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POLYNOMIAL INTERPOLATION OF HOLOMORPHIC FUNCTIONS IN C AND $\mathbf{C}^n$

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0. Introduction. This paper is concerned with polynomial interpolation, particularly Lagrange interpolation, of functions holomorphic in a neighborhood of a polynomially convex nonpluripolar compact set $K \subset \mathbf{C}^n$. The general framework is as follows: let $h_d = h_d(n) \equiv \binom{n+d}{d}$ denote the dimension of the complex vector space P_d of holomorphic polynomials of degree at most d in n variables $z_1, \ldots, z_n$. For each positive integer d, we choose h_d points $A_{d1}, \ldots, A_{dh_d}$ in K; thus we get a doubly-indexed array $\{A_{dj}\}_{\substack{j=1,\ldots,h_d \\ d=1,2,\ldots}}$ of points in K. Given a function f holomorphic in a neighborhood of K, we would like to know under what conditions on the array do the Lagrange interpolation polynomials $L_d f$ associated to f and $\{A_{dj}\}$ converge uniformly to f on K. In one variable ($n = 1$), Walsh [**23**] proved a necessary and sufficient condition on the array $\{A_{dj}\}$ in order to guarantee uniform convergence of the sequence $\{L_d f\}$ to f on K for each such f (see 1.4). In theorem 1.5 we give several other conditions—they are not equivalent to Walsh's condition and in theorem 1.5 we give the precise relation between the various conditions.

The proof of Walsh's condition depends on the Hermite remainder formula (1.3). No such simple formula is available in the case $n > 1$. We will show, via several examples, that many analogues of the one variable results do not hold. Theorem 4.1 summarizes our knowledge of the generalization of Theorem 1.5 to several variables. The results are not as complete as the one variable case (see, in particular, Problem 5.5).

Finally, in Section 5 we list a few open questions on polynomial interpolation in $\mathbf{C}^n$.

1. One-variable case. 1.1 Let $K \subset \mathbf{C}$ be compact, nonpolar, and polynomially convex. For simplicity, we assume K is regular for the

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exterior Dirichlet problem, although most of the results below remain true without this assumption. Since $n = 1$, $h_d = h_d(1) = d + 1$ and we can take as a basis for P_d the standard monomials $1, z, z^2, \ldots, z^d$. We denote a triangular array of points in K by $\{z_{dj}\}_{\substack{j=0,1,\ldots,d \\ d=1,2,\ldots}}$, i.e., for each degree d, we have $d+1$ points $z_{d0}, \ldots, z_{dd}$. Provided these points are distinct, the associated Vandermonde determinant

$$V(z_{d0}, \ldots, z_{dd}) \equiv \det [z_{dj}^k]_{j,k=0,\ldots,d} = \det \begin{bmatrix} 1 & \cdots & 1 \\ z_{d0} & \cdots & z_{dd} \\ \vdots & & \vdots \\ z_{d0}^d & \cdots & z_{dd}^d \end{bmatrix} \tag{1.1.1}$$

is nonzero and the *fundamental Lagrange interpolating polynomials* (FLIP)

$$l_j^{(d)}(z) \equiv \frac{V(z_{d0}, \ldots, \overset{\swarrow (j+1)\text{th}}{z}, \ldots, z_{dd})}{V(z_{d0}, \ldots, z_{dd})} = \begin{cases} 1 & \text{if } z = z_{dj} \\ 0 & \text{if } z = z_{dk},\ k \neq j \end{cases} \tag{1.1.2}$$

are well defined; this allows us to write down the *Lagrange interpolating polynomial* (LIP) L_d of degree d associated to a function f and the array $\{z_{dj}\}$ as

$$L_d f(z) \equiv \sum_{j=0}^{d} f(z_{dj}) l_j^{(d)}(z), \qquad d = 1, 2, \ldots \tag{1.1.3}$$

so that

$$L_d f(z_{dj}) = f(z_{dj}), \qquad j = 0, 1, \ldots, d. \tag{1.1.4}$$

In order to determine how well the polynomials $L_d f$ approximate f at other points in K, we recall the following concepts. The d^{th} *Lebesgue function* of the array $\{z_{dj}\}$ is the 'piecewise' polynomial

$$\Lambda_d(z) \equiv \sum_{j=0}^{d} |l_j^{(d)}(z)|. \tag{1.1.5}$$

The supremum norm of this function on K is called the d^{th} *Lebesgue constant* (by abuse of notation we use the same letter for the function $\Lambda_d(z)$ and the constant Λ_d), i.e.,

$$\Lambda_d = ||\Lambda_d||_K \equiv \sup\{|\Lambda_d(z)| : z \in K\}. \tag{1.1.6}$$

This number is the operator norm of the projection from the Banach space of functions continuous on K to the space P_d (with the sup norm). Thus the slower the rate of growth of the sequence $\{\Lambda_d\}$, the better the array is for uniform approximation. One cannot, however, expect the Lebesgue constants to be bounded. In fact, a result of Faber-Bernstein [**15**] shows that there is a constant $c > 0$ such that, for any array in $K = [0,1]$ we have $\Lambda_d \geq c \log d$.

The asymptotic behavior of the Vandermonde determinants $V(z_{d0}, \ldots, z_{dd})$ as $d \mapsto +\infty$ also give an indication of the approximation properties of the LIPs $L_d f$. For an arbitrary set of $d+1$ points $z_0, \ldots, z_d$ in K, $V(z_0, \ldots, z_d)$ is a polynomial in $d+1$ variables of degree $\sum_{k=1}^{d} k = \binom{d+1}{2}$; the number

$$D_{d+1}(K) = \max_{z_0, \ldots, z_d \in K} |V(z_0, \ldots, z_d)|^{\frac{1}{\binom{d+1}{2}}} \equiv V_{d+1}^{\frac{1}{\binom{d+1}{2}}} \tag{1.1.7}$$

is called the $d+1^{\text{st}}$ *order diameter* of K. It is easy to see that the sequence of numbers $\{D_{d+1}(K)\}_{d=1,2,\ldots}$ is nonincreasing; since $D_2(K) =$ 'ordinary' diameter of K and K is compact, the sequence has a limit

$$D(K) \equiv \lim_{d \to +\infty} D_{d+1}(K) \tag{1.1.8}$$

called the *transfinite diameter* of K (see, e.g., [**11**] or [**7**] for details). The assumption that K be nonpolar is equivalent to $D(K) > 0$; this implies that $D_{d+1}(K) > 0$ for each d and thus ensures the existence of triangular arrays $\{z_{dj}\}$ with $V(z_{d0}, \ldots, z_{dd}) \neq 0$. The relationship between the limit points of the sequence of numbers $\{|V(z_{d0}, \ldots, z_{dd})|^{\frac{1}{\binom{d+1}{2}}}\}_{d=1,2,\ldots}$ and $D(K)$ will be useful in what follows. We remark that the above definition of $D(K)$ is equivalent to the following notion of *Chebyshev constant* $T(K)$; for each $d = 1, 2, \ldots$, let P_d^M denote the class of monic polynomials of degree d, i.e.,

$$P_d^M = \{p_d(z) = z^d + a_1 z^{d-1} + \cdots + a_d; a_1, \ldots, a_d \in \mathbf{C}\}. \tag{1.1.9}$$

Set $T_d = T_d(K) \equiv [\inf\{||p_d||_K : p_d \in P_d^M\}]^{1/d}$, then $\lim_{d\to+\infty} T_d(K) \equiv T(K)$ exists; from the simple inequalities $T_d^d \le (V_d/V_{d-1}) \le dT_d^d$ (see, e.g., [**11**]), it follows that $T(K) = D(K)$.

Finally, given an array $\{z_{dj}\}$, we can form, for each d, the discrete probability measure $\mu_d \equiv (1/d+1)\sum_{j=0}^{d}[z_{dj}]$ where $[z_{dj}]$ denotes the unit Dirac measure at the point z_{dj}. We will study the set of weak-* limits of the family $\{\mu_d\}_{d=1,2,\dots}$ and the relationship with the *equilibrium measure* μ_K on K, i.e., the unique probability measure on K which minimizes the *energy integrals* $I(\mu) \equiv \int_K\int_K \log(1/|z-\xi|)\, d\mu(\xi)\, d\mu(z)$ over all probability measures μ supported on K. The associated *logarithmic potential function* $p_{\mu_K}(z) \equiv \int_K \log(1/|z-\xi|)\, d\mu_K(\xi)$, $z \in \mathbf{C}$, associated to μ_K is then a superharmonic function which is harmonic off K and is equal to the constant value $-\log D(K) = I(\mu_K)$ on K; indeed, $-p_{\mu_K}(z) - \log D(K) = u_K(z)$, the *Green function* for the complement of K, which will be defined in Theorem 1.4. We remark that

$$I(\mu_K) = \sup_{z\in K} p_{\mu_K}(z) < \sup_{z\in K} p_\mu(z) \tag{1.1.10}$$

for any other probability measure μ supported on K.

1.2 As a concrete example of a natural array of points $\{z_{dj}\}$, we can take, for each $d = 1, 2, \dots$, a set of $d+1$ *Fekete points*, i.e., points $z_{d0}, \dots, z_{dd}$ which achieve the maximum in (1.1.7) (such points are in general not unique). By construction

$$\lim_{d\to+\infty} |V(z_{d0}, \dots, z_{dd})|^{\frac{1}{\binom{d+1}{2}}} = D(K); \tag{A}$$

by definition of Fekete points, each FLIP satisfies $|l_j^{(d)}(z)| \le 1$ on K so that $\Lambda_d \le d+1$; hence

$$\lim_{d\to+\infty} \Lambda_d^{1/d} = 1. \tag{B}$$

Finally, it will follow from the proof of Theorem 1.5 that

$$\lim_{d\to+\infty} \mu_d = \mu_K (\text{weak-}^*). \tag{C}$$

Each of these three conditions (A), (B), (C) will imply that for any f holomorphic in a neighborhood of K, the LIPs $\{L_d f\}$ converge uniformly to f on K, (we write $L_d f \rightrightarrows f$ on K). A direct proof of this fact for arrays of Fekete points was proved by Shen [**19**]; his proof, as well as Walsh's more general result (to be stated below), follow from Hermite's remainder formula.

1.3 Hermite's Remainder Formula (HRF): Given $d+1$ points $z_{d0}, \dots, z_{dd} \in \mathbf{C}$, let Γ be a contour surrounding these points. If f is any function analytic in a neighborhood of the region bounded by Γ, then

$$f(z) = L_d f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{w_d(z)}{w_d(t)} \frac{f(t)}{t-z}\, dt \tag{1.3.1}$$

for z inside Γ where $w_d(z) \equiv (z - z_{d0}) \cdots (z - z_{dd})$.

HRF is an easy consequence of the residue theorem; however, a more enlightening proof is to notice that for any point $t \notin \{z_{d0}, \dots, z_{dd}\}$, the integrand $(w_d(z)/w_d(t))(1/(t-z))$ is exactly the difference between $g(z) \equiv (1/t - z)$ and its LIP $L_d g(z)$ at the point z; multiplying by $f(t)$ and integrating around Γ then yields the general formula. Clearly then, the behavior of the polynomials $w_d(z)$ associated with the array $\{z_{dj}\}$ determine the approximation properties of LIPs. We now state, without proof, the fundamental result of Walsh [**10**].

Theorem 1.4. (Walsh) *Let $K \subset \mathbf{C}$ be compact, nonpolar, polynomially convex and regular. Let $\{z_{dj}\}$ be an array of points in K. Then for any function f which is holomorphic in a neighborhood of K (which may depend on f), we have $L_d f \rightrightarrows f$ on K if and only if*

$$\lim_{d \to +\infty} ||w_d||_K^{\frac{1}{d+1}} = D(K) \tag{1.4.1}$$

or, equivalently,

$$\lim_{d \to +\infty} |w_d(z)|^{\frac{1}{d+1}} = D(K) e^{u_K(z)} \tag{1.4.2}$$

uniformly on compact subsets of the complement of K where

$$\begin{aligned} u_K(z) &= \sup \left\{ \frac{1}{\deg(p)} \log \frac{|p(z)|}{||p||_K} : p \textit{ polynomial} \right\} \\ &= \lim_{d \to +\infty} \sup \left\{ \frac{1}{d} \log \frac{|p_d(z)|}{||p_d||_K} : p_d \in P_d \right\} \end{aligned} \tag{1.4.3}$$

is the Green function for the complement of K with logarithmic pole at ∞.

We now state the main theorem of this section.

Theorem 1.5. *Let $K \subset \mathbf{C}$ be compact, nonpolar, polynomially convex, and regular. Let $\{z_{dj}\}$ be an array of points in K. Consider the following four properties which an array may or may not possess.*

1. $\lim_{d\to+\infty} \Lambda_d^{1/d} = 1$
2. $\lim_{d\to+\infty} |V(z_{d0},\dots,z_{dd})|^{\frac{1}{\binom{d+1}{2}}} = D(K)$
3. $\lim_{d\to+\infty} \mu_d = \mu_K$ *weak-**
4. $L_d f \rightrightarrows f$ *on K for each f holomorphic on a neighborhood of K.*

Then

I. $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$.

II. *None of the reverse implications in* I *are true.*

Proof. I. **1.6** ($1 \Rightarrow 2$) (see Bos [**6**]). Let $V_{d+1} = |V(\xi_{d0},\dots,\xi_{dd})|$, i.e., $\xi_{d0},\dots,\xi_{dd}$ are a set of $d+1$ Fekete points for K. Then $p_d^{(0)}(\xi) \equiv V(\xi,\xi_{d1},\dots,\xi_{dd})$ is a polynomial of degree d in ξ; hence

$$p_d^{(0)}(\xi) = L_d p_d^{(0)}(\xi) = \sum_{j=0}^{d} p_d^{(0)}(z_{dj}) l_j^{(d)}(\xi), \tag{1.6.1}$$

so, in particular, at $\xi = \xi_{d0}$, we have

$$p_d^{(0)}(\xi_{d0}) = V(\xi_{d0},\dots,\xi_{dd}) = \sum_{j=0}^{d} V(z_{dj},\xi_{d1},\dots,\xi_{dd}) l_j^{(d)}(\xi_{d0}) \tag{1.6.2}$$

so that

$$V_{d+1} = |V(\xi_{d0},\dots,\xi_{dd})| \le \Lambda_d \max_{j=0,\dots,d} |V(z_{dj},\xi_{d1},\dots,\xi_{dd})|. \tag{1.6.3}$$

For simplicity in notation, assume $\max_{j=0,\dots,d} |V(z_{dj},\xi_{d1},\dots,\xi_{dd})| = |V(z_{d0},\xi_{d1},\dots,\xi_{dd})|$ so that

$$V_{d+1} \le \Lambda_d |V(z_{d0},\xi_{d1},\dots,\xi_{dd})|. \tag{1.6.4}$$

Now $p_d^{(1)}(\xi) \equiv V(z_{d0}, \xi, \xi_{d2}, \dots, \xi_{dd})$ is a polynomial of degree d in ξ; repeating the above argument we get

$$|V(z_{d0}, \xi_{d1}, \xi_{d2}, \dots, \xi_{dd})| \leq \Lambda_d |V(z_{d0}, z_{d1}, \xi_{d2}, \dots, \xi_{dd})| \tag{1.6.5}$$

so that

$$V_{d+1} \leq \Lambda_d^2 |V(z_{d0}, z_{d1}, \xi_{d2}, \dots, \xi_{dd})|. \tag{1.6.6}$$

After $d-1$ more steps, we finally obtain

$$V_{d+1} \leq \Lambda_d^{d+1} |V(z_{d0}, z_{d1}, \dots, z_{dd})| \leq \Lambda_d^{d+1} V_{d+1} \tag{1.6.7}$$

by definition of Fekete points. Taking $\binom{d+1}{2}$ roots yields the result.

1.7 (2 $\Rightarrow$ 3). We first reformulate condition 2: If we write $|V(z_{d0}, \dots, z_{dd})| = c_{d+1} V_{d+1}$, then 2 is equivalent to

$$\lim_{d \to +\infty} c_d^{\frac{1}{d^2}} = \lim_{d \to +\infty} c_d^{\frac{1}{\binom{d}{2}}} = 1. \tag{1.7.1}$$

Now for each $d = 1, 2, \dots,$ we define a measure $\tilde{\mu}_d$ as follows: let $\Delta_{dj} = \{z : |z - z_{dj}| \leq 1/\sqrt{\pi(d+1)}\}$, $j = 0, \dots, d$; these are $d+1$ discs of radius $1/\sqrt{\pi(d+1)}$ and area $(1/(d+1))$. Let $\tilde{\mu}_d$ be Lebesgue measure dm on $\mathbf{C}$ restricted to $\cup_{j=0}^{d} \Delta_{dj}$. Then $\tilde{\mu}_d$ is a probability measure satisfying

$$\operatorname{supp} \tilde{\mu}_d = \bigcup_{j=0}^{d} \Delta_{dj} \subset \left\{ z : \operatorname{dist}(z, K) \leq \frac{1}{\sqrt{\pi(d+1)}} \right\}. \tag{1.7.2}$$

The proof of (1.7.1) $\Rightarrow$ 3 will consist of three steps:

Step 1. $\lim_{d \to +\infty} I(\tilde{\mu}_d) = I(\mu_K)$ $(= -\log(D(K)))$.

Proof of Step 1. Fix $\delta > 0$ and choose an open set $\mathcal{O} \supset K$ satisfying

$$D(\mathcal{O}) \leq D(K) + \delta. \tag{1.7.3}$$

Since $\mathcal{O}$ is open, $\{z : \operatorname{dist}(z, K) \le \rho\} \subset \mathcal{O}$ for $\rho > 0$ sufficiently small; hence, we can choose N large so that

$$\operatorname{supp} \tilde{\mu}_d \subset \mathcal{O} \qquad \text{for } d > N. \tag{1.7.4}$$

For such d,

(1.7.5)

$$\begin{aligned}\log \frac{1}{D(K)+\delta} &\le \log \frac{1}{D(\overline{\mathcal{O}})} = \inf\{I(r) : r(\overline{\mathcal{O}}) = 1, \operatorname{supp} r \subset \overline{\mathcal{O}}\} \\ &\le I(\tilde{\mu}_d) = \int_{\mathcal{O}} \int_{\mathcal{O}} \log \frac{1}{|z-\xi|}\, d\tilde{\mu}_d(z)\, d\tilde{\mu}_d(\xi) \\ &= \sum_{j=0}^{d} \int_{\Delta_{dj}} \left[\sum_{i=0}^{d} \int_{\Delta_{di}} \log \frac{1}{|z-\xi|}\, \mathrm{dm}\,(z) \right] \mathrm{dm}\,(\xi) \\ &\le \sum_{i,j=0}^{d} \int_{\Delta_{dj}} \frac{1}{d+1} \log \frac{1}{|z_{di}-\xi|}\, \mathrm{dm}\,(\xi)\end{aligned}$$

(by superharmonicity of $u(z) = \log(1/|z-\xi|)$)

$$\begin{aligned}&= \frac{1}{d+1}\left[\sum_{i\ne j} \int_{\Delta_{dj}} \log \frac{1}{|z_{di}-\xi|}\, \mathrm{dm}\,(\xi) + \sum_{i=j} \int_{\Delta_{di}} \log \frac{1}{|z_{di}-\xi|}\, \mathrm{dm}\,(\xi) \right] \\ &\le \frac{1}{d+1}\left[\sum_{i\ne j} \frac{1}{d+1} \log \frac{1}{|z_{di}-z_{dj}|}\, \mathrm{dm}\,(\xi) \right. \\ &\qquad \left. + (d+1) \int_{|\xi| \le \frac{1}{\sqrt{\pi(d+1)}}} \log \frac{1}{|\xi|}\, \mathrm{dm}\,(\xi) \right]\end{aligned}$$

where we have used superharmonicity of $u_i(z) = \log(1/|z_{di}-\xi|)$ on the integrals over Δ_{dj}, $i \ne j$, and a translation to integrals over $|\xi| \le (1/\sqrt{\pi(d+1)})$ for the others. By direct computation of the latter integral, we obtain

(1.7.6)

$$\begin{aligned}&\frac{2}{(d+1)^2} \sum_{i<j} \log \frac{1}{|z_{di}-z_{dj}|} + \frac{1}{(d+1)} \log \sqrt{\pi(d+1)} + \frac{1}{2(d+1)} \\ &\qquad = \frac{2}{(d+1)^2} \log \frac{1}{c_{d+1}} + \left(\frac{d}{d+1}\right) \log \frac{1}{V_{d+1}^{\frac{1}{\binom{d+1}{2}}}} + O\left(\frac{\log d}{d}\right)\end{aligned}$$

by definition of c_{d+1}. Since $\lim_{d\to+\infty} V_{d+1}^{1/\binom{d+1}{2}} = D(K)$ and $\lim_{d\to+\infty} c_{d+1}^{2/(d+1)^2} = 1$ by (1.7.1), $\log(1/(D(K)+\delta)) \le \underline{\lim}_{d\to+\infty} I(\tilde{\mu}_d) \le \overline{\lim}_{d\to+\infty} I(\tilde{\mu}_d) \le \log(1/D(K))$ for all $\delta > 0$ and the result follows.

Note that since $\operatorname{supp}\tilde{\mu}_d \subset \overline{\mathcal{O}}$ for $d > N$, we can choose a subsequence, which we still denote by $\{\tilde{\mu}_d\}$, that converges weak-* to a probability measure $\tilde{\mu}$ supported on K.

Step 2. $I(\tilde{\mu}) = -\log D(K)$.

Proof of Step 2. Since $I(\tilde{\mu}) \ge I(\mu_K) \equiv \inf\{I(v) : v(K) = 1, \operatorname{supp} v \subset K\}$ we need to show that $I(\tilde{\mu}) \le I(\mu_K)$. First of all, note that for any compactly supported measure v,

$$I(v) = \lim_{\rho\to 0^+} \iint \log \frac{1}{|z-\xi|_\rho}\, dv(z)\, dv(\xi) \tag{1.7.7}$$

where

(1.7.8)

$$\log \frac{1}{|z-\xi|_\rho} \equiv \begin{cases} \log \frac{1}{|z-\xi|} & \text{if } |z-\xi| \ge \rho \\ \log \frac{1}{\rho} & \text{if } |z-\xi| \le \rho \end{cases} \text{ is continuous in } (z,\xi).$$

The weak-* convergence of $\{\tilde{\mu}_d\}$ to $\tilde{\mu}$ implies the weak-* convergence of $\{\tilde{\mu}_d \times \tilde{\mu}_d\}$ to $\tilde{\mu} \times \tilde{\mu}$; since $\log(1/|z-\xi|_\rho)$ is continuous, we thus have

(1.7.9)

$$\begin{aligned} \int_K \int_K \log \frac{1}{|z-\xi|_\rho}\, d\tilde{\mu}(z)\, d\tilde{\mu}(\xi) &= \lim_{d\to+\infty} \iint \log \frac{1}{|z-\xi|_\rho}\, d\tilde{\mu}_d(z)\, d\tilde{\mu}_d(\xi) \\ &\le \lim_{d\to+\infty} \iint \log \frac{1}{|z-\xi|}\, d\tilde{\mu}_d(z)\, d\tilde{\mu}_d(\xi) \\ &= \lim_{d\to+\infty} I(\tilde{\mu}_d) = I(\mu_K) \end{aligned}$$

from Step 1. This inequality holds for all $\rho > 0$ and the result follows.

Note that the above argument actually shows that for any subsequence of $\{\tilde{\mu}_d\}$ which converges weak-* to (necessarily) a probability measure v supported on K, $I(v) = I(\mu_K)$; thus, by uniqueness of the equilibrium measure, $\tilde{\mu} = v = \mu_K$, i.e., $\{\tilde{\mu}_d\}$ has a *unique* weak-* limit, namely, μ_K. Thus, it suffices now to show that for any subsequence of $\{\mu_d\}$ which converges weak-*, we get the same weak-* limit from the corresponding subsequence of $\{\tilde{\mu}_d\}$. If we again use $\{\mu_d\}$, $\{\tilde{\mu}_d\}$ to denote the subsequences, we must prove the following.

Step 3. Let $\lim_{d\to+\infty} \mu_d = \mu$ weak-* and $\lim_{d\to+\infty} \tilde{\mu}_d = \tilde{\mu}$ weak-*. Then $\mu = \tilde{\mu}$.

Proof of Step 3. Given g continuous on a neighborhood $\mathcal{O}$ of K, we must show that $\lim_{d\to+\infty} \int_{\mathcal{O}} g d\mu_d = \lim_{d\to+\infty} \int_{\mathcal{O}} g d\tilde{\mu}_d$. Fix U open with

$$K \subset U \subset \overline{U} \subset \mathcal{O}$$

and choose N_1 so that if $d > N_1$, then $\operatorname{supp} \tilde{\mu}_d \subset \overline{U}$. Since g is uniformly continuous on $\overline{U}$, given $\varepsilon > 0$, we can find N_2 so that $|g(z) - g(\xi)| < \varepsilon$ if $z, \xi \in \overline{U}$ satisfy $|z - \xi| \leq (1/\sqrt{\pi(d+1)})$ for $d > N_2$. For $d > \max(N_1, N_2)$,

(1.7.10)

$$\begin{aligned} \left| \int_{\mathcal{O}} g\, d\mu_d - \int_{\mathcal{O}} g d\tilde{\mu}_d \right| &= \left| \sum_{j=0}^{d} \left[\frac{1}{d+1} g(z_{dj}) - \int_{\Delta_{dj}} g \,\mathrm{dm}\, \right] \right| \\ &= \left| \sum_{j=0}^{d} \int_{\Delta_{dj}} (g(z_{dj}) - g(z))\, \mathrm{dm}\,(z) \right| \quad (\text{since } m(\Delta_{dj}) = \frac{1}{d+1}) \\ &\leq \sum_{j=0}^{d} \int_{\Delta_{dj}} |g(z_{dj}) - g(z)|\, \mathrm{dm}\,(z) < \sum_{j=0}^{d} \varepsilon \left(\frac{1}{d+1} \right) = \varepsilon. \end{aligned}$$

Thus $\lim_{d\to+\infty} | \int_{\mathcal{O}} g\, d\mu_d - \int_{\mathcal{O}} g\, d\tilde{\mu}_d| = 0$. Since each of the limits

$$\lim_{d\to+\infty} \int_{\mathcal{O}} g\, d\mu_d, \qquad \lim_{d\to+\infty} \int_{\mathcal{O}} g\, d\tilde{\mu}_d$$

exists by hypothesis, the result follows and the proof of (1.7.1) $\Rightarrow$ 3 is complete.

1.8 (3 $\Rightarrow$ 4). For each point $z \notin K$, $u(\xi) \equiv \log(1/|z-\xi|)$ is continuous on a neighborhood of K. Thus, by weak-* convergence of $\{\mu_d\}$ to μ_K, the logarithmic potentials

$$(1.8.1) \qquad p_{\mu_d}(z) \equiv \int_K \log \frac{1}{|z-\xi|}\, d\mu_d(\xi) = \frac{1}{d+1} \sum_{j=0}^{d} \log \frac{1}{|z - z_{dj}|}$$

converge pointwise to $p_{\mu_K}(z) = \int_K \log(1/|z-\xi|)\, d\mu(\xi) = u_K(z) - \log D(K)$, i.e., exponentiating, we obtain

$$\lim_{d\to+\infty} |w_d(z)|^{\frac{1}{d+1}} = D(K)e^{u_K(z)} \tag{1.8.2}$$

pointwise for $z \notin K$. Clearly, for z lying in some fixed compact subset A of the complement of K, there exist constants $c_1, c_2 > 0$ depending only on A and K so that

$$c_1 < |z-\xi| < c_2 \tag{1.8.3}$$

for $z \in A$, $\xi \in K$; thus, the convergence above is uniform on compact subsets of the complement of K. By Walsh's theorem (Theorem 1.4), this is equivalent to (4). □

Proof. II. **1.9** ($2 \not\Rightarrow 1$). Suppose K is connected. Let $\{\xi_{dj}\}$ be an array of Fekete points, i.e., for each d, $|V(\xi_{d0}, \dots, \xi_{dd})| = V_{d+1}$. Choose a sequence $\{c_d\}$ of real numbers satisfying

(a) $0 < c_d < 1$, $d = 1, 2, \dots$

(b) $\lim_{d\to+\infty} c_d^{1/d^2} = 1$

(c) $\overline{\lim}_{d\to+\infty} c_d^{1/d} < 1$

(for example, for any $0 < v < 1$, the sequence $c_d = v^d$ will work). We define an array of points $\{z_{dj}\}$ in K which satisfies 2, but not 1 by modifying the first points ξ_{d0}, $d = 1, 2, \dots$, of our Fekete array. For each $d = 1, 2, \dots$, we choose $z_{dj} = \xi_{dj}$ for $j = 1, 2, \dots, d$; and we choose $z_{d0} \in K$ so that

$$|V(z_{d0}, \xi_{d1}, \dots, \xi_{dd})| = c_{d+1}V_{d+1}. \tag{1.9.1}$$

We can find such points $z_{d0} \in K$ since $|p_d(\xi)| \equiv |V(\xi, \xi_{d1}, \dots, \xi_{dd})|$ is a continuous function on the connected compact set K with $0 \le |p_d(\xi)| \le V_{d+1}$ on K, the extreme values are achieved, and $0 < c_d < 1$. Clearly, then, (1.9.1) and (b) imply that the array $\{z_{dj}\}$ satisfies (1.7.1). On the other hand, if we form the FLIP $l_0^{(d)}(z) \equiv (V(z, \xi_{d1}, \dots, \xi_{dd})/(V(z_{d0}, \xi_{d1}, \dots, \xi_{dd}))$, then $|l_0^{(d)}(\xi_{d0})| = (1/c_{d+1})$ so that $\Lambda_d \ge ||l_0^{(d)}|| \ge (1/c_{d+1})$, which, together with (c), contradicts 1.

1.10 ($3 \not\Rightarrow 2$). Take any array $\{z_{dj}\}$ satisfying 3. Replace, for each $d = 1, 2, \ldots$, the first point z_{d0} by z_{d1}, i.e., consider the new array

$$z_{d1}, z_{d1}, z_{d2}, \ldots, z_{dd}, \qquad d = 1, 2, \ldots .$$

Clearly, if $\mu_d \equiv (1/(d+1)) \sum_{j=0}^{d} [z_{dj}]$ converges weak-* to μ_K, then $\tilde{\mu}_d \equiv (2/(d+1))[z_{d1}] + \sum_{j=2}^{d} [z_{dj}]$ does also since $\mu_d - \tilde{\mu}_d = (1/(d+1))([z_{d0}] - [z_{d1}])$ and $\lim_{d \to +\infty} (1/(d+1)) = 0$. However, $V(z_{d1}, z_{d1}, z_{d2}, \ldots, z_{dd}) = 0$, for all d.

1.11 ($4 \not\Rightarrow 3$). By Theorem 1.4, it suffices to find an array $\{z_{dj}\}$ satisfying $\lim_{d \to +\infty} ||w_d||_K^{1/d+1} = D(K)$ but $\lim_{d \to +\infty} \mu_d \neq \mu_K$ weak-*. Take K to be a disc $K = \{z : |z| \leq R\}$ and $z_{dj} = 0$, for all $j = 0, 1, \ldots, d$; $d = 1, 2, \ldots$. Then $\mu_d = [0]$, for all $d = 1, 2, \ldots$, but $\mu_K \neq [0]$ and $w_d(z) = z^{d+1}$ so that

$$\lim_{d \to \infty} ||w_d||_K^{\frac{1}{d+1}} = \lim_{d \to \infty} (R^{d+1})^{\frac{1}{d+1}} = R = D(K). \square \tag{1.11.1}$$

2. Remarks. 2.1 The only results which appear to be new are $2 \not\Rightarrow 1$ and $2 \Rightarrow 3$. This latter result was stated but proved incorrectly for arrays of Fekete points in Hille ([**17**], Vol. II). We have essentially followed arguments in Tsuji's book [**22**] to obtain a correct proof of the more general statement. A result similar to $2 \Rightarrow 3$ of Theorem 1.5 of this paper appears in the proof of Theorem 1 in Blatt et al. [**3**]

2.2 We can give a direct proof of $1 \Rightarrow 4$ which generalizes to $\mathbf{C}^n$, $n > 1$. First of all, the polynomial convexity of K ensures the existence, for each f holomorphic in a neighborhood of K, of constants $A > 0$, $0 < \theta < 1$, and a sequence of polynomials $\{p_d\}$ with $\deg p_d \leq d$ satisfying $||f - p_d||_K \leq A\theta^d$, i.e., the polynomials $\{p_d\}$ converge uniformly to f on K at a geometric rate—indeed, the LIPs $L_d f = p_d$ associated to an array of Fekete points will work. Forming the LIP of degree d for $f - p_d$ associated with a given array $\{z_{dj}\}$ satisfying (1), since $L_d p_d = p_d$ and L_d is a linear operator, we have

$$L_d(f - p_d) = L_d f - L_d p_d = L_d f - p_d \tag{2.2.1}$$

so that

$$
\begin{aligned}
||f - L_d f||_K &= ||f - p_d + p_d - L_d f||_K \\
&= ||f - p_d + L_d p_d - L_d f||_K \\
&= ||f - p_d + L_d(f - p_d)||_K \\
&\le ||f - p_d||_K + \Lambda_d ||f - p_d||_K \\
&\le A\theta^d (1 + \Lambda_d).
\end{aligned} \tag{2.2.2}
$$

Hence, $\lim_{d\to+\infty} \Lambda_d^{1/d} = 1$ implies $L_d f \rightrightarrows f$ on K.

2.3 Since $\operatorname{supp} \mu_K = \partial K$ under the hypothesis of the main theorem, when discussing condition 3 it is natural to restrict attention to arrays $\{z_{dj}\} \subset \partial K$. Indeed, under this restriction, conditions 3 and 4 are equivalent. Suppose 3 does not hold, i.e., $\lim_{d\to+\infty} \mu_d \neq \mu_K$ weak-*. We can find a subsequence, which we still denote by $\{\mu_d\}$, such that $\lim_{d\to+\infty} \mu_d = \mu$ weak-* where μ is a probability measure with $\operatorname{supp} \mu \subset \partial K$ which is not the equilibrium measure μ_K. As in the proof of $3 \Rightarrow 4$, for $z \notin K$, the logarithmic potentials

$$
-\frac{1}{d+1} \log |w_d(z)| = p_{\mu_d}(z) \equiv \frac{1}{d+1} \int_K \log \frac{1}{|z-\xi|}\, d\mu_d(\xi) \tag{2.3.1}
$$

converge pointwise to $p_\mu(z) = \int_K \log(1/|z-\xi|)\, d\mu(\xi)$ and the convergence is uniform on compact subsets of $\mathbf{C} - K$. Since 4 is equivalent to $p_\mu(z) = p_{\mu_K}(z)$ for $z \notin K$, we show that there exists a point $z \notin K$ at which $p_\mu(z) \neq p_{\mu_K}(z)$ to reach a contradiction. Since $\mu \neq \mu_K$, $\sup_{z\in K} p_\mu(z) > \log D(K) = \sup_{z\in K} p_{\mu_K}(z)$ by (1.1.10). But since $\operatorname{supp} \mu \subset \partial K$, p_μ is harmonic in $K^0 =$ interior of K so that $\sup_{z\in K} p_\mu(z) = \sup_{z\in\partial K} p_\mu(z)$. Thus, there exists $z_0 \in \partial K$ such that

$$
p_\mu(z_0) > -\log D(K) = p_{\mu_K}(z_0). \tag{2.3.2}
$$

By lower-semicontinuity of $p_\mu(z) - p_{\mu_K}(z)$ (logarithmic potentials are superharmonic; p_{μ_K} is continuous by assumption), this inequality persists for z in a neighborhood of z_0, hence at points $z \in \mathbf{C} - K$.

2.4 Condition 1 implies that the Lebesgue functions can be used to obtain the Green function μ_K in the sense that

$$
\lim_{d\to+\infty} \Lambda_d(z) = e^{u_K(z)}. \tag{2.4.1}
$$

For clearly $\Lambda_d(z) \leq (d+1)\max_{j=0,1,\dots,d} |l_j^{(d)}(z)|$; since

$$|l_j^{(d)}(z)| \over \Lambda_d \leq \frac{|l_j^{(d)}(z)|}{||l_j^{(d)}||_K} \leq e^{du_K(z)} \tag{2.4.2}$$

by the definition of u_K (1.4.3), we obtain

$$\Lambda_d(z)^{\frac{1}{d}} \leq \left(\frac{d+1}{\Lambda_d}\right)^{\frac{1}{d}} e^{u_K(z)} \tag{2.4.3}$$

for all $z \in \mathbf{C}$. On the other hand, given a polynomial p_d of degree d, we have $p_d(z) = \sum_{j=0}^{d} p_d(z_{dj}) l_j^{(d)}(z)$ so that $|p_d(z)| \leq ||p_d||_K \Lambda_d(z)$. Hence, again from (1.4.3), we get $e^{u_K(z)} \leq \Lambda_d(z)^{1/d}$. Thus, 1 implies (2.4.1). However, (2.4.1) is not true under the weaker assumption 2. This follows from a modification of Fekete points similar to that used in showing $2 \not\Rightarrow 1$. Starting with Fekete points $\{\xi_{dj}\}$, i.e., for each d, $|V(\xi_{d0},\dots,\xi_{dd})| = V_{d+1}$, we fix a point $z \notin K$. Order the points $\xi_{d0},\dots,\xi_{dd}$ so that

$$\max_{j=0,1,\dots,d} |l_j^{(d)}(z)| = |l_0^{(d)}(z)|, \qquad j = 1, 2, \dots . \tag{2.4.4}$$

Now, given a sequence $\{c_d\} \subset (0,1)$ satisfying a, b, c of (1.9), we again modify ξ_{d0} to z_{d0} so that (1.9.1) holds. Calling the new FLIPs $\tilde{l}_j^{(d)}$, we clearly have that

$$\tilde{l}_0^{(d)}(\xi) = \frac{e^{i\theta}}{c_{d+1}} l_0^{(d)}(\xi) \qquad \text{for some } \theta \in [0, 2\pi]. \tag{2.4.5}$$

In particular, at $\xi = z$,

$$\tilde{\Lambda}_d(z) \equiv \sum_{j=0}^{d} |\tilde{l}_j^{(d)}(z)| \geq |\tilde{l}_0^{(d)}(z)| = \frac{1}{c_{d+1}} |l_0^{(d)}(z)| \geq \frac{1}{(d+1)c_{d+1}} \Lambda_d(z). \tag{2.4.6}$$

Thus,

$$\varliminf_{d\to+\infty} \tilde{\Lambda}_d(z)^{1/d} \geq \frac{1}{\overline{\lim}_{d\to+\infty} c_d^{1/d}} e^{u_K(z)} > e^{u_K(z)}$$

by b) and the fact that $\{\xi_{dj}\}$ satisfy 1.

2.5 We now give a well-known algorithm which can be used to construct arrays satisfying properties 2, 3 and 4, so-called *Leja sequences* [**13**]. We use the term 'sequence' instead of 'array' because $z_{dj} = z_j$, i.e., these 'arrays' have a desirable permanence property: to choose the $d+1$ points of the dth row of our array, we keep the d points of the previous row and add one more. Indeed, having chosen $z_0, z_1, \dots, z_{d-1} \in K$, we choose $z_d \in K$ so that

$$(2.5.1)\quad |z_d - z_0||z_d - z_1| \cdots |z_d - z_{d-1}| = \max_{z \in K} |z - z_0||z - z_1| \cdots |z - z_{d-1}|.$$

The choice of z_d is not necessarily unique; in any case, we show that such sequences satisfy 2 of the main theorem (and hence also 3 and 4). Note that, by definition of the point z_d, $||w_{d-1}||_K = |z_d - z_0||z_d - z_1| \cdots |z_d - z_{d-1}|$. Since for any monic polynomial p_d of degree d we have $||p_d||_K \geq D(K)^d$ (see [**7**] or [**11**]), we have

$$(2.5.2)\quad \begin{aligned} V_{d+1} \geq |V(z_0, \dots, z_d)| &= ||w_{d-1}||_K ||w_{d-2}||_K \cdots ||w_0||_K \\ &\geq D(K)^{d+(d-1)+\cdots+1} = D(K)^{\binom{d+1}{2}}. \end{aligned}$$

Hence, $V_{d+1}^{1/\binom{d+1}{2}} \geq |V(z_0, \dots, z_d)|^{1/\binom{d+1}{2}} \geq D(K)$; letting $d \to +\infty$ yields the result. We remark that Leja [**13**] proved directly that such sequences satisfy 4; he required a rather technical lemma relating the sequences $\{|V(z_0, \dots, z_d)|\}$ and $\{||w_d||_K\}$. As an exercise, the reader should verify that if $K = \{z : |z| \leq 1\}$ and z_0 is chosen with $|z_0| = 1$, then for each $d = 2^k - 1$, the 2^k points $z_0, \dots, z_d$ form a rotation of the dth roots of unity, i.e., $z_0, \dots, z_d$ form a set of Fekete points for K.

Leja sequences were first introduced in a paper of A. Edrei [**9**].

3. $\mathbf{C}^n$, $n > 1$. **3.1** Let $K \subset \mathbf{C}^n$, $n > 1$, be compact, nonpluripolar, and polynomially convex. When we refer to the set K as a *regular* set, this now means the following: as in the one-variable case, we can form a Green function u_K associated to K by setting

$$(3.1.1)\quad u_K(z) \equiv \sup \left\{ \frac{1}{\deg (p)} \log \frac{|p(z)|}{||p||_K} : p \text{ polynomial} \right\} \qquad z \in \mathbf{C}^n.$$

However, in general $u_K(z)$ need not be plurisubharmonic (psh), nor continuous. If we set $u_K^* = \overline{\lim}_{\xi \to z} u_K(\xi)$, then either $u_K^* \equiv +\infty$ in $\mathbf{C}^n$ (in case K is pluripolar) or $u_K^* \in L$ where

$$(3.1.2)\qquad L = \{u \in \text{psh}\,(\mathbf{C}^n) : u(z) \leq \log |z| + O(1) \text{ as } |z| \to +\infty\}.$$

We say that K is *regular* if $u_K = u_K^*$, i.e., u_K is continuous on $\mathbf{C}^n$. This occurs, for example, if K is regular for the exterior (R^{2n}) Dirichlet problem; for more general criteria for regularity, see [**17**].

To discuss LIPs, we continue using the notation from the introduction. For each positive integer d, we choose a basis $e_1, \dots, e_{h_d}$ for the vector space P_d of polynomials of degree at most d with the $\{e_j\}$ ordered so that '$i \leq j$' implies $\deg e_i \leq \deg e_j$. Given a set of h_d points $A_{d1}, \dots, A_{dh_d} \in K$ which do not lie on any algebraic surface of degree $\leq d$ (such points exist if K is not pluripolar), we form the generalized Vandermonde determinant

$$V(A_{d1}, \dots, A_{dh_d}) \equiv \det[e_i(A_{dj})]_{i,j=1,\dots,h_d} \neq 0. \tag{3.1.3}$$

As in the one variable case, we can construct the h_d FLIPs of degree d by setting

$$l_j^{(d)}(z) \equiv \frac{V(A_{d1}, \dots, \overset{\swarrow j\text{th}}{z}, \dots, A_{dh_d})}{V(A_{d1}, \dots, A_{dh_d})} = \begin{cases} 1 & \text{if } z = A_{dj} \\ 0 & \text{if } z = A_{di},\ i \neq j \end{cases} \tag{3.1.4}$$

for $j = 1, \dots, h_d$. Thus, the LIP $L_d f$ of degree d associated to a function f and the array $\{A_{dj}\}_{\substack{j=1,\dots,h_d \\ d=1,2,\dots}}$ is

$$L_d f(z) \equiv \sum_{j=1}^{h_d} f(A_{dj}) l_j^{(d)}(z), \qquad d = 1, 2, \dots. \tag{3.1.5}$$

Again, we have the Lebesgue functions $\Lambda_d(z) \equiv \sum_{j=1}^{h_d} |l_j^{(d)}(z)|$ and the Lebesgue constants $\Lambda_d \equiv ||\Lambda_d||_K$.

An important difference occurs with the generalized Vandermonde determinants (3.1.3) in the several variable case: in one variable, the ratio used to construct the FLIPs $l_j^{(d)}$ can be simplified:
(3.1.6)

$$l_j^{(d)}(z) \equiv \frac{V(z_{d0}, \dots, z, \dots, z_{dd})}{V(z_{d0}, \dots, z_{dd})} = \frac{\prod_{\substack{i=0 \\ i \neq j}}^{d} (z - z_{di})}{\prod_{\substack{i=0 \\ i \neq j}}^{d} (z_{dj} - z_{di})} = \frac{w_d(z)}{(z - z_{dj}) w_d'(z_{dj})}.$$

No 'cancellation' occurs in (3.1.4) for $n \geq 2$. Furthermore, in one variable, the dth order diameters $\{D_d(K)\}_{d=1,2,\dots}$ decrease with d which easily implies the existence of the limit $D(K) \equiv \lim_{d\to+\infty} D_d(K)$. In several variables, one can again define dth order diameters as

$$D_d(K) \equiv \max_{\xi_1,\dots,\xi_{h_d}\in K} |V(\xi_1,\dots,\xi_{h_d})|^{\frac{1}{l_d}} \tag{3.1.7}$$

where $l_d = \sum_{j=1}^d j(h_j - h_{j-1}) =$ degree of $V(\xi_1,\dots,\xi_{h_d})$ as a polynomial in $\xi_1,\dots,\xi_{h_d}$, but the existence of the limit

$$D(K) \equiv \lim_{d\to+\infty} D_d(K) \tag{3.1.8}$$

for arbitrary compact sets was not proved until 1975 by Zaharjuta [**24**].

3.2 We briefly indicate the idea behind Zaharjuta's proof since this will be used in what follows. We order the monomials $e_i(z) = z^{k(i)} \equiv z_1^{k_1(i)}\dots z_n^{k_n(i)}$ so that, as before, '$i < j$' implies $\deg e_i = |k(i)| = k_1(i) + \cdots + k_n(i) \le \deg e_j$. For each $i = 1,2,3,\dots$, we consider the class of normalized polynomials

$$P^i \equiv \{p_i(z) = e_i(z) + \sum_{j<i} c_j e_j(z) :\ c_j \in \mathbf{C}\} \tag{3.2.1}$$

We call $\tau_i \equiv M_i^{1/|k(i)|} \equiv [\inf\{||p_i||_K : p_i \in P^i\}]^{1/|k(i)|}$ the i^{th}-*Chebyshev constant* of K. Note that for each degree d, there are $h_d - h_{d-1}$ polynomial classes P^i with $|k(i)| = d$; Zaharjuta showed that the geometric mean of the corresponding Chebyshev constants, i.e.,

$$\tau_d^0 \equiv \left[\prod_{|k(i)|=d} \tau_i\right]^{1/h_d - h_{d-1}} \tag{3.2.2}$$

converge as $d \to +\infty$ by relating the τ_d^0's to an 'average' of *directional Chebyshev constants*

$$\tau(K,\theta) \equiv \varlimsup_{\substack{j\to+\infty \\ \frac{k(j)}{|k(j)|}\to\theta}} \tau_j; \tag{3.2.3}$$

where $\theta = (\theta_1,\dots,\theta_n)$ is a point on the standard n-simplex $\Sigma = \{\theta \in R^n : \sum_{j=1}^n \theta_j = 1,\ \theta_j \ge 0,\ j = 1,\dots,n\}$. Precisely, he showed that

$$\lim_{d\to+\infty} \ln \tau_d^0 = \frac{1}{\operatorname{meas}(\Sigma)} \int_\Sigma \ln \tau(K,\theta)\, d\theta \tag{3.2.4}$$

where $\operatorname{meas}(\Sigma) = \int_\Sigma d\theta$. In any case, using standard arguments analogous to the one-variable case, one can show that

$$\tau_i^{|k(i)|} \le \frac{V_i}{V_{i-1}} \le i\tau_i^{|k(i)|}, \qquad i = 1, 2, \dots ; \tag{3.2.5}$$

multiplying these inequalities together for i such that $|k(i)| = d$, one obtains

$$(\tau_d^0)^{r_d} \le \frac{V_{h_d}}{V_{h_{d-1}}} \le \frac{h_d!}{h_{d-1}!}(\tau_d^0)^{r_d}, \qquad d = 1, 2, \dots \tag{3.2.6}$$

where $r_d = d(h_d - h_{d-1})$. Letting $V_0 = 1$ and multiplying the inequalities (3.2.6) for $1, 2, \dots, d$, one obtains

$$\prod_{j=1}^{d} (\tau_j^0)^{r_j} \le V_{h_d} \le h_d! \prod_{j=1}^{d} (\tau_j^0)^{r_j}. \tag{3.2.7}$$

Taking $l_d = \sum_{j=1}^d j(h_j - h_{j-1}) = \sum_{j=1}^d r_j$ roots in (3.2.7) and using the fact that the geometric mean of the τ_j^0's converge yields the desired result. Note we have used the important fact that while $h_d = O(d^n)$, $l_d = O(d^{n+1})$ so that $\lim_{d\to+\infty}(h_d!)^{1/l_d} = 1$.

We now indicate a method for computing the transfinite diameter of a compact set K without explicitly calculating the maximal Vandermonde determinants, V_d.

Theorem 3.3. *Let $K \subset \mathbf{C}^n$ be compact and let μ be a nonnegative Borel measure with* $\operatorname{supp}\mu \subset K$ *such that the pair (K, μ) satisfies the Bernstein-Markov property:*
(BM)
$\forall\lambda > 1, \quad \exists M > 0 \ni ||p||_K \le M\lambda^{\deg(p)}||p||_2$ *for all polynomials p*

where $||p||_2^2 \equiv \int_K |p|^2\, d\mu$. Then $D(K) = \lim_{d\to+\infty} G_{h_d}^{1/2l_d}$ where $G_k =$ $\det[\int_K e_i\overline{e_j}d\mu]_{i,j=1,\dots,k}$ *is the kth Gram determinant associated with (K, μ).*

Remark. The proof is essentially due to Bos [**5**]. We note that if K is regular, then the Monge-Ampere measure $\mu_K \equiv (dd^c u_K^*)^n$ associated with the function $u_K = u_K^*$ is a measure with $\operatorname{supp}\mu_K \subset K$ such that (K, μ_K) satisfies (BM) (see [**16, 1**] for the definition and properties of $(dd^c u_K^*)^n$). Also, if, in this case, μ is a measure such that μ_K is absolutely continuous with respect to μ, then (K, μ) also satisfies (BM). If $K = \partial\Omega$ where Ω is a bounded pseudoconvex domain with C^1 boundary, then μ_K is absolutely continuous with respect to $dS_\Omega =$ surface area measure on $\partial\Omega$; also if K is a regular subset of $R^n \subset \mathbf{C}^n$ with nonempty interior, then μ_K is absolutely continuous with respect to Lebesgue measure on R^n restricted to K. Thus, in principle, one can compute the transfinite diameter of such sets by computing the integrals $\int_{\partial\Omega} z^{k(i)}\bar{z}^{k(j)}\, dS_\Omega(z)$ in the first case and $\int_K x^{k(i)}x^{k(j)}\,\mathrm{dm}\,(x)$ in the second case where $x = (x_1, \dots, x_n) = (\operatorname{Re} z_1, \dots, \operatorname{Re} z_n)$. Bos used the latter case to compute $D(K)$ for the real disc $K = B_2 \equiv \{(z_1, z_2) \in \mathbf{C}^2 : \operatorname{Im} z_1 = \operatorname{Im} z_2 = 0,\ x_1^2 + x_2^2 \le 1\}$; he obtained $D(B_2) = (1/\sqrt{2e})$ [**5**]; we will use this fact later.

Proof. We want to relate the Gram determinants G_i to the Chebyshev constants τ_i using (BM). Let

$$\mu_i \equiv K_i^{1/|k(i)|} \equiv [\inf\{||p_i||_2 : p_i \in P^i\}]^{1/|k(i)|}. \tag{3.3.1}$$

It follows that (see, e.g., [**8**, p. 181]) $K_i^2 = (G_i/G_{i-1})$. Furthermore, if $t_i \in P^i$ is a Chebyshev polynomial, i.e., $||t_i||_K = M_i$, and if $A_i \in P^i$ satisfies $||A_i||_2 = K_i$, then we have, assuming $\mu(K) = 1$ for simplicity,

$$K_i = ||A_i||_2 \le ||t_i||_2 \le ||t_i||_K = M_i \tag{3.3.2}$$

and, by (BM), given a fixed $\lambda > 1$, $\exists M$ so that

$$M_i = ||t_i||_K \le ||A_i||_K \le M\lambda^{|k(i)|}||A_i||_2 = M\lambda^{|k(i)|}K_i. \tag{3.3.3}$$

Hence,

$$\frac{1}{M^2}\left(\frac{1}{\lambda^2}\right)^{|k(i)|}(\tau_i^{|k(i)|})^2 \le K_i^2 = \frac{G_i}{G_{i-1}} \le (\tau_i^{|k(i)|})^2, \tag{3.3.4}$$

an inequality analogous to (3.2.5). Following Zaharjuta's reasoning, we multiply the inequalities (3.3.4) for each i with $|k(i)| = d$ to get

$$\left(\frac{1}{M^2}\right)^{r_d/d}\left(\frac{1}{\lambda^2}\right)^{r_d}(\tau_d^0)^{2r_d} \le \frac{G_{h_d}}{G_{h_{d-1}}} \le (\tau_d^0)^{2r_d}, \tag{3.3.5}$$

similar to (3.2.6). Again, letting $G_0 = 1$ and multiplying the inequalities above for $1, 2, \ldots, d$, we obtain

$$\left(\frac{1}{M^2}\right)^{\sum_{j=1}^{d} r_j/j} \left(\frac{1}{\lambda^2}\right)^{l_d} (\tau_d^0)^{2l_d} \leq G_{h_d} \leq (\tau_d^0)^{2l_d}. \tag{3.3.6}$$

Taking $2l_d$ roots in (3.3.6) and letting $d \to +\infty$, we obtain, via Zaharjuta,

$$\frac{1}{\lambda} D(K) \leq \varliminf_{d\to+\infty} G_{h_d}^{1/2l_d} \leq \varlimsup_{d\to+\infty} G_{h_d}^{1/2l_d} \leq D(K). \tag{3.3.7}$$

Since the above inequality holds for all $\lambda > 1$, the result follows. □

Remark **3.4** The Chebyshev polynomials $t_i \in P^i$ associated to a nonpluripolar compact set $K \subset \mathbf{C}^n$ $(n > 1)$ are not necessarily unique as occurs in the one-variable case. As a simple example, consider the simplex $K = \{(x_1, x_2) \in R^2 \subset \mathbf{C}^2 : 0 \leq x_1, x_2 \leq 1, 0 \leq x_1 + x_2 \leq 1\}$. Then $P^3 = \{z_2 + az_1 + b : a, b \in \mathbf{C}\}$; one can easily show that $M_3 = \inf\{||p_3||_K : p_3 \in P^3\} = 1/2$ and that for $a = 0$ or $a = 1$, $p_3(z_1, z_2) = z_2 + az_1 - 1/2$ satisfies $||p_3||_K = 1/2$. On the other hand, for any completely circular set $K \subset \mathbf{C}^n$, i.e., $(z_1, \ldots, z_n) \in K$ implies $(\alpha_1 z_1, \ldots, \alpha_n z_n) \in K$ for all $|\alpha_i| \leq 1$, any $p_i \in P^i$ satisfies $||p_i||_K \geq ||e_i||_K$ by the Cauchy estimates so that the Chebyshev polynomials t_i are precisely e_i (and hence are unique).

3.5 As an alternate method for computing $D(K)$ under the hypothesis of Theorem 3.3, note that from (3.3.4),

$$\left(\frac{1}{M}\right)^{1/|k(i)|} \frac{1}{\lambda}\tau_i \leq ||A_i||_2^{1/|k(i)|} \leq \tau_i \tag{3.5.1}$$

so that for $\theta \in \Sigma_0 \equiv \{\theta \in \Sigma : \theta_i > 0, i = 1, \ldots, n\}$

(3.5.2)

$$\frac{1}{\lambda}\tau(K,\theta) \leq \varliminf_{\substack{i\to+\infty \\ \frac{k(i)}{|k(i)|}\to\theta}} ||A_i||_2^{1/|k(i)|} \leq \varlimsup_{\substack{i\to+\infty \\ \frac{k(i)}{|k(i)|}\to\theta}} ||A_i||_2^{1/|k(i)|} \leq \tau(K,\theta)$$

for all $\lambda > 1$ since the limit in the definition of $\tau(K,\theta)$ exists for such θ (Lemma 1, [**24**]). Hence, for such θ,

$$\tau(K,\theta) = \lim_{\substack{i\to+\infty \\ \frac{k(i)}{|k(i)|}\to\theta}} ||A_i||_2^{1/|k(i)|}, \tag{3.5.3}$$

i.e., the L^2 norms of the orthogonal polynomials $\{A_i\}$ with respect to the measure μ can be used to compute the directional Chebyshev constants $\tau(K,\theta)$ and hence $D(K)$ by (3.2.4). If the geometry of the set K is relatively simple, e.g., if $K = B_2$, then the orthogonal polynomials A_i with respect to certain natural measures μ on K can be computed explicitly. In this regard, we mention that under the hypothesis of Theorem 3.3, Zeriahi [**25**] has shown that one can recover the extremal function u_K as a pointwise upper limit of the orthonormal polynomials $B_i(z) \equiv A_i(z)/||A_i||_2$.

Theorem 3.6. [**25**, Theorem 1]. *Let $K \subset \mathbf{C}^n$ be compact and let μ be a nonnegative Borel measure on K such that (K,μ) satisfies (BM). Then*

$$\overline{\lim_{i\to+\infty}} |B_i(z)|^{1/|k(i)|} = e^{u_K(z)}, \qquad z \in \mathbf{C}^n - \hat{K}.$$

We again utilize Zahajuta's proof to construct sequences of points which can be used to compute the transfinite diameter.

Proposition 3.7. *Let $K \subset \mathbf{C}^n$ be compact. Define the sequence $\{\xi_j\}_{j=1,2,\ldots}$ in K as follows. Having chosen $\xi_1,\ldots,\xi_{k-1}$, choose $\xi_k \in K$ so that $|V(\xi_1,\ldots,\xi_k)| = \max_{\xi\in K} |V(\xi_1,\ldots,\xi_{k-1},\xi)|$. Then*

$$\lim_{d\to+\infty} |V(\xi_1,\ldots,\xi_{h_d})|^{1/l_d} = D(K).$$

Remark. In analogy with the one-variable case, we call such a sequence of points a *Leja sequence* for K.

Proof. Let $L_k \equiv |V(\xi_1, \ldots, \xi_k)|$. We relate L_k to V_k. Clearly, $L_k \leq V_k$. On the other hand, since

(3.7.1)
$$\frac{V(\xi_1, \ldots, \xi_{k-1}, \xi)}{V(\xi_1, \ldots, \xi_{k-1})} = e_k(\xi) + \sum_{j<k} c_j e_j(\xi) \equiv p_k(\xi) \text{ and } ||p_k||_K = \frac{L_k}{L_{k-1}}$$

by definition of ξ_k, we have

$$\frac{L_k}{L_{k-1}} = ||p_k||_K \geq ||t_k||_K = \tau_k^{\deg(e_k)}. \tag{3.7.2}$$

Thus, again letting $L_0 = 1$, we have

$$\begin{aligned} V_{h_d} = \frac{V_{h_d}}{V_0} \geq \frac{L_{h_d}}{L_0} &= \frac{L_{h_d}}{L_{h_{d-1}}} \frac{L_{h_d-1}}{L_{h_d} - 1} \cdots \frac{L_1}{L_0} \\ &= L_{h_d} \geq \prod_{k=1}^{d} (\tau_k^0)^k. \end{aligned} \tag{3.7.3}$$

Taking l_d roots and using (3.2.7), we see that $\lim_{d\to+\infty} L_{h_d}^{1/l_d} = D(K)$. □

4. Theorem 1.5 in $\mathbf{C}^n$. Theorem 4.1. *Let $K \subset \mathbf{C}^n$ $(n \geq 2)$ be compact, nonpluripolar, polynomially convex, and regular. Let $\{A_{dj}\}$ be an array of points in K. Consider the following four properties which an array may or may not possess.*

1. $\lim_{d\to+\infty} \Lambda_d^{1/d} = 1$
2. $\lim_{d\to+\infty} |V(A_{d1}, \ldots, A_{dh_d})|^{1/l_d} = D(K)$
3. $\lim_{d\to+\infty} (1/h_d) \sum_{j=1}^{h_d} [A_{dj}] = c_n \mu_K \equiv c_n (dd^c u_K^*)^n$ weak-*
4. $L_d f \rightrightarrows f$ *on K for each f holomorphic on a neighborhood of K (in 3, c_n is a dimensional constant chosen so that $\mu_K(K) = 1/c_n$).*

Then

I. $1 \Rightarrow 2, 4.$

II. $2 \not\Rightarrow 1,\ 3 \not\Rightarrow 1, 2, 4.$

Proof. I. **4.2** $(1 \Rightarrow 2)$. This is a corollary 2.2 of [**5**]; the proof is as in the one-variable case. The only difference is that instead of inequality

(1.4), we obtain, after h_d steps,

$$V_{h_d} \le \Lambda_d^{h_d} |V(A_{d1}, \dots, A_{dh_d})| \le \Lambda_d^{h_d} V_{h_d}. \tag{4.2.1}$$

Since $h_d/l_d = O(1/d)$, taking l_d roots and using 1 yields 2 upon letting $d \to +\infty$. $(1 \Rightarrow 4)$. Since K is polynomially convex, from Siciak [**20**], given f holomorphic on a neighborhood of K, $\exists A > 0$, $0 < \theta < 1$, and a sequence $\{p_d\}$ of polynomials with $\deg p_d \le d$ so that $||f - p_d||_K \le A\theta^d$, $d = 1, 2, \dots$. Thus, from Remark 2.2 in Chapter 2, the result follows. □

Proof. II. **4.3** $(2 \not\Rightarrow 1)$. This also follows as in the one-variable case; indeed, we now have more flexibility with the sequence $\{c_d\}$ (1.9), we can relax condition (b) to $\lim_{d\to+\infty} c_d^{1/l_d} = 1$. This is now equivalent to 2 where $|V(A_{d1}, \dots, A_{dh_d})| = c_d V_{h_d}$.

$(3 \not\Rightarrow 1, 2, 4)$. Note that $3 \not\Rightarrow 2$ can be seen as in the one-variable case; also, since $1 \Rightarrow 4$, it suffices to construct an example of a compact set $K \subset \mathbf{C}^n$ and an array $\{A_{dj}\} \subset K$ satisfying 3 but not 4 to complete the proof. To get such an array, and also to get a more interesting example of $3 \not\Rightarrow 2$, we follow the technique used by Bos to construct 'natural' arrays $\{A_{dj}\}$ on the real disc $B_2 \subset \mathbf{C}^2$. Note in $\mathbf{C}^2$, $h_d = \binom{d+2}{2}$. For simplicity in describing Bos's scheme, we let $d = 2s$ be even. Choose $s + 1$ radii $R_{s0} < R_{s1} < \cdots < R_{ss} = 1$ and choose $4j + 1$ points on the circle $x_1^2 + x_2^2 = R_{sj}^2$. This gives us $\sum_{j=0}^{s}(4j + 1) = h_d$ points. The maximal Vandermonde determinant $|V(A_{d1}, \dots, A_{dh_d})|$ out of such configurations is achieved by choosing equally spaced points on the circles—the determinants are then independent of the location of the points on one circle relative to the location of the points on any other circle—and the value of the determinant $|V(A_{d1}, \dots, A_{dh_d})|$ becomes

$$\begin{aligned} |V(A_{d1}, \dots, A_{dh_d})| =& C_d \prod_{j=0}^{s} (R_{s,s-j}^2)^{(2s-2j+1)(2s-2j)/2} \\ & \cdot \prod_{i=1}^{s} \Big[\prod_{j=0}^{i-1} (R_{s,s-i}^2 - R_{s,s-j}^2) \Big]^{2(2s-2i)+1)} \end{aligned} \tag{4.3.1}$$

where C_d is a constant depending on d, n. The idea behind (4.3.1) (and the motivation for the scheme) is that by performing elementary row operations on the matrix $V(A_{d1}, \dots, A_{dh_d})$ one obtains a matrix consisting of $(4j+1)\times(4j+1)$ blocks which are, up to a constant, Vandermonde matrices in one variable for points on a circle. The dimension of the space of trigonometric polynomials, $a_0+\sum_{k=1}^{2j}(a_k\cos k\theta+b_k\sin k\theta)$, of degree $\le 2j$ is $4j+1$; these block determinants are maximized by choosing equally spaces points (see [**6**] for details).

We want to relate the distribution of radii $\{R_{sj}\}$ to a distribution of points $\{\gamma_{sj}\}$ on the unit interval. To make things precise, let $G : [0,1] \to [0,1]$ be a strictly increasing continuous function which satisfies $G(0) = 0$ and $G(1) = 1$, let $F = G^{-1}$. We want to realize F as the limiting cumulative distribution function associated with an array of points $\{\gamma_{sj}\}_{\substack{j=0,1,\dots,s\\ s=1,2,\dots}}$ on $[0,1]$—the γ_{sj} will just be given by $\gamma_{sj} = R_{sj}^2$. To achieve this, define, for $j = 0,1,\dots,s$,

$$\text{(4.3.2)}\quad \gamma_{sj} = G\left(\frac{j}{s+1}\right) = F^{-1}\left(\frac{j}{s+1}\right) \qquad \text{i.e., } F(\gamma_{sj}) = \frac{j}{s+1}.$$

Defining $F_s(x) = (j+1)/(s+1)$ for $x \in [\gamma_{sj}, \gamma_{s,j+1})$, $j = 0,1,\dots,s$, it follows easily that the step functions F_s converge uniformly to F on $[0,1]$. Setting $R_{sj}^2 = \gamma_{sj}$, $j = 0,1,\dots,s$, and following the above procedure of placing $4j+1$ equally spaced points on $x_1^2 + x_2^2 = R_{sj}^2$, we get an array $\{A_{dj}\}_{\substack{j=0,1,\dots,h_d\\ d=1,2,\dots}}$ on B_2 (for $d = 2s$ even; a similar procedure with slight modifications can be used for d odd to achieve our goal). The distributions F_s on $[0,1]$ give rise to radial distributions

$$\text{(4.3.3)}\quad H_s(R) \equiv \frac{2}{(2s+2)(2s+1)}\#\{j : |A_{dj}| \le R\}, \qquad 0 \le R \le 1$$

$(d = 2s)$ on B_2, it follows from the combinatorics and the fact that $F_s \rightrightarrows F$ on $[0,1]$ that

(4.3.4)

$$[\mathbf{4}\text{, Lemma 4.5}]\quad \lim_{s\to+\infty} H_s(R) = (F(R^2))^2 \text{ uniformly on } 0 \le R \le 1.$$

This gives us a mechanism for creating '*Bos arrays*' on B_2 from arrays on $[0,1]$. We now state several propositions without proof which will lead us to the construction of a Bos array satisfying 3 but not 2 or 4. The first proposition tells us what $D(B_2)$ is.

Proposition 4.4. [**5**, Theorem 2.6] $D(B_2) = 1/\sqrt{2e}$.

To prove this, Bos used a special case of Theorem 2.1 (which motivated this theorem) as mentioned previously.

Now we need to know how to compute $\lim_{d\to+\infty} |V(A_{d1}, \ldots, A_{dh_d})|^{1/l_d}$ for Bos arrays constructed above.

Proposition 4.5. *Given* $G : [0,1] \to [0,1]$ *continuous, onto, strictly increasing, define* $R^2_{s,s-j} \equiv G(s-j)/(s+1)$ *(this is just* (4.3.2) *with* $R^2_{sj} = \gamma_{sj}$*),* $j = 0, 1, \ldots, s$*,* $s = 1, 2, \ldots$ *and construct a corresponding Bos array* $\{A_{dj}\}$*. Then*

(4.5.1) $$\lim_{d\to+\infty} |V(A_{d1}, \ldots, A_{dh_d})|^{1/l_d} = \frac{1}{\sqrt{2}} \exp\left(\frac{3}{4} L(G)\right)$$

where

(4.5.2) $$L(G) \equiv \int_0^1 x^2 \log G(x)\, dx + 2\int_0^1 \int_x^1 x \log[G(y) - G(x)]\, dy\, dx.$$

The proof of Proposition 2.4 follows as in Bos's [**5**, Theorem 3.2]. Setting $R^2_{s,s-j} = G((s-j)/(s+1))$ in (4.3.2), taking logarithms and using (4.3.3), we recognize the sums as Riemann sums for the sum of the two improper Riemann integrals in (4.5.2). Letting $s \to +\infty$ yields the result.

We need to relate the function G to property 4.

Proposition 4.6. [**4**, Section 4]. *Given* G *as above and a corresponding Bos array* $\{A_{dj}\}$*, if* $G(x) \neq (1 - \cos \pi x)/2$*, then* 4 *does not hold.*

The reason is that if $G(x) \neq (1 - \cos \pi x)/2$, then the array $\{g_{sj}\}$ on $[0,1]$ cannot have the Chebyshev distribution $F(x) = 1/2 + (1/\pi)\sin^{-1}(2x-1)$ as its limiting distribution; it is known in this case that one can find a function g analytic in a neighborhood of $[0,1]$ in $\mathbf{C}$ such that the LIPs $L_s g(z) \equiv \sum_{j=0}^{s} g(\gamma_{sj}) l_j^{(s)}(z)$ do not converge uniformly to g on $[0,1]$. The function $f(z_1,z_2) \equiv g(z_1^2+z_2^2)$ is then analytic in a neighborhood of B_2 in $\mathbf{C}^2$, and as a function on R^2, it is a radial function ($f(x_1,x_2) = g(x_1^2+x_2^2) \equiv g(r^2)$); for such functions, the two-variable LIPs $L_d f$ reduce to one-variable radical LIPs (see [**5**]) so that $L_d f \rightrightarrows f$ on B_2.

Finally, we need to know the Monge-Ampere measure $\mu_{B_2} = (dd^c u_{B_2}^*)^2$.

Proposition 4.7. [**18**]. *$c_2\mu_{B_2} = (1/2\pi)(r dr d\theta/\sqrt{1-r^2})$ where (r,θ) are polar coordinates on R^2.*

We now have all the ingredients we need. Set $G(x) = 1-(x^2-1)^2$. This gives $(F(R^2))^2 = 1-\sqrt{1-R^2} = \int_0^{2\pi}\int_0^R (1/2\pi)(r/\sqrt{1-r^2})\,dr\,d\theta$, i.e., 3 holds. Since $G(x) \neq (1-\cos\pi x)/2$, by Proposition 4.6, 4 does not hold. Finally, by setting $G(x) = 1-(x^2-1)^2$ in Proposition 4.5 and by direct (but tedious!) calculation, one computes $L(G) < -2/3$. Since $L(G) = -2/3$ if and only if $\lim_{d\to+\infty} |V(A_{d1},\dots,A_{dh_d})|^{1/l_d} = 1/\sqrt{2e} = D(B_2)$ by Proposition 4.4, 2 does not hold. Thus, the construction, the example, and hence the theorem are done. □

We close this section with a positive result, due to Bloom and Siciak, for product sets in $\mathbf{C}^2$ [see **18; 4**; Section 4.8].

Theorem 4.8. *Let $K_1, K_2 \subset \mathbf{C}$ be compact, nonpolar, and polynomially convex. Then there exist arrays $\{A_{dj}\} \subset K \equiv K_1 \times K_2$ satisfying $2, 3$, and 4.*

Proof. We give the construction of $\{A_{dj}\}$ and outline the proofs that the array satisfies 2, 3, and 4. Let (z,w) be the variables in $\mathbf{C}^2$. We choose one-variable Leja sequences $\{z_j\}_{j=0,1,2,\dots} \subset K_1$ and $\{w_k\}_{k=0,1,2,\dots} \subset K_2$. Then for each d, the h_d points $A_{d1},\dots,A_{dh_d}$ will consist of the triangular array

$$\{A_{dj}\}_{j=1,\dots,h_d} \equiv \{(z_j,w_k) : j+k \leq d\}. \tag{4.8.1}$$

The array $\{A_{dj}\}$ satisfies 2. Schiffer and Siciak showed [**18**] that for such 'intertwined' arrays, the two-variable Vandermonde determinant $|V(A_{d1},\dots,A_{dh_d})|$ can be written as the product of one-variable

Vandermonde determinants, i.e.,
(4.8.2)

$$V(A_{d1},\dots,A_{dh_d})| = \prod_{j=1}^{d} |V(z_0,z_1,\dots,z_j)||V(w_0,w_1,\dots,w_j)|.$$

Since $|V(z_0,z_1,\dots,z_j)| \geq D(K_1)^{\binom{j+1}{2}}$ and $|V(w_0,\dots,w_j)| \geq D(K_2)^{\binom{j+1}{2}}$ by (2.5.2), using the fact that $l_d = \sum_{j=1}^{d} j(j+1)$ in $\mathbf{C}^2$, we get, from (4.8.2) that

$$|V(A_{d1},\dots,A_{d_{h_d}})|^{\frac{1}{l_d}} \geq [D(K_1)D(K_2)]^{\frac{1}{2}}. \tag{4.8.3}$$

However, Schiffer and Siciak also proved that

$$\overline{\lim_{d\to+\infty}} D_d(K) \leq [D(K_1)D(K_2)]^{\frac{1}{2}}. \tag{4.8.4}$$

This, combined with (4.8.3) yields 2 (and $D(K) = [D(K_1)D(K_2)]^{1/2}$).

The array $\{A_{dj}\}$ *satisfies 3.* First of all, since one variable Leja sequences satisfy 2 and hence 3 $\lim_{d\to+\infty}(1/(d+1))\sum_{j=0}^{d}[z_j] = \mu_{K_1}$ weak-* and $\lim_{d\to+\infty}(1/(d+1))\sum_{k=0}^{d}[w_k] = \mu_{K_2}$ weak-*. It follows from a general measure theory lemma due to Assani [**4**, Lemma 4.9] that $\lim_{d\to+\infty}(1/h_d)\sum_{j=1}^{h_d}[A_{dj}] = \mu_{K_1}\times\mu_{K_2}$ weak-*. Finally, from work of Bedford and Taylor [**2**, Proposition 2.2], $(dd^c u_K^*)^2 = \mu_K \equiv c\mu_{K_1}\times\mu_{K_2}$ and 3 holds.

The array $\{A_{dj}\}$ *satisfies 4.* To see this, given f holomorphic in a neighborhood of $K_1\times K_2$, we write the LIPs $L_d f$ in a different manner, namely as *Newton polynomials*: write

$$L_d f(z,w) = \sum_{j+k\leq d} a_{jk}(z-z_0)\cdots(z-z_j)(w-w_0)\cdots(w-w_k) \tag{4.8.5}$$

where, in order to achieve $L_d f(z_j,w_k) = f(z_j,w_k)$ for $j+k\leq d$, we choose
(4.8.6)

$$a_{jk} = \frac{1}{(2\pi i)^2}\int_{\Gamma_z}\int_{\Gamma_w} \frac{f(z,w)}{(z-z_0)\cdots(z-z_j)(w-w_0)\cdots(w-w_k)}\,dz\,dw$$

where Γ_z is a contour surrounding $z_0, z_1, \dots, z_j$ and Γ_w is a contour surrounding $w_0, w_1, \dots, w_k$. We thus have, in this case, an explicit formula for the remainder $|f(z,w) - L_d f(z,w)|$ as an iterated contour integral; using the properties of our one-variable Leja sequences $\{z_j\}$ and $\{w_k\}$ (precisely, property 4), we have that in the denominator $|z-z_0|\cdots|z-z_j|^{1/j+1} \rightrightarrows D(K_1)e^{u_{K_1}(z)}$ on compact subsets of $\mathbf{C} - K_1$ and $|w-w_0|\cdots|w-w_k|^{1/k+1} \rightrightarrows D(K_2)e^{u_{K_2}(w)}$ on compact subsets of $\mathbf{C} - K_2$; these facts imply that $L_d f \rightrightarrows f$ on $K = K_1 \times K_2$. We refer the reader to [**21**] for details. □

5. Open questions. We end the discussion with a list of open problems involving polynomial interpolation, both in $\mathbf{C}^1$ and $\mathbf{C}^n$, $n > 1$. The list is 'chronological' in the sense of following the order of presentation in the first two chapters. Many problems listed below are first mentioned by others; we make no claims of originality nor do we make judgments as to the difficulty of the questions.

(5.1) Do the one-variable Leja sequences $\{z_j\}$ defined in Section 2, Remark 2.5, satisfy 1?

(5.2) If $K \subset \mathbf{C}^n$ $(n \geq 1)$ is compact, non (pluri)-polar and *not* regular, does there exist a measure μ with $\operatorname{supp}\mu \subset K$ such that (K,μ) satisfies (BM) (see Theorem 3.3)? More specifically, does μ_K satisfy (BM) in this case?

(5.3) Do the normalized Chebyshev polynomials $\{\tilde{t}_i \equiv (t_i(z)/||t_i||_k)\}$ satisfy $\overline{\lim}_{i\to+\infty} |\tilde{t}_i(z)|^{1/k(i)|} = e^{u_K(z)}$, $z \in \mathbf{C}^n - \hat{K}$, if K is regular? If K is completely circular (see the remark after Theorem 3.3), then the result is true [**20**, Prop. 5.4].

(5.4) Do the several-variable Leja sequences $\{\xi_j\}$ defined in Proposition (3.10) satisfy 1? Do they satisfy 4?

(5.5) For a compact, polynomially convex, nonpluripolar, regular compact set $K \subset \mathbf{C}^n$ $(n > 1)$ does $2 \Rightarrow 3$? Does $2 \Rightarrow 4$?

(5.6) For sets as in (5.5), does an array $\{A_{dj}\} \subset K$ of Fekete points satisfy 3?

(5.7) Find an example of a compact set $K \subset \mathbf{C}^n$, $n > 1$, and an *explicit* array $\{A_{dj}\} \subset K$ which satisfies 4.

(5.8) Find an example of a compact set $K \subset \mathbf{C}^n$, $n > 1$, for which one can *explicitly* construct an array of Fekete points.

(5.9) Does there exist a Bos array in B_2 with $L(G) = -2/3$ (if so, we get a negative answer to (5.5))?

(5.10) Do the arrays in Theorem 4.8 satisfy 1?

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