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RIMCOMPACTNESS AND SIMILAR PROPERTIES IN PREIMAGES

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ABSTRACT. It is shown that the properties of rimcompactness, almost rimcompactness and having a compactification with totally disconnected remainder are preserved in preimages under closed maps whose point preimages have the appropriate property and have compact zero-dimensional boundary. Examples indicate that the hypotheses on the maps cannot be weakened significantly.

All spaces in this paper will be completely regular and Hausdorff. A space X is *rimcompact* if X has a base of open sets with compact boundaries, *almost rimcompact* if X has a compactification KX in which each point of the remainder $KX\backslash X$ has a base of open sets of KX whose boundaries lie in X, and *TDI* (for totally disconnected at infinity) if X has a compactification with totally disconnected remainder. Note that an almost rimcompact space is TDI.

Various authors have considered the preservation of the above properties in images and preimages, with an interesting duality developing. "Map" will mean continuous surjection; a map $f : X \to Y$ is *perfect* (*rimperfect*, respectively) if f is closed and $f^{\leftarrow}(y)$ ($bd_x f^{\leftarrow}(y)$ respectively) is compact for $y \in Y$, and *monotone* if $f^{\leftarrow}(y)$ is connected for $y \in Y$. It is shown in [**1**] that rimcompactness is preserved in images under rimperfect monotone maps. Following on this work, we showed in [**3**] that the properties of almost rimcompactness and having a compactification with 0-dimensional remainder are preserved in rimperfect monotone images. Also, the properties of rimcompactness and almost rimcompactness are preserved in preimages of such spaces under rimperfect maps whose point preimages are zero-dimensional. Since the perfect monotone preimage of a rimcompact space need not be rimcompact [**4**], zero-dimensionality in some form is important in this last result. In this paper we prove the stronger result stated in the abstract.

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We denote by $\mathbf{C}(\beta X)$ the decomposition of βX consisting of $\{\{x\} : x \in X\} \cup \{C : C$ is a connected component of $\beta X \backslash X\}$. It is known that X is TDI if and only if $\mathbf{C}(\beta X)$ consists of compact sets and is an upper semicontinuous decomposition of βX, in which case $\beta X / \mathbf{C}(\beta X)$ yields the largest compactification of X with totally disconnected remainder [**9**, or **8**]. If X is TDI, then X is almost rimcompact if and only if each connected component of $\beta X \backslash X$ has a base in βX of open sets whose boundaries lie in X, in which case the compactification $\beta X \backslash \mathbf{C}(\beta X)$ is the largest compactification of X witnessing the fact that X is almost rimcompact [**7, 2**] . If X is TDI, $F_0 X$ will denote $\beta X / \mathbf{C}(\beta X)$.

Although the result for rimcompact spaces can be written as a corollary to the result for the larger classes of almost rimcompact and TDI spaces, there is a straightforward proof of the result in this case that makes more direct use of the structure of rimcompact spaces. In the following, an open set U of X is *π-open* in X if $bd_X U$ is compact.

Theorem 1. *Suppose that $f : X \to Y$ is rimperfect, and for $y \in Y$, $f^{\leftarrow}(y)$ is rimcompact and has 0-dimensional boundary in X. If Y is rimcompact, then X is rimcompact.*

Proof. If $x \in \mathrm{int}_X f^{\leftarrow}(y)$ for some $y \in Y$, x has a base in X of π-open sets. Suppose then that $x \in bd_X f^{\leftarrow}(y)$ for some $y \in Y$ and that $x \notin T$ for T closed in X. Let $T' = T \cap bd_X f^{\leftarrow}(y)$; since $bd_X f^{\leftarrow}(y)$ is zero-dimensional and compact, there are open sets U_1 and U_2 of X with $x \in U_1$, $T' \subseteq U_2$, $\mathrm{cl}_X U_1 \cap \mathrm{cl}_X U_2 = \varnothing$, and $bd_X f^{\leftarrow}(y) \subseteq U_1 \cup U_2$. Without loss of generality, $U_1 \cap T = \varnothing$. Now $f^{\leftarrow}(y) \subseteq U_1 \cup U_2 \cup \mathrm{int}_X f^{\leftarrow}(y)$, f is closed and Y is rimcompact, so there is a π-open set W of Y with $f^{\leftarrow}(y) \subseteq f^{\leftarrow}[W] \subseteq f^{\leftarrow}[\mathrm{cl}_Y W] \subseteq U_1 \cup U_2 \cup \mathrm{int}_X f^{\leftarrow}(y)$. According to Lemma 3 of [**7**], $f^{\leftarrow}[W]$ is π-open in X. Now $f^{\leftarrow}(y) \cap [T \backslash (U_1 \cup U_2)]$ is closed in $f^{\leftarrow}(y)$ and $f^{\leftarrow}(y) \cap [T \backslash (U_1 \cup U_2)] \cap bd_X f^{\leftarrow}(y) = \varnothing$. Since $f^{\leftarrow}(y)$ is rimcompact, there is a π-open set V of $f^{\leftarrow}(y)$ with $bd_X f^{\leftarrow}(y) \subseteq V$ and $\mathrm{cl}_X V \cap f^{\leftarrow}(y) \cap [T \backslash (U_1 \cup U_2)] = \varnothing$. The sets $V' = V \cup (X \backslash f^{\leftarrow}(y))$ and $V' \cap f^{\leftarrow}[W]$ are π-open in X. Because $V' \cap f^{\leftarrow}[W] \subseteq U_1 \cup U_2$, and $\mathrm{cl}_X U_1 \cap \mathrm{cl}_X U_2 = \varnothing$, $V' \cap f^{\leftarrow}[W] \cap U_1$ is π-open in X and is the desired neighborhood of x in X. □

For a map $f : X \to Y$, the extension map from βX into βY will be denoted by f^β, and the composition of $f^\beta : \beta X \to \beta Y$ and $g : \beta Y \to F_0 Y$ by $f^0 : \beta X \to F_0 Y$.

Theorem 2. *Suppose that $f : X \to Y$ is rimperfect and that for $y \in Y$, $f^{\leftarrow}(y)$ is a TDI space with 0-dimensional boundary in X. If Y is a TDI space, $\mathbf{C}(\beta X)$ is an upper semicontinuous decomposition of βX consisting of compact sets. That is, X is a TDI space.*

Proof. We first show that $\mathbf{C}(\beta X)$ is a decomposition of βX into compact sets. Since f is a closed map, it follows from 1.1 of [**5**] that $f^{\beta\leftarrow}(y) = \mathrm{cl}_{\beta X} f^{\leftarrow}(y)$ for $y \in Y$. According to the corollary to Lemma 1 of [**9**], $bd_{\beta X}\mathrm{cl}_{\beta X} f^{\leftarrow}(y) = \mathrm{cl}_{\beta X} bd_X f^{\leftarrow}(y)$. Since $bd_X f^{\leftarrow}(y)$ is compact for $y \in Y$, $bd_{\beta X}\mathrm{cl}_{\beta X} f^{\leftarrow}(y) = bd_X f^{\leftarrow}(y)$, so that $\mathrm{cl}_{\beta X} f^{\leftarrow}(y) \cap (\beta X \backslash X) = \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} f^{\leftarrow}(y) \cap (\beta X \backslash X)$ and is clopen in $\beta X \backslash X$. That $bd_X f^{\leftarrow}(y)$ is compact also implies that $f^{\leftarrow}(y)$ is C^*-embedded in X, hence $\mathrm{cl}_{\beta X} f^{\leftarrow}(y) = \beta(f^{\leftarrow}(y))$; in particular the connected components of $\mathrm{cl}_{\beta X} f^{\leftarrow}(y) \backslash f^{\leftarrow}(y)$ are the connected components of $\beta(f^{\leftarrow}(y))$.

Let C be a connected component of $\beta X \backslash X$. If $C \cap f^{\beta\leftarrow}(y) \neq \varnothing$ for some $y \in Y$, then $C \subseteq f^{\beta\leftarrow}(y)$ and thus is a connected component of $f^{\beta\leftarrow}(y) \backslash f^{\leftarrow}(y)$ and is compact. If $C \subseteq f^{\beta\leftarrow}[\beta Y \backslash Y]$, then the set $f^\beta[C]$ is a connected subset of $\beta Y \backslash Y$ and thus is contained in some compact connected component D of $\beta Y \backslash Y$. Then $C \subseteq f^{\beta\leftarrow}[D]$, and as a connected component of the compact set $f^{\beta\leftarrow}[D]$, is compact.

We now show that each element of $\mathbf{C}(\beta X)$ has a base in βX of open sets saturated with respect to $\mathbf{C}(\beta X)$. For $y \in Y$, $f^{\beta\leftarrow}(y)$ is saturated with respect to $\mathbf{C}(\beta X)$. Also, $f^{\leftarrow}(y)$ is a TDI space, and hence $\mathbf{C}(\beta(f^{\leftarrow}(y)))$ is an upper semicontinuous decomposition of $\beta(f^{\leftarrow}(y))$; collapsing connected components of $\beta X \backslash X$ whose intersection with $f^{\beta\leftarrow}(y)$ is nonempty is equivalent to collapsing connected components of $f^{\beta\leftarrow}(y) \backslash f^{\leftarrow}(y) = \beta(f^{\leftarrow}(y)) \backslash f^{\leftarrow}(y)$. This implies that if $P \in \mathbf{C}(\beta X)$ is contained in $\mathrm{int}_{\beta X} f^{\beta\leftarrow}(y)$, P has a base of open sets of βX saturated with respect to $\mathbf{C}(\beta X)$. Also, the above comments imply that if T is closed in $f^{\leftarrow}(y)$ with $T \cap bd_X f^{\leftarrow}(y) = \varnothing$, the set

$S = \cup\{C \in \mathbf{C}(\beta X) : C \cap \mathrm{cl}_{\beta X} T \neq \varnothing\}$ is a closed subset of βX saturated with respect to $\mathbf{C}(\beta X)$. Then $\beta X \backslash S$ is an open saturated subset of βX

with $bd_X f^{\leftarrow}(y) \subseteq \beta X \backslash S$ and $T \cap (\beta X \backslash S) = \varnothing$, a fact we shall use later to show that any $x \in bd_X f^{\leftarrow}(y)$ has a base of saturated open sets.

If C is a connected component of $\beta X \backslash X$ contained in $\beta X \backslash f^{\beta \leftarrow}[Y]$, then $f^0[C]$ is a connected subset of the 0-dimensional space $F_0 Y \backslash Y$, hence $|f^0[C]| = 1$. It follows that if W is open in $F_0 Y$, $f^{0\leftarrow}[W]$ is saturated with respect to $\mathbf{C}(\beta X)$. Suppose then that C is a connected component of $\beta X \backslash X$ contained in $\beta X \backslash f^{\beta \leftarrow}[Y]$, and that $C \cap T = \varnothing$, where T is closed in βX. If D is the connected component of $\beta Y \backslash Y$ with $f^{\beta}[C] \subseteq D$, let $T' = T \cap f^{\beta \leftarrow}[D]$. Now Y is TDI, so that D, and thus $f^{\beta \leftarrow}[D]$, is compact. Since C is a connected component of $f^{\beta \leftarrow}[D]$, there is a clopen set K of $f^{\beta \leftarrow}[D]$ such that $C \subseteq K$ and $K \cap T' = \varnothing$. Choose open sets U_1 and U_2 of βX having disjoint closures and such that $K \subseteq U_1$ while $f^{\beta \leftarrow}[D] \backslash K \subseteq U_2$. Without loss of generality, $U_1 \cap T = \varnothing$. Now f^0 is a closed map, and $f^{0\leftarrow}[g[D]] \subseteq U_1 \cup U_2$, thus there is an open set W of $F_0 Y$ with $f^{\beta \leftarrow}[D] = f^{0\leftarrow}[g[D]] \subseteq f^{0\leftarrow}[W] \subseteq U_1 \cup U_2$. If $V = f^{0\leftarrow}[W] \cap U_1$, then $C \subseteq V \subseteq \beta X \backslash T$. Because the only nontrivial elements of $\mathbf{C}(\beta X)$ are connected sets, $f^{0\leftarrow}[W]$ is saturated, and since $U_1 \cap U_2 = \varnothing$, V is saturated with respect to $\mathbf{C}(\beta X)$.

Finally, suppose that $x \in bd_X f^{\leftarrow}(y)$ for some $y \in Y$, and that $x \notin T$, where T is closed in βX. Once again, let $T' = T \cap bd_X f^{\leftarrow}(y)$; since $bd_X f^{\leftarrow}(y)$ is 0-dimensional and compact, there are open sets U_1 and U_2 of βX with $x \in U_1$, $T' \subseteq U_2$, $\mathrm{cl}_{\beta X} U_1 \cap \mathrm{cl}_{\beta X} U_2 = \varnothing$, and $bd_X f^{\leftarrow}(y) \subseteq U_1 \cup U_2$. Without loss of generality, $U_1 \cap T = \varnothing$. Now $bd_{\beta X} f^{\beta \leftarrow}(y) = bd_{\beta X} \mathrm{cl}_{\beta X} f^{\leftarrow}(y) = bd_X f^{\leftarrow}(y)$, so that $f^{\beta \leftarrow}(y) \subseteq U_1 \cup U_2 \cup \mathrm{int}_{\beta X} f^{\beta \leftarrow}(y)$. Since $f^{\beta \leftarrow}(y) = f^{0\leftarrow}(y)$, there is W open in $F_0 Y$ with $f^{0\leftarrow}(y) \subseteq f^{0\leftarrow}[W] \subseteq U_1 \cup U_2 \cup \mathrm{int}_{\beta X} f^{\beta \leftarrow}(y)$. As previously discussed, it is possible to choose V saturated and open in βX so that $x \in V$ and $V \cap [f^{\beta \leftarrow}(y) \backslash (U_2 \cup U_2)] = \varnothing$. The set $V \cap f^{0\leftarrow}[W] \cap U_1$ is a saturated open neighborhood of x having empty intersection with T. □

The result for Y almost rimcompact follows easily.

Corollary 1. *Suppose that $f : X \to Y$ is rimperfect, and for $y \in Y$, $f^{\leftarrow}(y)$ is almost rimcompact with 0-dimensional boundary. If Y is almost rimcompact, then X is almost rimcompact.*

Proof. According to Theorem 2, X is TDI, so that it suffices to show that each connected component C of $\beta X \backslash X$ has a base in βX of open sets whose boundaries lie in X. Recall that $bd_{\beta X} f^{\beta \leftarrow}(y) = bd_X f^{\leftarrow}(y) \subseteq X$ for $y \in Y$. If $C \cap f^{\beta \leftarrow}(y) \neq \varnothing$ for some $y \in Y$, then C is a connected component of $f^{\beta \leftarrow}(y) \backslash f^{\leftarrow}(y) = \beta(f^{\leftarrow}(y)) \backslash f^{\leftarrow}(y)$. The set $f^{\leftarrow}(y)$ is almost rimcompact, thus C has a base in $f^{\beta \leftarrow}(y)$ of open sets whose boundaries lie in $f^{\leftarrow}(y)$. Since $C \subseteq \text{int}_{\beta X} f^{\beta \leftarrow}(y)$, C has the desired base in βX.

If $C \subseteq \beta X \backslash f^{\beta \leftarrow}[Y]$, then $f^0[C]$, as a single point of $F_0 Y \backslash Y$, has a base in $F_0 Y$ of open sets whose boundaries lie in Y. But if W is open in $F_0 Y$ with $bd_{F_0 Y}[W] \subseteq Y$, then $bd_{\beta X} f^{0 \leftarrow}[W] \subseteq f^{0 \leftarrow}[bd_{F_0 Y} W] \backslash \cup \{\text{int}_{\beta X} f^{0 \leftarrow}(y) : y \in bd_{F_0 Y} W\} = \cup \{bd_{\beta X} f^{0 \leftarrow}(y) : y \in bd_{F_0 Y} W\} = \cup \{bd_{\beta X} f^{\beta \leftarrow}(y) : y \in bd_{F_0 Y} W\} \subseteq X$. □

As was mentioned earlier, the perfect monotone preimage of a rimcompact space need not be rimcompact. Examples 3.1–3.3 of [**3**] indicate additional limitations in weakening the hypotheses of the above results.

We cannot show that Theorem 2 holds if 'rimcompact' is replaced everywhere by 'has a compactification with 0-dimensional remainder,' even in the case in which $f^{\leftarrow}(y)$ is 0-dimensional for $y \in Y$.

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