ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 2, Spring 1992

EXISTENCE OF SOLUTIONS TO BOUNDARY VALUE PROBLEMS FOR IMPULSIVE SECOND ORDER DIFFERENTIAL INCLUSIONS

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ABSTRACT. We consider nonlinear boundary value problems for second order differential inclusions $y'' \in F(t,y,y')$ where the solution undergoes an impulse at certain points t_k. The technique used is an adaptation of the topological transversality method to systems with impulses.

1. Introduction. In this paper we shall study the following boundary value problem for a system of second order impulsive differential inclusions:

$$\text{(1.1)} \qquad \begin{cases} y'' \in F(t,y,y') & \text{for a.e. } t \in [a_0,a_1] \\ y(t_k^+) = I_k(y(t_k)) & \\ y'(t_k^+) = N_k(y(t_k),y'(t_k)) & k = 1,\dots,m \\ G_i(\tilde{y}) = 0 & i = 0,1, \end{cases}$$

where

(i) $F : [a_0,a_1] \times \mathbf{R}^n \times \mathbf{R}^n \to 2^{\mathbf{R}^n}$ is a multifunction,

(ii) $I_k : \mathbf{R}^n \to \mathbf{R}^n$ is a homeomorphism for $k = 1,\dots,m$,

(iii) $N_k : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ is continuous, $k = 1,\dots,m$

(iv) $\tilde{y} = (y(a_0), y'(a_0), y(a_1), y'(a_1))$ and

$$G_i : \mathbf{R}^{4n} \longrightarrow \mathbf{R}^n \text{ is continuous, } i = 0,1.$$

(v) $a_0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = a_1$.

We look for a solution to (1.1) which is a piecewise C^1-function with points t_k of discontinuity, $k = 1,\dots,m$, of the first type for y and y' at which they are left continuous.

Received by the editors on March 5, 1989, and in revised form on September 8, 1989.
Research of the second author supported by NSERC, Canada.

We shall mention that differential inclusions, which generalize the notion of differential equations have been studied by many authors, for example, [**1, 7, 8, 9, 14, 24, 31, 32, 33**] (see [**32**] for an historical outline and an extensive list of resources).

There are many physical problems, in particular in optimal control theory, which involve impulsive behavior (see, e.g., [**2, 3, 6, 26, 29, 30**]. The fact that solutions to such problems are discontinuous renders the classical methods somewhat ineffective. We shall apply the topological transversality method of Granas based on the existence of *a priori* bounds for solutions to (1.1), which is appropriately modified to deal with the impulsive nature. The results presented in this paper could be obtained with the use of the topological degree but the simplicity of the topological transversality method seems to make this approach more desirable. Furthermore, the topological transversality method of Granas has been applied before in the study of boundary value problems in [**12, 13, 14, 16, 17, 18, 20, 21, 22, 23**].

To the best of our knowledge, we are not aware of any other attempts to apply these methods to systems of impulsive differential equations.

We first present two motivating examples involving impulses.

Example 1. A market model with price expectations and governmental price adjustment policy:

We consider a dynamic market model with n commodities. The supply and demand are denoted by the vectors $Q_s = (Q_s^1, \dots, Q_s^n)$ and $Q_d = (Q_d^1, \dots, Q_d^n)$, respectively. The price vector is denoted by $P = (P^1, \dots, P^n)$. We assume that the supply and demand are given by functions which depend on time, prices, actual changes of prices, and on the expectation of the price rising. We denote this by

$$Q_d = D(t, P(t), P'(t), P''(t)) \tag{1.2}$$

and

$$Q_s = S(t, P(t), P'(t), P''(t)). \tag{1.3}$$

We also assume that the process of price adjustment is characterized by the equation

$$\frac{dP}{dt} = q(Q_d, Q_s). \tag{1.4}$$

For example, $(dP^k/dt) = \alpha_k(Q_d^k - Q_s^k)$, $k = 1, \dots, n$. We are interested in the evolution of the prices according to this model. Putting (1.2) and (1.3) into (1.4) and solving this equation with respect to $P''(t)$, we obtain the following second order equation

$$P''(t) = F(t, P(t), P'(t)). \tag{1.5}$$

We suppose next that at certain times $0 < t_1 < \cdots < t_m < T$ the government, according to its market control policy, changes the actual prices at t_k, $k = 1, \dots, m$. This action may be viewed as an impulse at time t_k at which the price vector $P(t)$ is replaced by the new price vector $I_k(P(t_k))$, where $I_k : \mathbf{R}^n \to \mathbf{R}^n$ is the mapping which yields the "new prices." Evidently, such an intervention into the market will also cause some changes in the process of price adjustment. It therefore seems reasonable to assume that the vector $P'(t)$ is replaced by $N_k(P(t_k), P'(t_k))$. We call the market control policy *limited* if it does not change the actual trend in price adjustment. We can express this condition as follows: for every $k = 1, \dots, m$, we assume that if

$$P(t_k) \cdot P'(t_k) \geq 0 \text{ then } I_k(P(t_k)) \cdot N_k(P(t_k), P'(t_k)) \geq 0. \tag{1.6}$$

We may then consider the following boundary value problem for equation (1.5): $P(0) = P_0$, $P(t) = P_1$, where $P_0, P_1 \in \mathbf{R}^n$ are two given vectors. Thus, given an initial price level and a desired terminal price and assuming the market adjustments at times t_k, $k = 1, \dots, m$, one wishes to know if (for the governing equation) there is a solution to the BVP. (Such government interventions in the market actually occur in certain countries, especially in Eastern Europe.)

With slight modifications, a similar model illustrating the incomes of n independent producers may be given. In this model, taxes are imposed or subventions are granted at certain fixed times, which result in an impulsive differential system.

Example 2. A model of a cruise missile with deceiving impulses.

Suppose that $y(t) \in \mathbf{R}^3$ denotes the position function of a cruise missile. The missile is launched from a point $y_0 \in \mathbf{R}^3$ and is supposed to hit a target at a point $y_1 \in \mathbf{R}^3$. According to the laws of motion, we can consider the trajectory of the missile $y(t)$ as a solution to the following second order differential system of equations with Dirichlet boundary conditions:

$$\begin{cases} y''(t) = f(t, y(t), y'(t)) \\ y(0) = y_0, y(T) = y_1. \end{cases} \tag{1.7}$$

A solution to the BVP (1.7) represents the simplest and shortest trajectory from y_0 to y_1. Therefore, from a tactical point of view, this may be the least desirable since detection of the missile would lead to its destruction by the adversary. Therefore, it seems to be natural in this situation to change the direction of the missile several times in order to deceive the adversary and minimize the possibility of its destruction. The simplest way to do this is to change appropriately the direction of the missile at the time $0 < t_1 < \cdots < t_m < T$ according to the actual position of the missile and its direction. That is, at the time t_k, $k = 1, \dots, m$, an impulse $y'(t_k^+) = N_k(y(t_k), y'(t_k))$ changes the velocity vector of the missile. The function N_k consists of commands, depending on the position and velocity of the missile, for the guidance system. One may include in N_k additional information concerning the location of the adversary's radar and missile detectors in such a way that the actual trajectory of the missile perceived by the adversary will be very confusing. It seems reasonable that one should impose some restrictions on the impulses N_k. The condition

$$\text{if} \quad \|y\| > R \quad \text{and} \quad y \cdot y' \geq 0 \quad \text{then} \quad y \cdot N_k(y, y') \geq 0 \tag{1.8}$$

can be interpreted as the requirement that the impulses N_k do not perturb "too much," the trajectory of the missile when it is still at a safe distance from the target, i.e., $\|y\| > R$. The condition (1.8) means that the change of the direction can be done only in some limited range, which is plausible from physical considerations. With these considerations and restrictions in mind, it seems nevertheless to be of interest to study such models and to interpret the consequences of the introduction of impulses on the existence of solutions to the BVP (1.7).

2. Existence results. In what follows, we will consider the following functional Banach spaces: $C([a, b]; \mathbf{R}^m) = \{u : [a, b] \to \mathbf{R}^m : u$ is continuous on $[a, b]\}$ with the norm $\|u\|_\infty = \sup_{t\in[a,b]} \|u(t)\|$ where

$\|\cdot\|$ will denote the usual Euclidean norm in $\mathbf{R}^m$. $C^k([a,b];\mathbf{R}^m) = \{u : [a,b] \to \mathbf{R}^m : u^{(i)} \in C([a,b];\mathbf{R}^m), 0 \le i \le k\}$ with the norm

$$\|u\|_{\infty;k} = \max\{\|u^{(i)}\|_\infty : 0 \le i \le k\}.$$

$L^2([a,b];\mathbf{R}^m) = \{u : [a,b] \to \mathbf{R}^m : \|u(t)\| \text{ is } L^2\text{-integrable}\}$ with the norm $\|u\|_2 = (\int_a^b \|u(t)\|^2\,dt)^{1/2}$. $H^k([a,b];\mathbf{R}^m) = \{u : [a,b] \to \mathbf{R}^m : u \text{ has weak derivatives } u^{(i)} \in L^2([a,b];\mathbf{R}^m) \text{ for } 0 \le i \le k\}$ with the norm

$$\|u\|_{2;k} = \max\{\|u^{(i)}\|_2 : 0 \le i \le k\}.$$

The spaces $H^k([a,b];\mathbf{R}^m|$ are the usual Sobolev spaces of vector functions denoted also by $W^{k,2}([a,b];\mathbf{R}^m)$ (for more details, see [**5**]).

We introduce the following definition which will allow us to avoid repetition of technical assumptions in the sequel.

Definition. Suppose that $F : [a_0,a_1] \times \mathbf{R}^m \to 2^{\mathbf{R}^n}$ is a multifunction with nonempty convex and compact values. We say that F is *admissible* if the multivalued map (Nemitsky operator)

$$N_F : C([a_0,a_1];\mathbf{R}^m) \longrightarrow L^2([a_0,a_1];\mathbf{R}^n)$$

given by

$$N_F(u) := \{w \in L^2([a_0,a_1];\mathbf{R}^n) : w(t) \in F(t,u(t)) \text{ for a.e. } t \in [a_0,a_1]\}$$

is well defined with nonempty convex values and such that the composed multivalued map $(\mathcal{J} \circ N_F)(u) := \mathcal{J}(N_F(u))$ is upper semi-continuous and *completely continuous* (i.e., $\mathcal{J} \circ N_{F|X}$ is compact for every bounded set X), where $\mathcal{J} : L^2([a_0,a_1] : \mathbf{R}^n) \to C([a_0,a_1];\mathbf{R}^m)$ is an arbitrary completely continuous linear operator.

Our consideration of admissible multifunctions will be restricted to Carathéodory multifunctions (cf. [**7, 31, 32**]).

Let us recall that a multifunction $F : \overline{\Omega} \times \mathbf{R}^m \to 2^{R^n}$, $\Omega \subset \mathbf{R}^k$, with nonempty compact convex values is said to be a *Carathéodory multifunction* if it satisfies the following conditions:

(a) for each $u \in \mathbf{R}^m$ the mapping $F(\cdot,u)$ is measurable;

(b) for each $x \in \overline{\Omega}$ the mapping $F(x,\cdot)$ is upper semi-continuous.

The following lemma is a direct consequence of Proposition 1.7 in [**31**].

(2.1) Lemma. *Suppose that a Carathéodory multifunction* $F : [a,b] \times \mathbf{R}^m \to 2^{\mathbf{R}^n}$ *satisfies the following growth condition*

(G) *for any bounded* $B \subset \mathbf{R}^m$ *there exists* $\varphi_\beta \in L^2[a,b]$ *such that*

$$\|F(t,u)\| \le \varphi_B(t) \quad \textit{for a.e. } t \in [a,b] \quad \textit{and all } u \in B$$

where $\|F(t,u)\| := \sup\{\|v\| : v \in F(t,u)\}$.

Then F *is admissible.*

In what follows, we will suppose that the multifunction F appearing in the system (1.1) is admissible. We define the following Banach space:

$$C := \prod_{k=0}^{m} C([t_k, t_{k+1}]; \mathbf{R}^n)$$

where the norm $\|\{u_k\}_{k=0}^m\|_\infty$ is defined by $\|\{u_k\}_{k=0}^m\|_\infty := \max\{\|u_k\|_\infty; 0 \le k \le m\}$. We introduce also the following Banach spaces:

$$H^2 := \prod_{k=0}^{m} H^2([t_k, t_{k+1}]; \mathbf{R}^n)$$

and

$$L^2 := \prod_{k=0}^{m} L^2([t_k, t_{k+1}]; \mathbf{R}^n)$$

where the norms are defined in a similar manner as in C.

A solution to (1.1) may be identified with a unique element of the space H^2.

We shall make the following hypotheses in what follows:

(H1) There exists a constant $R > 0$ such that if $\|y_0\| > R$ and $y_0 \cdot y_1' = 0$, then there is a $\delta > 0$ such that

$$\operatorname*{ess\,inf}_{t \in [a_0, a_1]} \inf\{y \cdot w + \|y'\|^2 : w \in F(t, y, y'), (y, y') \in D_\delta\} > 0$$

where $D_\delta := \{(y, y') \in \mathbf{R}^{2n} : \|y - y_0\| + \|y' - y'_0\| < \delta\}$.

We put

$$S_{i,i+k} = \sup\{\|x\| : x \in (I_{i+k} \circ \cdots \circ I_i)(B(0,R))\}$$
$$R_{i,i+k} = \sup\{\|x\| : x \in (I_i^{-1} \circ \cdots \circ I_{i+k}^{-1})(B(0,R))\}$$

where $1 \le i$, $i + k \le m$, $k \ge 0$, $B(0,R) = \{x \in \mathbf{R}^n; \|x\| \le R\}$, and

$$M = \max\{R_{i,i+k}; S_{i,i+k}; R : 1 \le i, i+k \le m; k \ge 0\}.$$

The following well-known Nagumo conditions will also be assumed;

(H2) There is a function $\varphi : [0,\infty) \to (0,\infty)$ such that

$$\frac{s}{\varphi(s)} \in L^\infty_{\mathrm{loc}}[0,\infty), \ \int_0^\infty \frac{s\,ds}{\varphi(s)} = \infty \quad \text{and} \quad \|F(t,y,y')\| \le \varphi(\|y'\|)$$

for a.e. $t \in [a_0, a_1]$ and all $(y, y') \in D := \{(x, x') \in \mathbf{R}^n \times \mathbf{R}^n : \|x\| \le M\}$.

(H3) There exist constants k, $\alpha > 0$, such that

$$\|f(t,y,y')\| \le 2\alpha(y \cdot w + \|y'\|^2) + k$$

for a.e. $t \in [a_0, a_1]$, all $(y, y') \in D$ and $w \in F(t, y, y')$.

In the case where F is a continuous function the condition (H1) reduces to the classical Nagumo-Hartman condition [**25**]. The conditions (H2) and (H3) are related to the usual Bernstein-Nagumo growth conditions. We refer also to [**11, 12, 13, 14, 16, 17, 18, 20, 21, 22, 23**].

The following conditions were introduced in [**12**]:

Let $G_i : \mathbf{R}^{4n} \to \mathbf{R}^n$, $i = 0, 1$, be continuous functions and let $A_0, A_1 \in GL(n; \mathbf{R})$.

For a fixed function G_i, $i = 0, 1$, we introduce the following conditions:

(N1) One of the following two inequalities is satisfied for all $u_0, u'_0, u_1, u'_1 \in \mathbf{R}^n$:

$$(-1)^i[u_i \cdot u'_i \pm A_i(G_i(u_0, u'_0, u_1, u'_1) \cdot u'_i] \ge 0.$$

(N2) One of the following two inequalities is satisfied for all $u_0, u_0', u_1, u_1' \in \mathbf{R}^n$:

$$(-1)^i[u_i \cdot u_1' \pm A_i G_i(u_0, u_0', u_1, u_1') \cdot u_1] \geq 0.$$

(N3) One of the following two inequalities is satisfied for all $\lambda \in [0,1]$ and all $u_0, u_0', u_1, u_1' \in \mathbf{R}^n$ such that $\|u_i\| > R$: (where the constant $R > 0$ is the same as in the hypothesis (H1) introduced above)

$$u_i \neq \lambda[u_i \pm A_i G_i(u_0, u_0', u_1, u_1')].$$

Our last assumption concerns the impulses N_k and I_k:

(IN) For every $k = 1, \dots, m$ if $y \cdot y' \geq 0$ and $\|y\| > R$, then

$$I_k(y) \cdot N_k(y, y') \geq 0.$$

The condition (IN) says that the monotone character of the function $r(t) = \|y(t)\|^2$, where $y(t)$ is a solution to (1.1), is not changed after the impulse is applied, provided that $\|y(t)\|$ is large enough, i.e., $\|y(t)\| > R$.

Our main result is the following

(2.2) Theorem. *Under the hypotheses* (H1), (H2), (H3) *and* (IN) *if* $G_i : \mathbf{R}^{4n} \to \mathbf{R}^n$, $i = 0, 1$, *satisfy one of the conditions* (N1), (N2) *or* (N3), *the problem* (1.1) *has at least one solution* $y \in H^2$.

Proof. Let us introduce the following notation:

$$C_2 := \prod_{k=0}^{m} \{C([t_k, t_{k+1}]; \mathbf{R}^n) \times C([t_k, t_{k+1}]; \mathbf{R}^n)\}.$$

If the function G_i satisfies the condition (N1) or (N3), we put

$$w_i(\tilde{y}) := y(a_i); \quad g_i(\tilde{y}) := y(a_i) \pm A_i G_i(\tilde{y});$$

otherwise

$$w_i(\tilde{y}) := y'(a_i); \quad g_i(\tilde{y}) = y'(a_i) \pm A_i G_i(\tilde{y})$$

where $\tilde{y} = (y(a_0), y'(a_0), y(a_1), y'(a_1))$, $a_0 = t_0$, $a_1 = t_{m+1}$.

We consider the following family of nonlinear boundary value problems:

$$(1.1_\lambda)\qquad \begin{cases} y'' - \varepsilon y \in \lambda\{F(t,y,y') - \varepsilon y\} & \text{a.e. } t \in [a_0, a_1] \\ y(t_k) = I_k^{-1}(y(t_k^+)) \\ y'(t_k^+) = \lambda N_k(y(t_k), y'(t_k)) \\ w_i(\tilde{y}) = \lambda \cdot g_i(\tilde{y}) & i = 0,1;\ \lambda \in [0,1]. \end{cases}$$

We define the linear operator $\tilde{L} : H^2 \to L^2 \times \mathbf{R}^n \times \mathbf{R}^{m\cdot n} \times \mathbf{R}^n \times \mathbf{R}^{m\cdot n}$ by

$$\tilde{L}(\{u_k\}_{k=0}^m) := (\{u_k'' - \varepsilon u_k\}_{k=0}^m, w_0(\tilde{u}_0), \{u_k(t_{k+1})\}_{k=0}^{m-1}, w_1(\tilde{u}_m), \{u_k'(t_k)\}_{k=1}^m)$$

where $\tilde{u}_0 = (u_0(a_0), u_0'(a_0))$, $\tilde{u}_m = (u_m(a_1), u_m'(a_1))$, $a_0 = t_0$, $a_1 = t_{m+1}$. Let us observe that the operator $\tilde{L}$ is invertible, where $\varepsilon > 0$ will be restricted later. Indeed, the problem of finding the inverse for $\tilde{L}$ is equivalent to $m+1$ *independent* linear boundary value problems:

$$(P^0)\begin{cases} u_0'' - \varepsilon u_0 = f_0 \\ w_0(\tilde{u}_0) = x_0 \\ u_0(t_1) = z_0 \end{cases} \qquad (P^k)\begin{cases} u_k'' - \varepsilon u_k = f_k \\ u_k'(t_k) = x_k \\ u_k(t_{k+1}) = z_k \end{cases} \quad k = 1, \dots, m-1$$

and

$$(P^m)\begin{cases} u_m'' - \varepsilon u_m = f_m \\ u_m'(t_m) = x_m \\ w_1(\tilde{u}_m) = z_m \end{cases}$$

where

$$\{f_k\}_{k=0}^m \in L^2; (x_0, \{z_k\}_{k=0}^{m-1}, z_m) \in \mathbf{R}^n \times \mathbf{R}^{m\cdot n} \times \mathbf{R}^n \text{ and } \{x_k\}_{k=1}^m \in \mathbf{R}^{m\cdot n}.$$

Any of these linear boundary value problems $(P^0), (P^k), k = 1, \dots, m-1$, and (P^m), is uniquely solvable; thus, $\tilde{L}^{-1}$ exists and is continuous.

We put $j : H^2 \to C^2$; $j(\{u_k\}_{k=0}^m) = \{(u_k, u_k')\}_{k=0}^m$. It is clear that j is a completely continuous one-to-one linear operator. We define a family of multivalued maps

$$\Phi_\lambda : C_2 \to L^2 \times \mathbf{R}^n \times \mathbf{R}^{m\cdot n} \times \mathbf{R}^n \times \mathbf{R}^{m\cdot n}$$

by

$$\Phi_\lambda(\{u_k, v_k\}_{k=0}^m) = \lambda\cdot\Gamma(\{u_k, v_k\}_{k=0}^m)\times\{(\lambda\cdot g_0(\tilde{u},\tilde{v}), \{I_k^{-1}(u_k(t_k))\}_{k=1}^m,$$
$$\lambda g_1(\tilde{u},\tilde{v})\} \times \{\lambda N_k(v_{k-1}(t_k), u_{k-1}(t_k))\}_{k=1}^m$$

where $(\tilde{u},\tilde{v}) = (u_0(a_0), v_0(a_0), u_m(a_1), v_m(a_1))$ and

$$\Gamma(\{u_k, v_k\}_{k=0}^m) := \{w \in L^2 : w_k(t) \in F(t, u_k(t), v_k(t)) - \varepsilon u_k(t)$$
$$\text{a.e. } t \in [t_k, t_{k+1}], k = 0, \dots, m\}.$$

We consider the following diagram

$$\begin{array}{ccc} C_2 & \xrightarrow{\Phi_\lambda} & L^2 \times \mathbf{R}^{(2m+2)n} \\ & {\nwarrow}^{j} & \downarrow \tilde{L}^{-1} \\ & & H^2 \end{array}$$

The problem (1.1_λ) is equivalent to the following fixed point problem

$$w \in \mathcal{F}_\lambda(w); \quad w \in C_2 \tag{2.1}$$

where $w = \{(u_k, v_k)\}_{k=0}^m$ and $\mathcal{F}_\lambda := j \circ \tilde{L}^{-1} \circ \Phi_\lambda$ is a u.s.c. multivalued completely continuous map with nonempty convex values. It is easy to see that $\mathcal{F}_\lambda$ is a homotopy, with respect to $\lambda \in [0,1]$, of such multivalued maps. For $\lambda = 1$, the problem (2.1) is equivalent to (1.1) and for $\lambda = 0$, the problem (2.1) is the following second order impulsive system:

$$\begin{cases} y'' - \varepsilon y = 0 & \text{a.e. } t \in [a_0, a_1] \\ y(t_k) = I_k^{-1}(y(t_k^+)) & \\ y'(t_k^+) = 0 & k = 1, \dots, m \\ w_i(\tilde{y}) = 0 & i = 0, 1. \end{cases} \tag{2.2}$$

We will show that the map $\mathcal{F}_0$ is *essential* (in the sense of Granas) on some ball $U = B(0, \tilde{M})$ in the space C_2, if the radius $\tilde{M}$ is sufficiently large. Therefore, the existence of a solution to (1.1) will follow from the existence of *a priori* bounds for solutions to (1.1_λ).

For definitions and facts concerning the *topological transversality method* and the proofs of those results, we refer to [**11, 12, 13, 14, 18, 27**].

Suppose now that $y(t)$ is a solution to (1.1_λ), for $\lambda \in [0,1]$. The function $y(t)$ can be represented as a sequence $\{y_k\}_{k=0}^m$, $y_k : [t_k, t_{k+1}] \to \mathbf{R}^n$. We put $r(t) = \|y(t)\|^2$. Although the function $y(t)$ has discontinuities, the function $r(t)$ can also be represented as a sequence of functions $\{r_k(t)\}_{t=0}^m$; $r_k(t) = \|y_k(t)\|^2$, $k = 0, \dots, m$.

The constant $\varepsilon > 0$ can be chosen appropriately small in order that the conditions (H2), (H3) still imply the existence of a constant $M_1 > 0$ such that for any solution $y(t)$ of (1.1_λ), $\|y'(t)\| \le M_1$, provided $r(t) \le M^2$ (see [**12**] for details).

We need the following lemma for establishing the existence of *a priori* bounds for solutions to (1.1_λ).

(2.3) Lemma. *Under the above hypotheses, if y is a solution to the differential inclusion $y'' - \varepsilon y \in \lambda[F(t,y,y') - \varepsilon y]$, $\lambda \in [0,1]$ such that the function $r(t) = \|y(t)\|^2$ achieves its local maximum m_0^2 at a point $s_0 \in [a_0, a_1]$ such that $s_0 \neq t_0, t_1, \dots, t_m, t_{m+1}$, or $s_0 \in \{t_{k-1}, t_k\}$ and $r_k(s_0) = m_0^2$, $r_k'(s_0) = 0$, then it follows that $m_0^2 \le R^2$.*

Proof. Let us remark that the multifunction $F_\lambda(t,y,y') := \lambda F(t,y,y') + (1-\lambda)\varepsilon y$ also satisfies the hypothesis (H1) for $\lambda \in [0,1]$ (see [**12**, Lemma 4.1] for more details).

Suppose that $r_k(s_0) = m_0^2 > R^2$ and $r_k'(s_0) = 2y_k(s_0) \cdot y_k'(s_0) = 0$. Since $(y_k(t), y_k'(t)) \to (y_k(s_0), y_k'(s_0))$ as $t \to s_0$, thus there exist an $\alpha > 0$ and an $\eta > 0$ such that for almost every $t \in A_\eta := \{t \in \operatorname{Dom} y_k\colon |t - s_0| < \eta\}$

$$\inf\{y_k(t) \cdot w + \|y'\|^2 : w \in F_\lambda(t, y(t), y'(t))\} > \alpha > 0$$

and, therefore,

$$\frac{1}{2} r_k''(t) = y_k(t) \cdot y_k''(t) + \|y_k'(t)\|^2 > 0$$

for almost all $t \in A_\eta$. But this is a contradiction to the maximum principle (see [**4**]). $\square$

Let us now suppose that $y(t)$ is a solution to (1.1_λ) and let $r_k(s_0) = \sup \|y(t)\|^2$. If $s_0 \neq t_0, t_1, \dots, t_m, t_{m+1}$, then it follows that $r(s_0) =$

$r_k(s_0)$, and the *a priori* bound $\|y(t)\| \le R$ follows from Lemma (2.3). Suppose now that $s_0 = a_i, i = 0$ or 1. In this case we obtain by the assumptions (N1), (N2) or (N3), that $r'(a_i) = 0$. Indeed, if G_i satisfies (N1), we have

$$\begin{aligned} 0 &\ge (-1)^i r'(a_i) = (-1)^i 2y(a_i) \cdot y'(a_i) \\ &= (-1)^i 2\lambda [y(a_i) \cdot y'(a_i) \pm A_i G_i(\tilde{y}) \cdot y'(a_i)] \ge 0 \end{aligned}$$

and thus $r'(a_i) = 0$. If G_i satisfies (N2), we have

$$\begin{aligned} 0 &\ge (-1)^i r'(a_i) = (-1)^i 2y(a_i) \cdot y'(a_i) \\ &= (-1)^i 2\lambda [y(a_i) \cdot y'(a_i) \pm A_i G_i(\tilde{y}) \cdot y(a_i)] \ge 0 \end{aligned}$$

and thus $r'(a_i) = 0$.

In the last case where we suppose that G_i satisfies (N3) we have that $y(a_i) = \lambda[y(a_i) \pm A_i G_i(\tilde{y})]$, $\lambda > 0$, thus $\|y(t)\| \le \|y(a_i)\| \le R$ and the required estimate is obtained.

Since $r'(s_0) = 0$, in the case where G_i satisfies (N1) or (N3), it is sufficient to apply Lemma (2.3) and the *a priori* bound $\|y(t)\| \le R$ follows.

Suppose now that $s_0 = t_k$, $k \in \{1, \dots, m\}$. We will show that $\|y(t)\| \le M$.

There are only two cases to consider:

1) the supremum of $\|y(t)\|$ is achieved by $\|y_{k-1}(t)\|$ at the point t_k, or

2) the supremum of $\|y(t)\|$ is achieved by $\|y_k(t)\|$ at the point t_k where we identify the solution $y(t)$ with the sequence $\{y_i(t)\}_{i=0}^m$.

Case 1. Since the supremum of $r(t)$ is achieved at t_k by $r_{k-1}(t)$, thus $r'_{k-1}(t_k) \ge 0$. If $r'_{k-1}(t_k) = 0$ or $r_{k-1}(t_k) \le R^2$, the *a priori* bound $\|y(t)\| \le R$ follows. Suppose that $r'_{k-1}(t_k) > 0$ and $r_{k-1}(t_k) > R^2$. By the assumption (IN) we have that $r'_k(t_k) = 2\lambda \cdot I_k(y_{k-1}(t_k))$. $N_k(y_{k-1}(t_k), y'_{k-1}(t_k)) \ge 0$. If $r_k(t_k) \le R^2$, then the *a priori* bound

$$\|y(t)\| \le \|y_{k-1}(t_k)\| = \|I_k^{-1}(y_k(t_k))\| \le R_{k,k} \le M$$

follows. Otherwise, $r'_k(t_k) \ge 0$ and $r_k(t_k) > R^2$. By the hypothesis (H1) and Lemma (2.3), $r_k(t)$ cannot achieve its local maximum at a point $s_0 \in [t_k, t_{k+1}]$ such that $r_k'(s_0) = 0$ (this would imply that $r_k(t_k) \le r_k(s) \le R^2$); thus, the function $r_k(t)$ is increasing on the interval $[t_k, t_{k+1}]$ and $r_k'(t_{k+1}) > 0$. By the hypothesis (IN), we obtain that

$$r_{k+1}'(t_{k+1}) = 2\lambda I_{k+1}(y_k(t_{k+1}) \cdot N_{k+1}(y_k(t_{k+1}), y_k(t_{k+1})) \ge 0$$

and again by the same arguments we obtain that either $r_{k+1}(t_{k+1}) \le R^2$, and thus $\|y(t)\| \le R_{k,k+1}$, or the function $r_{k+1}(t)$ is strictly increasing on the interval $[t_{k+1}, t_{k+2}]$. By repeating this argument successively, eventually we will arrive to the last interval $[t_m, a_1]$, where $r_m'(t_m) \ge 0$. If $r_m(t_m) \le R^2$, then we get $\|y(t)\| \le R_{k,m}$; otherwise, $r_m(t)$ is strictly increasing on the interval $[t_m, a_1]$ and it achieves its maximum at the point $t = a_1$. But again we can use the assumptions (N1), (N2) or (N3) in order to obtain that $r_m(a_1) \le R^2$, thus $r_m(t_m) \le r_m(a_1) \le R^2$, a contradiction. This means that $\|y(t)\| \le M$.

Case 2. Suppose now that the supremum of $r(t)$ is achieved at the point $s = t_k$, by the function $r_k(t) = \|y_k(t)\|^2$. Then $r_k'(t_k) \le 0$. If $r_k(t_k) \le R^2$ or $r_k'(t_k) = 0$, then the *a priori* bound $\|y(t)\| \le R$ will follow. Suppose then that $r_k'(t_k) < 0$ and $r_k(t_k) > R^2$. By the assumption (IN), $r_{k-1}'(t_k) < 0$. If $r_{k-1}(t_k) \le R^2$, then

$$\|y(t)\| \le \|y_k(t_k)\| = \|I_k(y_{k-1}(t_k))\| \le S_{k,k} \le M.$$

Otherwise, by Lemma (2.3), the function $r_{k-1}(t)$ has to be strictly decreasing on the interval $[t_{k-1}, t_k]$ and $r_{k-2}'(t_{k-1}) < 0$. Now, if we repeat the same argument, we will obtain that $\|y(t)\| \le S_{k-1,k} \le M$ or the function $r_{k-2}(t) > R^2$ is decreasing on the interval $[t_{k-2}, t_{k-1}]$. Eventually, we will arrive to the first interval $[a_0, t_1]$, and by the same argument, $r_0(t_1) \le R^2$ and, thus $\|y(t)\| \le S_{1,k} \le M$, or $r_0(t) > R^2$ and $r_0(t)$ is a strictly decreasing function on the interval $[a_0, t_1]$. But this means that $r_0(t)$ achieves its maximum at the point $t = a_0$, thus the assumptions (N1), (N2) or (N3) will imply that $r_0'(a_0) = 0$, and by Lemma (2.3), $r_0(a_0) \le R^2$, thus $r_0(t_1) \le R^2$, a contradiction. Therefore, the *a priori* bound $\|y(t)\| \le M$ is proved.

We put $U = \{\{(u_k, v_k)\}_{k=0}^m \in C_2 : \|u_k\| < \tilde{M}, \|v_k\| < \tilde{M}\}$ where $\tilde{M} = \max\{M+1, M_1+1\}$. The obtained *a priori* bounds imply that all fixed points of $\mathcal{F}_\lambda$ belong to the interior of $\overline{U}$, thus $\mathcal{F}_\lambda$ is a homotopy in the class $\mathcal{K}_{C_2}, (\overline{U}; \partial U)$ (see [**12**] or [**23**] for more details). We shall show that $\mathcal{F}_0$ is an essential map in $\mathcal{K}_{C_2}(\overline{U}; \partial U)$. We consider next the family of impulsive differential systems:

$$(2.2_\lambda) \qquad \begin{cases} y'' - \varepsilon y = 0 \\ y(t_k) = \lambda I_k^{-1}(y(t_k^+)), & k = 1, \dots, m \\ y'(t_k^+) = 0 \\ w_i(\tilde{y}) = 0, & i = 0, 1, \lambda \in [0, 1]. \end{cases}$$

It is sufficient to prove the existence of *a priori* bounds for solutions to (2.2_λ). The system (2.2_λ) is a particular case of the system (1.1_λ), thus the same arguments apply. Suppose that $y(t)$ is a solution to (2.2_λ) and assume that the supremum of $r(t) = \|y(t)\|^2$ is achieved by $\|y_k(t)\|^2$ at the point $s_0 \in [t_k, t_{k+1}]$. If $s_0 \neq t_k, t_{k+1}$, then, by Lemma (2.3), $\|y(t)\| \leq R$. If $k = 0$ or $k = m$ and $s_0 = a_0$ or $s_0 = a_1$, respectively, then it is easy to observe that $r_k'(s_0) = 0$ and again the estimate $\|y(t)\| \leq R$ follows from Lemma (2.3). Now, suppose that $s_0 = t_k$ or t_{k+1}. The condition $y'(t_k^+) = 0$ means that $y_k'(t_k) = 0$ and therefore $r_k'(t_k) = 0$. This shows that $\|r_k(t_k)\| \leq R^2$, in the case $s_0 = t_k$. Suppose next that $s_0 = t_{k+1}$. By assumption that $y(t)$ is a solution to (2.2_λ), we have that

$$\|y_k(t_{k+1})\| = \{\lambda I_k^{-1}(y_{k+1}(t_{k+1}))\| \leq \|I_k^{-1}(y_{k+1}(t_{k+1}))\|$$

and $y_{k+1}'(t_{k+1}) = 0$, thus $r_{k+1}'(t_{k+1}) = 0$. We show similarly, as in the case of the system (1.1_λ), that either $\|y_{k+1}(t_{k+1})\| \leq R$ or the function $r_{k+1}(t)$ is strictly increasing on the interval $[t_{k+1}, t_{k+2}]$. By repeating the same arguments we obtain that $\|y(t)\| \leq M$.

We observe that the family (2.2_λ) defines a deformation of the map $\mathcal{F}_0$ to the constant map $\mathcal{F}^* \equiv 0$ in the class $\mathcal{K}_{C_2}(\overline{U}; \partial U)$. Since $\mathcal{F}^*$ is an essential map in $\mathcal{K}_{C_2}(\overline{U}; \partial U)$, by the topological transversality theorem (cf. [**11, 12, 14, 27, 28**]) $\mathcal{F}_0$ and consequently $\mathcal{F}_1$ are essential maps in $\mathcal{K}_{C_2}(\overline{U}; \partial U)$. This means that $\mathcal{F}_1$ has a fixed point in U, i.e., there is $w \in U$ such that $w \in \mathcal{F}_1(w)$ and the existence of a solution to the problem (1) follows. □

(2.4) Theorem. *Under the hypotheses* (H1), (H2), (H3) *and* (IN) *the following impulsive system of differential inclusions with the periodic* *boundary value conditions:*

$$(2.3)\qquad \begin{cases} y'' \in F(t,y,y') & \text{for a.e. } t \in [a_0,a_1] \\ y(t_k^+) = I_k(y(t_k)) & \\ y'(t_k^+) = N_k(y(t_k), y'(t_k)), & k = 1,\dots,m \\ y(a_0) = y(a_1) & \\ y'(a_0) = y'(a_1) & \end{cases}$$

has at least one solution $y \in H^2$.

Proof. The proof of Theorem (2.4) is similar to the proof of Theorem (2.2) but it contains some essential differences and, therefore, we give some details. We use the same notation as before. Put

$$H_p^2 := \{(u_k)_{k=0}^m \in H^2 : u_0(a_0) = u_m(a_1), u_0'(a_0) = u_m'(a_1)\}$$

and let

$$\tilde{L}^0 : H_p^1 \to L^2 \times \mathbf{R}^{m\cdot n} \times \mathbf{R}^{m\cdot n}$$

be defined by

$$\tilde{L}^0(\{u_k\}_{k=0}^m) := (\{u_k'' - \varepsilon u_k\}_{k=0}^m, \{u_k(t_{k+1})\}_{k=0}^{m-1}, \{u_t'(t_k)\}_{t=1}^m).$$

Let us observe that the operator $\tilde{L}^0$ is invertible for all $\varepsilon > 0$, where ε is sufficiently small in the sense explained earlier. Indeed, the problem of finding the inverse $\tilde{L}^0$ is equivalent to m independent linear boundary value problems:

$$(2.4)\qquad \begin{cases} u_k'' - \varepsilon u_k = f_k & \text{a.e. } t \in [t_k, t_{k+1}] \\ u_k'(t_k) = x_k & \\ u_k(t_{k+1}) = z_k, & k = 1,\dots,m-1 \end{cases}$$

and

$$(2.5)\qquad \begin{cases} v'' - \varepsilon v = \tilde{f} & \text{a.e. } t \in [t_m, a_1 + (t_1 - a_0)] \\ v'(t_m) = x_m & \\ v(a_1 + (t_1 - a_0)) = z_0 & \end{cases}$$

where

$$\tilde{f}(t) = \begin{cases} f_m(t) & \text{for } t \in [t_m, a_1] \\ f_0(t - a_1 + a_0) & \text{for } t \in [a_1, a_1 + (t_1 - a_0)], \end{cases}$$
$$u_m = v|_{[t_m, a_1]}, u_0(t) = v(t - a_1 + a_0) \quad \text{for } t \in [a_1, a_1 + (t_1 - a_0)]$$

and $\{f_k\}_{k=0}^m \in L^2$, $\{z_k\}_{k=0}^m \in \mathbf{R}^{m \cdot n}$, $\{x_k\}_{k=0}^m \in \mathbf{R}^{m \cdot n}$.

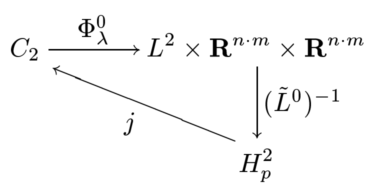
It is clear that the boundary value problems (2.4) and (2.5) are uniquely solvable, thus the operator $(\tilde{L}_0)^{-1}$ exists and is continuous. Now we define

$$\Phi_\lambda^0 : C_2 \to L^2 \times \mathbf{R}^{m \cdot n} \times \mathbf{R}^{m \cdot n}$$

by

$$\Phi_\lambda^0(\{u_{k_1} v_k\}_{k=0}^m) = \lambda \cdot \Gamma(\{u_k, v_k\}_{k=0}^m) \times (\{I_k^{-1}(u_k(t_k))\}_{k=1}^m \\ \times \{\lambda N_k(u_{k-1}(t_k), v_{k-1}(t_k))\}.$$

We consider the following diagram:



The problem (2.3) is equivalent to the fixed point problem

$$w \in \mathcal{F}_1^0(w), \quad w \in C_2, \quad w = \{(u_k, v_k)\}_{k=0}^m$$

where

$$\mathcal{F}_\lambda := j \circ (\tilde{L}^0)^{-1} \circ \Phi_\lambda^0.$$

The remaining part of the proof is similar to the proof of Theorem (2.2). □

3. Some existence results for impulsive systems on the interval $[0, \infty]$**.** In this section we discuss boundary value problems for differential inclusions on $[0, \infty)$. We note that such problems for systems without impulses have been considered (using similar methods) in [**12, 13, 14, 18, 23**]; (see also [**8, 9, 33**]).

Let $F : [0,\infty) \times \mathbf{R}^n \times \mathbf{R}^n \to 2^{\mathbf{R}^n}$ be a multifunction. We assume that F is *admissible*, i.e., $F|_{[0,r]}$ is admissible for all $r > 0$. In this section we study the problem of existence of a solution to the following second order impulsive differential system on the interval $[0, \infty)$:

$$(3.1)\qquad \begin{cases} y'' \in F(t, y, y') & \text{for a.e. } t \in [0,\infty) \\ y(t_k^+) = I_k(y(t_k)) & \\ y'(t_k^+) = N_k(y(t_k), y'(t_k)), & k = 1, 2, \dots \\ G(y(0), y'(0)) = 0 & \end{cases}$$

where $\lim_{k\to\infty} t_k = \infty$ and $t_{k+1} > t_k > 0$, $k = 1, 2, \dots$; $I_k : \mathbf{R}^n \to \mathbf{R}^n$ is a homeomorphism; $N_k : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ is a continuous map, $k = 1, 2, \dots$. We put

$$R_{i,i+k} = \sup\{\|x\| : x \in (I_i^{-1} \circ \dots \circ I_{i+k}^{-1})(B(0,R))\}$$
$$S_{i,i+k} = \sup\{\|x\| : x \in (I_{i+k} \circ \dots \circ I_i)(B(0,R))\}.$$

We make the following assumption

$$(*)\qquad \tilde{R} = \sup\{R_{i,i+k}, S_{i,i+k} : i = 1, 2, \dots, k = 0, 1, 2, \dots\} < \infty.$$

Let us suppose that $R > 0$ is a fixed number. We put $M = \max\{R, \tilde{R}\}$. Let us denote by $(\mathcal{B})$ one of the following boundary conditions:

(B1) $y(0) = r$; $\|r\| \le R$

(B2) $Ay(0) - By'(0) = r$, where A and B are symmetric nonnegative definite $n \times n$ matrices such that if $r = 0$ then at least one of these matrices is nonsingular; otherwise, both of them are nonsingular and $\|B^{-1}\|\|A^{-1}B\|\|r\| \le R$.

(B3) $G(y(0), y'(0)) = 0$, where $G : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ is a continuous function which satisfies one of the conditions (N1) or (N3) for $i = 0$, where $a_0 = 0$.

(B4) $y'(0) = 0$.

(B5) $G(y(0), y'(0)) = 0$, where $G : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ is a continuous function which satisfies the condition (N2) for $i = 0$, where $a_0 = 0$.

Let us remark that the condition (B1), for $r \neq 0$, is a special case of the condition (B3) and (B4) is a special case of (B5).

We shall state the following hypotheses:

(H1)$'$ If $\|y_0\| > R$ and $y_0 \cdot y_0' = 0$, then for all compact subsets $[0,r] \subset [0,\infty)$, $r > 0$, there is a $\delta > 0$ such that

$$\operatorname*{ess\,inf}_{t\in[0,r]} \inf\{y \cdot w + \|y'\|^2 : w \in F(t,y,y'), (y,y') \in D_\delta\} > 0$$

where $D_\delta := \{(y,y') : \|y - y_0\| + \|y' - y_0\| < \delta\}$.

(H2)$'$ There exists a function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi \in L^\infty_{\mathrm{loc}}[0,\infty)$,

$$\frac{s}{\varphi(s)} \in L^\infty_{\mathrm{loc}}[0,\infty), \qquad \int_0^\infty \frac{sds}{\varphi(s)} = \infty$$

and

$$\|F(t,y,y')\| \le \varphi(\|y'\|) \quad \text{for a.e. } t \in [0,\infty) \quad \text{and all } (y,y') \in D,$$

where $D = \{(x,x') : \|x\| \le R\}$.

(H3)$'$ There exist constants α and k such that

$$\|F(t,y,y')\| \le 2\alpha(y \cdot w + \|y'\|^2) + k \quad \text{for a.e. } t \in [0,\infty)$$
$$\text{and all } (y,y') \in D \text{ and } w \in F(t,y,y').$$

We assume also that the impulses N_k and I_k satisfy the condition (IN) for all $k = 1,2,\dots$, i.e.,

(IN) For every $k = 1,2,\dots$, if $y \cdot y' \ge 0$, then $I_k(y) \cdot N_k(y,y') \ge 0$, provided $\|y\| > R$.

(3.1) Theorem. *Under the hypotheses* (H1)$'$, (H2)$'$, (H3)$'$ *and* (IN), *the following system of impulsive differential equations*

$$\text{(3.3)} \qquad \begin{cases} y'' \in F(t,y,y') & \text{for a.e. } t \in [0,\infty) \\ y(t_k) = I_k^{-1}(y(t_k^+)) \\ y'(t_k^+) = N_k(y(t_k), y'(t_k)), & k = 1,2,\dots \\ y \in \mathcal{B} \end{cases}$$

has a bounded solution.

Proof. We consider the following family of impulsive differential inclusions:

$$(3.3_\mu) \qquad \begin{cases} y'' \in F(t,y,y') & \text{for a.e. } t \in [0,\mu] \\ y(t_k) = I_k^{-1}(y(t_k^+)) \\ y'(t_k^+) = N_k(y(t_k), y'(t_k)), & 0 < t_1 < t_2 < \cdots < t_m < \mu \\ y \in \mathcal{B}_\mu \end{cases}$$

where $\mathcal{B}_\mu$ denotes the set of all functions satisfying either (B1), (B2) or (B3) and the condition $y(\mu) = 0$, or (B4) and (B5) and the condition $y'(\mu) = 0$. It follows from Theorem (2.2) that (3.3_μ) has a solution $y_\mu \in H_\mu^2 := H^2$, where we assume $a_0 = 0$, $a_1 = \mu$. It is clear that for every $\mu \in \mathbf{N}$ the sequence $\{y_{\mu+k}\}_{k=1}^\infty$, restricted to the space H_μ^2 is bounded and thus it contains a subsequence convergent in the C^1-norm. By standard arguments there exists a subsequence $\{y_{\mu(k)}\} \subset \{y_\mu\}$ such that $y_{\mu(k)|[0,\mu]} \to y_{|[0,\mu]}$ in the C^1-topology for all $\mu \in \mathbf{N}$ and

$$y'' \in F(t,y,y') \quad \text{for a.e. } t \in [0,\infty)$$

(see [**12**] for details). It is also clear that y will satisfy the condition

$$y(t_k) = I_k^{-1}(y(t_k^+))$$

and

$$y'(t_k^+) = N_k(y(t_k), y'(t_k)) \quad \text{for all } k = 1, 2, \dots .$$

Now, since $\varphi \in L_{\text{loc}}^\infty[0,\infty)$, then

$$\|y''(t)\| \le \varphi(\|y'(t)\|) \le \text{ess sup}\,\{\varphi(s) : s \in [0, \|y'\|_0]\} < \infty$$

and this proves that y is bounded. □

Acknowledgment. The authors would like to acknowledge several helpful discussions with Professor B. Von Hohenbalken of the University of Alberta.

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