\to\infty} u^{-\beta} g(u,p) = 0 \tag{6}$$

$$\overline{\lim}_{u\to 0} u^{-1} g(u,p) < \lambda_1. \tag{7}$$

In [**2**], the existence of solutions to (4) was proved under the assumption that $g = g(x,u,p)$ is continuous. We assume here that g does not depend on x (see Remark 4 for a possible relaxation of this assumption); however, g can be discontinuous in (u,p) with finite jumps, so that

$$0 \le g(u,p) \le \varphi(u,p) \tag{8}$$

for some continuous function φ. For such a g, the problem (4) may have no solutions. By an *optimal solution* of (4), or, solution in the sense of Filippov [**10**], we mean a solution of the pair of inequalities

$$\underline{g}(u,p) \le \mathrm{L}u \le \bar{g}(u,p), \qquad \text{a.e. } x\in\Omega; \qquad u=0, \qquad x\in\delta\Omega \tag{9}$$

where

$$\underline{g}(u,p) := \underline{\lim}_{(v,q)\to(u,p)} g(v,q)$$

and

$$\bar{g}(u,p) := \overline{\lim}_{(v,q)\to(u,p)} g(v,q).$$

Theorem 4. *Let $g : R^+ \times R^N \to R^+$ be a nonnegative, possibly discontinuous function satisfying conditions* (5), (6), (7) *and* (8)*. Then the problem* (4) *has an optimal solution $\bar{u} \in C^1(\overline{\Omega}, R^+)$ such that $\bar{u} \in W^{2,q} \cap W_0^{1,q}$ for all $q > N$.*

Proof. We define $F(u,p) := [\underline{g}(u,p), \bar{g}(u,p)]$ (a closed interval in R^+). Then $F : R^+ \times R^N \to CR^+$ is u.s.c. Note that F satisfies the assumptions of Theorem 1 with $p = q$, $K = n+1$, $L = 0$, $c = 0$, and $\phi_B \equiv \max\{\phi(u,p) | (u,p) \in B\}$, where ϕ is a continuous function from

(8). We let $\{f_i\}$ be a sequence of continuous ε_i-approximations of F with $\varepsilon_i \to 0$, given by Theorem 2. We first verify that every f_i satisfies he same conditions as g does. The verification of (5) and (6), based on standard epsilon-delta arguments, is routine. For (7), let us notice that, in particular, $\lim_{u\to 0} g(u,p) = 0$. Given an ε_i-approximation f_i of F, we may obtain a continuous $2\varepsilon_i$-approximation $\tilde{f}_i$ of F by modifying f_i to a function which is identically zero for $0 \le u < \delta$, for a sufficiently small δ. Then (7) automatically holds for any such $\tilde{f}_i$.

It now follows from [**2**, Theorem 3.1] that there is a sequence of $u_i \in W^{2,q} \cap W_0^{1,q}$ with $\mathrm{L}u_i = f_i(u_i, Du_i)$. Moreover, the analysis of a priori norm estimates in the proofs of Lemma 2.3, Lemma 2.4 and Theorem 3.1 [**2**] shows that bounds appearing there can be chosen independently on ε_i so that $\{u_i\}$ is bounded in C^1.

We now consider L as a bijective operator from $W^{2,q} \cap W_0^{1,q}$ to L^q and F^* as a mapping from C^1 to CL^q. Since $q > N$, the inclusion $i : W^{2,q} \to C^1$ is compact. The conclusion now follows from Theorem 3 and Remark 3 applied for F^* and for $T = i\mathrm{L}^{-1}$. $\square$

Remark 4. The right-hand side of equation (4) may be allowed to depend on x provided we can separate the variable x from the "discontinuity" in (u, Du), in some sense. More explicitly, the conclusion of Theorem 4 will remain true for an x-dependent g if g can be written as

$$g(x,u,Du) = g_1(x,u,Du) + \alpha(x) g_2(u,Du),$$

where g_1 and α are continuous, α is bounded, and g_2 only may be discontinuous with finite jumps. In this case, we would be only approximating

$$F(u,p) = [\underline{g}_2(u,p), \bar{g}_2(u,p)]$$

by continuous functions f_{ε_i} and letting

$$f_i(x,u,p) = g_1(x,u,p) + \alpha(x) f_{\varepsilon_i}(u,p).$$

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