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SEPARATION THEOREMS FOR NONSELFADJOINT DIFFERENTIAL SYSTEMS

E.C. TOMASTIK

ABSTRACT. Conditions are given that identify certain solutions of the system of differential equations $x^{(n)} - (-1)^{n-k} \cdot q(t)x = 0$ that must have at least one component that vanishes. Here $q(t)$ is an $m \times m$ matrix of continuous functions that is positive with respect to a certain cone. The results presented are new even for second order self-adjoint systems and for the general scalar equation.

1. Introduction. This paper is concerned with separation theorems for the differential equation

$$x^{(n)} - (-1)^{n-k} q(t) x = 0, \tag{1}$$

where $n \geq 2$ and k is an integer with $1 \leq k \leq n-1$ and where $q(t)$ is an $m \times m$ matrix of functions continuous on the interval $[a, b]$ with $a \geq 0$, subject to the conjugate point type boundary conditions

$$\begin{cases} x^{(i)}(a) = \zeta^i, & i = 0, \dots, k-1, \\ x^{(i)}(b) = \eta^i, & i = 0, \dots, n-k-1. \end{cases} \tag{2}$$

(Also considered is the second order system given by (11) in Section 3, which is more general than (1) for $n = 2$.) More specifically, conditions will be given that identify certain solutions of (1) that must have at least one component that vanishes. Since no assumptions are made on the integer k or on the symmetry of $q(t)$, (1) will in general be nonself-adjoint. But even if (1) is self-adjoint, the results presented here are new. The results are new for the second order case also since the hypothesis on $q(t)$ given in this paper is not as restrictive as that given by the author in [**4**]. The results are also new in the general scalar case.

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Throughout this paper it is assumed that some partition $\{I, J\}$ of the integers $\{1, \dots, m\}$ has been given, i.e., $I \cup J = \{1, \dots, m\}$ with $I \cap J = \phi$, and that the cone K is given by

$$K = \{(z_1, \dots, z_m) : i \in I \Rightarrow z_i \geq 0, \qquad i \in J \Rightarrow z_i \leq 0\},$$

and that $q(t)$ satisfies the following "positivity" condition

$$q(t) : K \to K, \qquad t \in [a, b].$$

It is easy to see that this condition simply implies that

$$q_{ij}(t) = |q_{ij}(t)|\delta_i \delta_j \tag{3}$$

where $q = (q_{ij})$ and where $\delta_\mu = 1$ if $\mu \in I$ and $\delta_\mu = -1$ if $\mu \in J$.

A point $b = c_{k,n-k}(a) > a$ is called the (first) $(k, n-k)$-conjugate point of a if there exists a nontrivial solution $x(t)$ of (1) such that

$$\begin{cases} x^{(i)}(a) = 0, & i = 0, \dots, k-1, \\ x^{(i)}(b) = 0, & i = 0, \dots, n-k-1 \end{cases} \tag{4}$$

and there is no nontrivial solution $z(t)$ of (1) with $z^{(i)}(a) = 0$, $i = 0, \dots, k-1$ and $z^{(i)}(\beta) = 0$, $i = 0, \dots, n-k-1$, where $a < \beta < c_{k,n-k}(a) = b$.

There are very few separation theorems in the literature for (1) of the sort that assures that a given solution of (1) will vanish or that some component will vanish. The author [**4**] has given a separation theorem for (1) in the $n = 2$ case but used more restrictive hypotheses than given here.

Another such separation theorem has been given by Ahmad and Lazer [**1**] for the case $n = 2$. They assume that $q_{ij}(t) \geq 0$ if $i \neq j$. Under these conditions, they show that if b is the first conjugate point of a, then any solution of (1) with all components strictly positive at $t = a$ must have at least one component that vanishes on $(a, b]$. In this paper Corollary 2 gives a stronger conclusion for a (in general) different class of differential equations. Namely, that if K is the cone given above and $q(t)$ satisfies the positivity condition given above and $x(t)$ is any solution of (1) in the second order case, with $x(t) \in K^0$, the interior of K, in some right deleted neighborhood of a, then at least one component of $x(t)$ must vanish on $(a, b]$. Thus, Corollary 2 permits a solution to have zero components at $t = a$ whereas the result in [**1**] does not. Both results apply when one assumes that $q_{ij}(t) \geq 0$ for all $i, j = 1, \dots, m$. In this context, K is the first quadrant.

Morse [**2**] has given a separation theorem in the second order self-adjoint case, i.e., the case when the matrix $q(t)$ is symmetric. Although this theorem is a very important result, it does not actually permit one to conclude that a *particular* solution of (1) vanishes or that a component vanishes. Rather, the theorem only permits one to conclude that some unknown linear combination of m certain solutions will vanish. To be more precise, notice that if X is an $m \times m$ matrix solution of the matrix equation $X'' + q(t)X = 0$, where $q(t)$ is assumed to be symmetric, then a very easy calculation shows that

$$[X^*(t)X'(t) - X^{*\prime}(t)X(t)]' \equiv 0,$$

where "*" indicates transpose. Thus,

$$X^*(t)X'(t) - X^{*\prime}(t)X(t) \equiv C,$$

where C is an $m \times m$ constant matrix. One then says that X is a *prepared* matrix solution if C is the zero matrix. Two important examples are the two unique matrix solutions $U(t)$ and $V(t)$ given by the initial conditions $U(a) = 0$, $U'(a) = E$, and $V(a) = E$, $V'(a) = 0$, where E is the identity matrix. Morse's separation theorem says that the number of zeros, counting multiplicity, of $\det X_1(t)$ and $\det X_2(t)$, can differ by at most m, if $X_1(t)$ and $X_2(t)$ are two prepared matrix solutions and "det" denotes the determinant.

Now it is well known that the first conjugate point of a for this second order self-adjoint differential equation is the first zero after a of $\det U(t)$, where $U(t)$ has been defined in the previous paragraph. Thus, if b is the first conjugate point of a, then $\det U(t)$ has $m+1$ zeros on $[a, b]$. By the Morse separation theorem, the matrix solution $V(t)$, for example, defined above must have $\det V(t) = 0$ in $[a, b]$. The only thing that can be concluded from this about *vector* solutions of the corresponding second order self-adjoint equation is that some linear combination of the solutions that make up the columns of the matrix $V(t)$ must vanish on $[a, b]$. Although this is certainly useful information, one does not know which particular solution will vanish. In contrast, this paper will identify certain particular solutions that will have a component that vanishes.

The following example is instructive in what one may expect to happen. Consider the second order self-adjoint differential equation $x'' + qx = 0$, where q is the constant 2×2 symmetric matrix with all entries equal to one. Then four linearly independent solutions are $x_1^*(t) = (1, -1)$, $x_2^*(t) = (t, -t)$, $x_3^*(t) = (\sin\sqrt{2}t, \sin\sqrt{2}t)$, $x_4^*(t) = (\cos\sqrt{2}t, \cos\sqrt{2}t)$. Notice that the first solution does not vanish at all and the second solution vanishes only once, while the other two solutions oscillate. Thus, even in the second order self-adjoint case, one can have a solution that oscillates, while another solution never even has one component that vanishes.

For the example in the previous paragraph, one can readily calculate the prepared matrix solution $U(t)$ and easily discover that the first conjugate point of $a = 0$ is $\pi/\sqrt{2}$. One consequence of the results presented in this paper is that any solution of the equation given in the previous paragraph that starts out in the positive first quadrant must have one component that vanishes on $(0, \pi/\sqrt{2})$. Consider then as an example the specific solution $x_5^*(t) = (2t + \cos\sqrt{2}t, -2t + \cos\sqrt{2}t)$ of the second order self-adjoint differential equation of the previous paragraph. One sees that $x_5^*(0) = (1, 1)$ and that the second component vanishes on $(0, \pi/\sqrt{2})$ but that the first component does not vanish at all. This demonstrates that, even in the self-adjoint case, only one component needs to vanish. Thus, in the results presented in this paper, one cannot in general expect anything more than this.

Consider now the differential operator

$$(-1)^{n-k} y^{(n)} = 0,$$

subject to the $(k, n-k)$-conjugate point boundary condition (4). Then the Green's function of this operator subject to (4) is given by

$$g(t,s) = H\left(\frac{(t-a)(b-s)}{b-a}, \frac{(s-a)(b-t)}{b-a}\right)$$

where

$$H(v,u) = \frac{1}{(n-k-1)!(k-1)!} \int_0^{\delta} (v-\xi)^{k-1}(u-\xi)^{n-k-1}\, d\xi, \\ \delta = \min\{u, v\}.$$

Thus, $y(t)$ is a solution of (1) subject to (4) if and only if

$$y(t) = \int_a^b g(t,s)q(s)y(s)\,ds.$$

Notice that it is possible for a *component* of $y(t)$ to be identically equal to zero. For example, if (1) were the self-adjoint second order equation with $q(t)$ diagonal with diagonal elements equal to 4 and 1 and $a = 0$, then $y^*(t) = (\sin 2t, 0)$.

It is said that the vector function $p^*(t) = (p_1(t), \dots, p_m(t))$ is an interpolating polynomial for (2) if $p_1(t), \dots, p_m(t)$ are polynomials of degree $(n-1)$ and

$$\begin{aligned} p^{(i)}(a) &= \zeta^i, \qquad i = 0, \dots, k-1, \\ p^{(i)}(b) &= \eta^i, \qquad i = 0, \dots, n-k-1. \end{aligned}$$

It is well known that (2) has a unique interpolating polynomial (cf. [**3**]).

Now define the possibly empty sets M and N by the following

$$\text{(5)} \qquad \begin{cases} M = \{\mu \in \{1, \dots, m\} : \zeta^i_\mu = 0, \quad i = 0, \dots, k-1, \\ N = \{\mu \in \{1, \dots, m\} : \eta^i_\mu = 0, \quad i = 0, \dots, n-k-1. \end{cases}$$

The following further assumptions will be made on $p(t)$.

$$\text{(6)} \qquad \begin{cases} \mu \in M \Rightarrow \delta_\mu p_\mu^{(k)}(a) > 0, \\ \nu \in N \Rightarrow (-1)^{n-k}\delta_\nu p_\nu^{(n-k)}(b) > 0, \end{cases}$$

where M and N are the same as in (5).

2. A general separation theorem. A general separation theorem can now be given.

Theorem 1. *Assume that* $c_{k,n-k}(a) = b$ *and that* $p(t)$ *is the unique interpolating polynomial for* (2) *and that* $p(t) \in K^0$ *for all* $t \in (a, b)$ *and* $p(t)$ *satisfies* (6). *If* $x(t)$ *is a solution of* (1) *that satisfies* (2), *and if* $x(t) \in K^0$ *in some right deleted neighborhood of* a *or* $x(t) \in K^0$

in some left deleted neighborhood of b*, then* $x(t)$ *has a component that vanishes on* (a,b).

In order to prove the theorem, notice that since $c_{k,n-k}(a)=b$, there exists a nontrivial solution $y(t)$ of (1) that satisfies (4) and also satisfies

$$y(t)=\int_a^b g(t,s)q(s)y(s)\,ds.$$

It should be emphasized that since the hypothesis on $q(t)$ is weaker than that given in [**5**] for $n=2$ or in [**4**] for $n=n$, it cannot be concluded that $y(t)$ is in the *interior* of the cone, K, for $t\in(a,b)$, although this could be the case. As was pointed out earlier, some components of $y(t)$ could even be identically equal to zero.

If $x(t)$ given in the theorem exists, then it must satisfy

$$x(t)=p(t)+\int_a^b g(t,s)q(s)x(s)\,ds.$$

Now assume contrary to the conclusion of the theorem that no component of $x(t)$ vanishes on (a,b), i.e., $x(t)\in K^0$ for all $t\in(a,b)$.

Of course, $y(t)$ has a zero at $t=a$ of order at least equal to k and a zero at $t=b$ of order at least equal to $n-k$.

If $i\notin M$, then $x_i(t)$ has a zero at $t=a$ of order at most equal to $k-1$. Suppose now that $i\in M$ and consider

$$x^{(k)}(a)=p^{(k)}(a)+\int_a^b \frac{\partial^k}{\partial t^k}g(a,s)q(s)x(s)\,ds, \tag{7}$$

and

$$x_i^{(k)}(a)=p_i^{(k)}(a)+\sum_{j=1}^m\int_a^b \frac{\partial^k}{\partial t^k}g(a,s)q_{ij}(s)x_j(s)\,ds. \tag{8}$$

It is well known that $(\partial^k/\partial t^k)g(a,s)>0$, $a<s<b$. Also, since $x(s)\in K$, $q(s)x(s)\in K$ for all $s\in[a,b]$. It follows that the last term in (7) is in K. It follows then that the sum in (8) has sign equal to

the sign of δ_i or is zero. Now, since $p_i^{(k)}(a)$ has by hypothesis the same sign as δ_i and is *not* zero, $x_i^{(k)}(a)$ is not zero (and has the same sign as δ_i). Thus, $x_i(t)$ has a zero at $t = a$ of at most equal to k.

If $i \notin N$, then $x_i(t)$ has a zero at $t = b$ of order at most equal to $n - k - 1$. If $i \in N$, then one can show in a manner similar to that in the previous paragraph that $x_i(t)$ has a zero at $t = b$ of order at most equal to $n - k$. The proof follows since it is well known that $(-1)^{n-k}(\partial^{n-k}/\partial t^{n-k})g(b, s) > 0$ on (a, b).

As a consequence, the following terms are certainly finite numbers:

$$||y_i|| = \sup_{t \in (a,b)} |y_i(t)|/|x_i(t)|, \qquad ||y|| = \max_{i=1,\dots,m} ||y_i||.$$

Since the assumption that $x(t) \in K^0$ for $t \in (a, b)$ implies that $x_i(t) = |x_i(t)|\delta_i$, it follows that

$$\begin{aligned} |y_i(t)| &= \left| \sum_{j=1}^m \int_a^b g(t,s) q_{ij}(s) y_j(s)\, ds \right| \\ &\le \sum_{j=1}^m \int_a^b g(t,s) |q_{ij}(s)|\, |x_j(s)|\, |y_j(s)|\, |x_j^{-1}(s)|\, ds \\ &\le \sum_{j=1}^m \int_a^b g(t,s) |q_{ij}(s)|\, |x_j(s)|\, ds ||y|| \\ &= \sum_{j=1}^m \int_a^b g(t,s) q_{ij}(s) \delta_i \delta_j x_j(s) \delta_j\, ds ||y|| \\ &= \delta_i \sum_{j=1}^m \int_a^b g(t,s) q_{ij}(s) x_j(s)\, ds ||y||. \end{aligned}$$

Thus,

$$\frac{|y_i(t)|}{|x_i(t)|} \le \frac{1}{x_i(t)} \sum_{j=1}^m \int_a^b g(t,s) q_{ij}(s) x_j(s)\, ds ||y|| \tag{9}$$

$$= \frac{1}{x_i(t)} [x_i(t) - p_i(t)] ||y||. \tag{10}$$

The last term is strictly less than $||y||$ for all $t \in (a,b)$. This follows since for $t \in (a,b)$, $p_i(t)$ and $x_i(t)$ are of the same sign due to the fact that both $p(t)$ and $x(t)$ are in K^0. If $i \notin M$, then a simple application of L'Hospital's rule (if necessary) assures that $(p_i/x_i)(a+) = 1$. If $i \notin N$, then also $(p_i/x_i)(b-) = 1$. If $i \in M$, then $(p_i/x_i)(a+) = p_i^{(k)}(a)/x_i^{(k)}(a) \neq 0$. Also, $i \in N$ implies $(p_i/x_i)(b-) = p_i^{(n-k)}(b)/x_i^{(n-k)}(b) \neq 0$. Thus, it follows that the term in front of $||y||$ in (10) is bounded strictly away from 1 on (a,b). It can then be concluded that $||y_i|| < ||y||$ for $i = 1, \dots, m$. Thus, $||y|| < ||y||$. This contradiction then establishes the theorem.

The case $n = 2$ is certainly the most important. For this case, one can say considerably more than Theorem 1. Suppose, for example, that $x(a) \in K^0$. If now $x(b) \notin K^0$, then $x(t)$ must have left K^0, and therefore, by continuity considerations, $x(t)$ must have one component that vanishes. Thus, one can assume that $x(b) \in K^0$. But for $n = 2$, the components of the interpolating polynomial $p(t)$ are simply linear functions; thus, the assumption that $x(a)$ and $x(b)$ both are in K^0 *automatically* implies that $p(t) \in K^0$ for all $t \in [a,b]$. Then Theorem 1 assures that one component of $x(t)$ must vanish on (a,b). If, for some μ, $x_\mu(a) = 0$, then, in the terminology of Theorem 1, $\mu \in M$. If $\mu \in I$ and one assumes that $x_\mu(b) > 0$, then necessarily $p'_\mu(a) > 0$, since $p_\mu(t)$ is a linear function. If $\mu \in J$ and one assumes that $x_\mu(b) < 0$, then necessarily $p'_\mu(a) < 0$. Thus, it has been established that under certain conditions $\mu \in M \Rightarrow \delta_\mu p'_\mu(a) > 0$. The following corollary has thus been established, where $c_{1,1}(a)$ has been set equal to $c(a)$ since there is only one possible type of conjugate point when $n = 2$.

Corollary 2. *If $n = 2$ and $c(a) = b$ and $x(t)$ is a solution of* (1) *with $x(a) \in K^0$ or if $x(a) = 0$, then $x'(a) \in K^0$, then $x(t)$ has a component that vanishes on $(a,b]$.*

Recall that the solution $x_t(t)$ to the second order self-adjoint example given in the introduction for which $x_5(a) \in K^0$ but only one component of $x(t)$ vanished on $(a, c(a))$. Consider also the solution $x_6^*(t) = (t + \sin\sqrt{2}t, -t + \sin\sqrt{2}t)$. Then one sees that $x_6(t) \in K^0$ in some right deleted neighborhood of 0, since $x_6'(0) \in K^0$ and that the second component vanishes on $(0, \pi/\sqrt{2})$ but that the first component does not vanish after $t = 0$. Thus, one realizes that in general the conclusion in Corollary 2 cannot be strengthened.

An analogous corollary can be given for general n if $k = n - 1$. But first a more general result can be given based on the observation that for certain boundary conditions the interpolating polynomial is *automatically* in K^0 for all $t \in (a, b)$, just as was observed for the $n = 2$ case. The following is such a result.

Corollary 3. *Suppose that* $c_{k,n-k}(a) = b$ *and that* $x(t)$ *is a solution of* (1) *with* $x^{(i)}(t) \in K^0$ *in some right deleted neighborhood of* a *for* $i = 0, \dots, k-1$, *and* $(-1)^i x^{(i)}(t) \in K^0$ *in some left deleted neighborhood of* b *for* $i = 1, \dots, n-k-1$. *Then* $x(t)$ *has a component that vanishes on* $(a, b]$.

To prove the corollary, notice that the hypothesis assures that $x(t) \in K^0$ for t sufficiently close and to the right of a. Thus, if $x(b) \notin K^0$, then $x(t)$ must have left K^0 and, therefore, by continuity considerations, $x(t)$ must have one component that vanishes. Thus, one can assume that $x(b) \in K^0$.

Let $p(t)$ be the unique interpolating polynomial for (2). Then it will be shown that $p(t) \in K^0$ for all $t \in (a, b)$. Assume $i \in \{1, \dots, m\}$ has been given and that $\phi(t)$ denotes $p_i(t)$. Consider the case that $i \in I$, the other case being similar.

Now suppose that $j < k - 1$ and there exists $\tau \in (a, b)$ such that $\phi^{(j)}(\tau) = 0$. Then there exists $c \in (a, \tau)$ such that $\phi^{(j+1)}(c) = 0$, otherwise $\phi^{(j+1)}(t) > 0$ on (a, τ). This latter fact follows since $j + 1 \leq k - 1$ and, therefore, by the hypothesis, $\phi^{(j+1)}(t) > 0$ in some right deleted neighborhood of a. Then $\phi^{(j)}(t)$ is strictly increasing on (a, τ). But this is impossible since $\phi^{(j)}(t) > 0$ in some right deleted neighborhood of a and $\phi^{(j)}(\tau) = 0$.

Now suppose that $j < n - k - 1$ and there exists $\tau \in (a, b)$ such that $\phi^{(j)}(\tau) = 0$. Then there exists $c \in (\tau, b)$ such that $\phi^{(j+1)}(c) = 0$, otherwise $(-1)^j \phi^{(j+1)}(t) < 0$ on (τ, b). This latter fact follows since $j+1 \leq n-k-1$ and, therefore, by the hypothesis $(-1)^{j+1} \phi^{(j+1)}(b-) > 0$ in some left deleted neighborhood of b. Then $(-1)^j \phi^{(j)}(t)$ is strictly decreasing on (τ, b). But this is impossible since $(-1)^j \phi^{(j)}(\tau) = 0$ and $(-1)^j \phi^{(j)}(t) > 0$ in some left deleted neighborhood of b.

Now assume that $k-1 \geq n-k-1$, the other case being similar. Suppose that $j < n-k-1$ and that $\phi^{(j)}(t_i) = 0$ for $i = 1, \dots, \rho$, where $a < t_1 < \cdots < t_\rho < b$. Then, by Rolle's theorem, $\phi^{(j+1)}(t)$ has a zero on each of the intervals (t_i, t_{i+1}), for a total of at least $\rho - 1$ zeros on $[t_1, t_\rho]$. But, by the results in the previous two paragraphs, $\phi^{(j+1)}(t)$ has a zero on (a, t_1) and also on (t_ρ, b). Thus, $\phi^{(j+1)}(t)$ has at least $\rho + 1$ zeros on (a, b).

If $\phi(t)$ has a zero on (a,b) of order exactly one, then $\phi(t)$ must become negative on (a,b) and then, since $\phi(t)$ is positive to the right of a and to the left of b, $\phi(t)$ must have two zeros on (a,b). Then, by the previous discussion, $\phi'(t)$ must have at least three zeros on (a,b). If, on the other hand, $\phi(t)$ has a zero of order two or larger at $\tau \in (a,b)$, then this means that $\phi'(t)$ has a zero at $\tau \in (a,b)$. Then, by the previous discussion, $\phi'(t)$ will once again have three zeros on (a,b).

Then $\phi''(t)$ will have at least four zeros on $(a,b), \dots$, and $\phi^{(n-k-1)}(t)$ will have $n-k+1$ zeros on (a,b). Now this is already a contradiction in the case that $k-1 = n-k-1$, i.e., $n = 2k$, since the degree of $\phi^{(n-k-1)}(t)$ equals $(n-1)-(n-k-1) = k$, whereas $\phi^{(n-k-1)}(t)$ vanishes $n-k+1 = 2k-k+1 = k+1$ times.

So now assume that $k-1 > n-k-1$. Then, by Rolle's theorem, $\phi^{(n-k)}(t)$ vanishes at least $n-k$ times on (a,b) since it has already been established that $\phi^{(n-k-1)}(t)$ vanishes at least $n-k+1$ times on (a,b). It further follows that $\phi^{(n-k)}(t)$ vanishes between a and the first zero of $\phi^{(n-k-1)}(t)$ since $n-k-1 < k-1$. Thus, $\phi^{(n-k)}(t)$ vanishes $n-k+1$ times on (a,b). One can proceed in a similar fashion until one has that $\phi^{(k-1)}(t)$ vanishes $n-k+1$ times on (a,b). But this contradicts the fact that the degree of $\phi^{(k-1)}(t)$ is $(n-1)-(k-1) = n-k$.

It has thus been show that the interpolating polynomials is in K^0 for all $t \in (a,b)$ and, therefore, from Theorem 1, $x(t)$ must have one component that vanishes on (a,b). This concludes the proof of Corollary 3.

Returning now to the case that $k = n-1$, Corollary 3 leads to the following result.

Corollary 4. *Assume $c_{n-1,1}(a) = b$ and let $x(t)$ be a solution of* (1) *with $x^{(i)}(t) \in K^0$ in some right deleted neighborhood of a for*

$i = 0, \dots, n-2$. *Then* $x(t)$ *must have a component that vanishes in* $(a, b]$.

Solutions such as those described in Corollary 4 must exist by the fundamental existence theory.

Corollary 4 does not allow for the possibility that some $x^i(t)$ are not in the interior of K in some right deleted neighborhood of a. The following corollary allows for this possibility and takes a different approach.

Corollary 5. *Suppose* $c_{n-1,1}(a) = b$ *and let* $x(t)$ *be a solution of* (1) *such that*

$$x(a) = \zeta^0 \in K^0,$$

$$\delta_i x_i^{(j)}(a) \geq (-1)^j \frac{(n-1)\dots(n-j)}{(b-a)^j} \zeta_i^0, \ j = 1, \dots, n-2, \ i = 1, \dots, m.$$

Then $x(t)$ *has a component that vanishes in* $(a, b]$.

To prove the corollary, define the $(n-1)$-degree polynomials $\phi_j(t)$, $j = 0, \dots, n-2$, by

$$\phi_j^i(a) = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases}$$

and

$$\phi_j(b) = 0.$$

Also define

$$\phi_{n-1}(t) = \left(\frac{t-a}{b-a}\right)^{n-1}.$$

These polynomials are unique [**3**] and are positive on (a, b). If $\phi(t)$ is a polynomial of degree $(n-1)$ with prescribed values for $\phi^{(i)}(a)$, $i = 0, \dots, n-2$, and $\phi(b)$, then $\phi(t)$ can be written uniquely as

$$\phi(t) = \sum_{i=0}^{n-2} \phi^{(i)}(a)\phi_i(t) + \phi(b)\phi_{n-1}(t).$$

If $x(b) \notin K^0$, then by continuity one component of $x(t)$ must vanish on $(a, b]$. Assume then that $x(b) \in K^0$. Let $i \in I$, the case $i \in J$

being similar. Let $p(t)$ be the interpolating polynomial for (1) and let $p_i(t) = \phi(t)$. Define

$$r(t) = \zeta \left(\frac{b-t}{b-a}\right)^{n-1},$$

where $\zeta = \zeta_i^0$. Notice that

$$r^{(j)}(a) = (-1)^j \frac{(n-1)\dots(n-j)}{(b-a)^j}\zeta, \qquad j = 1, \dots , n-2,$$

and, therefore,

$$r(t) = \zeta\phi_0(t) + \sum_{j=1}^{n-2} r^{(j)}(a)\phi_j(t).$$

Then

$$\phi(t) - r(t) = \sum_{j=1}^{n-2}\left[\phi^{(j)}(a) - (-1)^j \frac{(n-1)\dots(n-j)}{(b-a)^j}\zeta\right]\phi_j(t) + \phi(b)\phi_{n-1}(t)$$

and this term is positive on $(a, b]$ by hypothesis, the assumption that $\phi(b) = x_i(b) > 0$, and the fact that $\phi_0(t), \dots , \phi_{n-1}(t)$ are all positive on (a, b), while $\phi_0(t), \phi_1(t), \dots , \phi_{n-2}(t)$ are zero at $t = b$ and $\phi_{n-1}(b) = 1$. Thus, $\phi(t) > 0$ on $(a, b]$ and Theorem 1 assures that $x(t)$ has a component that vanishes on $(a, b]$.

3. A general second order system. In this section a separation theorem is given for the second order differential equation

$$(r(t)x')' + q(t)x = 0, \tag{11}$$

where $r(t)$ is an $m\times m$ matrix of continuous functions on $[a, b]$, $r(t)$ and $\int_a^t r^{-1}(s)\,ds$ are both nonsingular for all $t \in [a, b]$. It is also assumed that $r^{-1}(t)$ satisfies the "positivity" condition

$$r^{-1}(t) : K \to K,\ t \in [a, b],$$

which is the same condition satisfied by $q(t)$. Furthermore, define the sets K_t and D_t by

$$K_t = \left(\int_a^t r^{-1}(s)\,ds\right)(K),$$

$$D_t = \left(\int_a^t r^{-1}(s)\,ds\right)^{-1}(K),$$

and assume throughout the rest of this paper that

$$K_t \subset K_b,\ D_b \subset D_t \quad \text{for all } t \in (a,b].$$

For more details on these conditions, see [**6**]. Considered here will be only how such conditions can easily be checked. For simplicity, consider the case $m = 2$ and K the positive first quadrant. Then the boundary of K_t is composed of the two rays determined by $(\int_a^t r^{-1}(s)\,ds)\vec{\imath}$ and $(\int_a^t r^{-1}(s)\,ds)\vec{\jmath}$, and these vectors are just the columns of $\int_a^t r^{-1}(s)\,ds$. Using these two vectors, one can easily determine if the cone K_t, which is a subset of the cone K, is in K_b for all $t \in (a,b]$. In a similar fashion, the boundary of D_t is determined by the columns of $(\int_a^t r^{-1}(s)\,ds)^{-1}$. Using these two column vectors one can easily determine if the set D_t, which contains the cone K, contains D_b for all $t \in (a,b]$.

Consider now the differential operator

$$D_r x(t) = -(r(t)x'(t))'$$

subject to the conjugate point boundary condition

$$x(a) = x(b) = 0. \tag{12}$$

It is easy to see that the Green's matrix for this differential operator subject to the conjugate point boundary conditions given above is given by

(13)

$$G(t,s) = \begin{cases} \int_t^b r^{-1}(\xi)\,d\xi(\int_a^b r^{-1}(\xi)\,d\xi)^{-1}\int_a^s r^{-1}(\xi)\,d\xi, & a \le s \le t \le b \\ \int_a^t r^{-1}(\xi)\,d\xi(\int_a^b r^{-1}(\xi)\,d\xi)^{-1}\int_s^b r^{-1}(\xi)\,d\xi, & a \le t \le s \le b. \end{cases}$$

Thus, $y(t)$ is a solution of (11) and satisfies (12) if and only if

$$y(t) = \int_a^b G(t,s)q(s)y(s)\,ds.$$

The following separation theorem can now be given for (11).

Theorem 6. *If $c(a) = b$ and $x(t)$ is a solution of (11) with $x(t) \in K^0$ in some right deleted neighborhood of a, then $x(t)$ has a component that vanishes on $(a,b]$.*

The proof of this theorem is the same as the proof of Theorem 1, once one realizes that the assumptions $K_t \subset K_b$ and $D_b \subset D_t$ for all $t \in (a, b]$ implies that

$$G(t,s) : K \to K$$

for all $s, t \in [a, b]$. This has been established in [**6**].

There is no other separation theorem in the literature for the differential equation (11) in the case that $r(t)$ is not the identity matrix.

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Department of Mathematics, University of Connecticut, Storrs, Connecticut 06269