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ON STRONG LIFTING COMPACTNESS FOR THE WEAK* TOPOLOGY

W. STRAUSS

Dedicated to Prof. B. Volkmann on the occasion of his 60th birthday

Introduction. The notion of lifting compactness was introduced in [**3**], that of strong lifting compactness in [**1**], both for completely regular Hausdorff spaces. It turned out in [**1**] that the strong lifting compactness of a Banach space X under its weak topology is equivalent with each of the following strong properties.

(SL) For every Baire measure μ on X any lifting of $\mathfrak{L}^\infty(\mu)$ is almost strong, respectively there exists an almost strong lifting for $\mathfrak{L}^\infty(\mu)$.

(SB) Every scalarly measurable function from a complete probability space into X is Bochner measurable.

It is therefore natural to ask whether these equivalences hold also for other locally convex topologies. In this paper, we check the weak* topology on conjugates of Banach spaces. In Theorem 3.4 we give a characterization of such conjugate Banach spaces which satisfy condition (SB), from which it becomes obvious that condition (SB) is neither equivalent with condition (SL) nor with strong lifting compactness of the conjugate under its weak* topology (see also the examples in 3.8). We call Banach spaces which satisfy the equivalent conditions of Theorem 3.4 SB*-spaces. These spaces form a strong counterpart to the well-known class of Asplund spaces, since they are definable by a strict equivalence instead of a weak equivalence for measurable functions which characterizes Asplund spaces in the sense of [**20**, **18**]. As a preparation, this weak equivalence characterization for Asplund spaces is derived in Theorem 2.4, and at the same time an equivalent lifting invariance condition. We also introduce L*-spaces, M*-spaces, and (strict) W*-spaces which are related to SB*-spaces and to Banach spaces whose conjugate has the weak Radon Nikodym property of [**17**]. Weak* strongly lifting compact spaces, i.e., such

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spaces whose conjugates are strongly lifting compact under their weak* topologies, are characterized by a lifting condition for vector valued functions in Theorem 3.1, and M*-spaces are as well in Theorem 2.10. These characterizations are basic for the permanence properties which are discussed in Section 4. The techniques of this paper rely on lifting properties for vector valued functions. The classes of spaces considered here may also have interesting geometrical properties, but here we are only interested in measure theoretical problems and properties.

In [**11**] A. Bellow proved the powerful separation property for Banach spaces, a property which is obviously related to lifting invariance. In Section 2 we consider this property for the weak* topology, and if this property is fulfilled we call the Banach space a space with the weak* separation property. A characterization of these spaces in terms of lifting invariance conditions is given in Theorem 2.3. Any Banach space with the weak* separation property is an Asplund space, and the converse holds for separable spaces. This makes it easy to see that the weak* separation property does not hold in general in contrast to the separation property of A. Bellow.

1. Preliminaries. Throughout, (Ω, Σ, μ) denotes a complete probability space, $\mathfrak{L}^0(\mu)$ is the space of all Σ-measurable real valued functions on Ω, $\mathfrak{L}^\infty(\mu)$ is the space of all bounded Σ-measurable functions on Ω, $\mathfrak{L}^p(\mu)$ is the space of all p-integrable functions with respect to (Ω, Σ, μ), and $L^p(\mu)$ denotes the space of all classes $\hat{f}$ of functions $f \in \mathfrak{L}^p(\mu)$ modulo null functions for $0 \leq p \leq \infty$. For a Banach space Y we write $\mathfrak{L}^0_Y(\mu)$ for the space of all Bochner measurable functions from Ω into Y and $\mathfrak{L}^\infty_Y(\mu)$ for all $\psi \in \mathfrak{L}^0_Y(\mu)$ such that $||\psi||_\infty = \sup\{||\psi(\omega)|| : \omega \in \Omega\} < \infty$. For a separated duality (X, Y) and $0 \leq p \leq \infty$, let $\mathfrak{L}^p_{(X,Y)}(\mu)$ be the space of all functions ϕ from Ω into X such that $(\phi, y) \in \mathfrak{L}^p(\mu)$ for all $y \in Y$. If Y is a Banach space with dual space Y', then $\mathfrak{L}^0_{(Y,Y')}(\mu)$ is the space of all scalarly measurable functions from Ω into Y and $\mathfrak{L}^0_{(Y',Y)}(\mu)$ is the set of all weak* measurable functions from Ω into Y'. For $f, g \in \mathfrak{L}^0_{(X,Y)}(\mu)$ we write $f = g$ a.e. (μ) if $\mu(\{f \neq g\}) = 0$, and we define $f \equiv g\sigma(X, Y)$, the weak equivalence of f and g, by means of $(f, y) = (g, y)$ a.e. (μ) for all $y \in Y$. If Y is a Banach space and Γ a linear subspace of Y' total on Y, then $\mathfrak{B}_{(Y,\Gamma)}(\mu)$, the space of all Γ-uniformly bounded functions, is the set of all $\phi \in \mathfrak{L}^0_{(Y,\Gamma)}(\mu)$ for which there exists a constant $M < \infty$ such that $|(\phi, y')| \leq M||y'||$ a.e. (μ) for every $y' \in \Gamma$.

All unexplained topological measure theoretic notions will be those of [**12, 15** and **22**]. Those concerning lifting theory may be found in [**10**], those concerning (strong) lifting compactness, Baire liftings, Baire measurability, and Bochner measurability in [**1**]. We will frequently use the following remark which is an immediate consequence of [**21**, 3-3-3].

Remark 1.1. If Γ is a linear subspace of Y', total on the Banach space Y, then for any $\phi \in \mathfrak{L}^0_{(Y,\Gamma)}(\mu)$ there exists a sequence of Γ-uniformly bounded functions $\phi_n \in \mathfrak{B}_{(Y,\Gamma)}(\mu)$ having pairwise disjoint supports $\operatorname{supp}(\phi_n)$ $\phi = \sum_{n=1}^{\infty} \phi_n$, and moreover, there exist pairwise disjoint $S_n \in \sum$ such that $\operatorname{supp}(\phi_n) \subseteq S_n$ for $n \in \mathbf{N}$, $\cup_{n=1}^{\infty} S_n = \Omega$. Besides the (multiplicative) lifting for $\mathfrak{L}^\infty(\mu)$ in the sense of [**10**, Chapter III, Definition 2] we need in addition a rather weak type of lifting for $\mathfrak{L}^0(\mu)$ which we call vector lifting. This is defined to be a linear map λ from $\mathfrak{L}^0(\mu)$ into $\mathfrak{L}^0(\mu)$ such that $\lambda(f) = f$ a.e. (μ) and $\lambda(f) = \lambda(g)$ if $f = g$ a.e. (μ) for $f, g \in \mathfrak{L}^0(\mu)$. Applying the axiom of choice, we get the following result which implies the existence of vector liftings.

Lemma 1.2. *If $\mathfrak{L}$ is a linear subspace of $\mathfrak{L}^0(\mu)$ and $\Omega_0 \in \Sigma$ such that $\mu(\Omega_0) = 1$, and $f = g$ a.e. (μ) implies $f(\omega) = g(\omega)$ for $\omega \in \Omega_0$ and $f, g \in \mathfrak{L}$, then there exists a vector lifting λ for $\mathfrak{L}^0(\mu)$ such that $\lambda(f)(\omega) = f(\omega)$ for $\omega \in \Omega_0$ and $f \in \mathfrak{L}$.*

Proof. For $\hat{g}$ in the linear subspace $S = \{\hat{g} \in L^0(\mu) : L \cap \hat{g} \neq \varnothing\}$ of $L^0(\mu)$ we define a map λ_0 from S into $\mathfrak{L}^0(\mu)$ unambiguously by means of

$$\lambda_0(\hat{g}) = f\chi_{\Omega_0} \quad \text{if} \quad f \in \mathfrak{L} \cap \hat{g}.$$

We then choose a Hamel basis $(\hat{g}_i)_{i\in I}$ for the linear complement C of S in $L^0(\mu)$ and then $h_i \in \hat{g}_i$ for $i \in I$. For $\hat{g} \in C$ we can write $\hat{g} = \Sigma_{i\in I}\alpha_i\hat{g}_i$ with $\alpha_i \in \mathbf{R}$ for $i \in I$ and $\alpha_i \neq 0$ for at most finitely many $i \in I$. We then put $\lambda_1(\hat{g}) = \Sigma_{i\in I}\alpha_i h_i$ and $\hat{\lambda}(h) = \lambda_0(\hat{h}_0) + \lambda_1(\hat{g})$ if $\hat{h} = \hat{h}_0 + \hat{g}$ uniquely for $\hat{h} \in L^0(\mu)$ with $\hat{h}_0 \in S$ and $\hat{g} \in C$. If r is the canonical map of $\mathfrak{L}^0(\mu)$ onto $L^0(\mu)$, then put $\lambda = \hat{\lambda} \circ r$. □

Let Y be a Banach space. For a lifting ρ of $\mathfrak{L}^\infty(\mu)$, we define for $\phi \in \mathfrak{B}_{(Y',Y)}(\mu)$ a map $\rho^*\phi$ from Ω into Y' by means of

$$(y, \rho^*\phi) = \rho(y, \phi) \qquad \text{for} \quad y \in Y,$$

and for $\phi \in \mathfrak{B}_{(Y,Y')}(\mu)$ a map $\rho'\phi$ from Ω into Y'' by means of

$$(\rho'\phi, y') = \rho(\phi, y') \qquad \text{for} \quad y' \in Y'.$$

If λ is a vector lifting for $\mathfrak{L}^0(\mu)$ then let $\lambda^*\phi$ for $\phi \in \mathfrak{L}^0_{(Y',Y)}(\mu)$ be defined in analogy with $\rho^*\phi$, accordingly $\lambda'\phi$ for $\phi \in \mathfrak{L}^0_{(Y,Y')}(\mu)$ in analogy with $\rho'\phi$. Then $\lambda^*\phi$ is a map from Ω into Y^*, the linear dual space of Y, and $\lambda'\phi$ is a map from Ω into Y'^*, the linear dual space of Y'.

For a topological space (Ω, τ) with topology τ we denote by Baire (Ω, τ) the Baire σ-field, and by Borel (Ω, τ) the Borel σ-field of Ω with respect to the topology τ. We write $(\Omega, \Sigma, \mu)^\sim = (\Omega, \tilde{\Sigma}, \tilde{\mu})$ for the Carathéodory completion of the measure space (Ω, Σ, μ).

2. The weak* separation property. For a Banach space Y and a complete probability space (Ω, Σ, μ) we call a function $\phi \in \mathfrak{L}^0_{(Y,Y')}(\mu)$ separating if there exists a set $\Omega_0 \in \Sigma$ with $\mu(\Omega_0) = 1$ such that $(\phi, y') = (\phi, z')$ a.e. (μ) for all y', $z' \in Y'$ implies $(\phi(\omega), y') = (\phi(\omega), z')$ for all $\omega \in \Omega_0$.

Lemma 2.1. *Any $\phi \in \mathfrak{L}^0_Y(\mu)$ is separating. If, in addition, $\phi \in \mathfrak{B}_{(Y,Y')}(\mu)$, then $\phi = \rho'\phi$ a.e.(μ) for any lifting ρ of $\mathfrak{L}^\infty(\mu)$ and $||\rho'\phi||_\infty < \infty$.*

Proof. Let us first assume that $\phi \in \mathfrak{B}_{(Y,Y')}(\mu)$ in addition. We then may choose a sequence $(\phi_n)_{n\in\mathbf{N}}$ of simple functions such that $\lim_{n\to\infty} \phi_n = \phi$ a.e. (μ). By Egorov, there exists for any $m \in \mathbf{N}$ a set $\Omega_m \in \Sigma$ such that $\mu(\Omega \backslash \Omega_m) < 1/m$ and $\lim_{n\to\infty} \phi_n = \phi$ uniformly on Ω_m, and in addition $\Omega_m \subseteq \Omega_{m+1}$ for $m \in \mathbf{N}$. If $\phi_n = \sum_{i=1}^{k_n} y_{in}\chi_{A_{in}}$ for $y_{in} \in Y$, $A_{in} \in \Sigma$, we may assume $A_{in} = \rho(A_{in})$ for $1 \le i \le k_n$, $n \in \mathbf{N}$, and a lifting ρ of $\mathfrak{L}^\infty(\mu)$. Then $\rho'\phi_n = \phi_n$ for $n \in \mathbf{N}$. For any $\varepsilon > 0$ there exists an $n_0 \in \mathbf{N}$ such that $||(\phi_n - \phi, y')\chi_{\Omega_m}||_\infty \le \varepsilon$ for $n \ge n_0$, $||y'|| \le 1$. Then $|(\rho(\phi_n, y') - \rho(\phi, y'))\chi_{\rho(\Omega_m)}||_\infty \le \varepsilon$

for $n \geq n_0$, $||y'|| \leq 1$, i.e., we have $||(\phi_n - \rho'\phi)\chi_{\rho(\Omega_m)}||_\infty \leq \varepsilon$ for $n \geq n_0$. If $\Omega_0 := \cup_{m=1}^\infty (\Omega_m \cap \rho(\Omega_m))$, then $\mu(\Omega_0) = 1$ and $(\rho'\phi)(\omega) = \lim_{n\to\infty} \phi_n(\omega) = \phi(\omega)$ for $\omega \in \Omega_0$.

If only $\phi \in \mathfrak{L}^0_Y(\mu)$, let $\Omega_n := \{||\phi|| < n\} \in \Sigma$. Then $\Omega_0 = \cup_{n=1}^\infty \Omega_n$, $\Omega_n \in \Sigma$ since $||\phi|| \in \mathfrak{L}^0(\mu)$. Put $A_n := \Omega_n \setminus \cup_{i=1}^{n-1} \Omega_i$, $\phi_n := \phi\chi_{A_n}$ for $n \in \mathbf{N}$. Since $||\phi_n|| \leq n$ we find by the last paragraph sets $B_n \in \Sigma$ such that $B_n \subseteq A_n$, $\mu(B_n) = \mu(A_n)$, and $(\rho'\phi_n)(\omega) = \phi_n(\omega) = \phi(\omega)$ for $\omega \in B_n$, $n \in \mathbf{N}$. If $(\phi, y') = (\phi, z')$ a.e. (μ) for $y', z' \in Y'$, then $(\rho'\phi_n, y') = (\rho'\phi_n, z')$; therefore, $(\phi(\omega), y') = ((\rho'\phi_n)(\omega), y') = ((\rho'\phi_n)(\omega), z') = (\phi(\omega), z')$ for $\omega \in \Omega_1 := \cup_{n=1}^\infty B_n \in \Sigma$, $\mu(\Omega_1) = 1$. □

We call $\phi \in \mathfrak{L}^0_{(Y',Y)}(\mu)$ weak* separating if there is some $\Omega_0 \in \Sigma$ with $\mu(\Omega_0) = 1$ such that $(y, \phi) = (z, \phi)$ a.e. (μ) for $y, z \in Y$ implies always $(y, \phi(\omega)) = (z, \phi(\omega))$ for all $\omega \in \Omega_0$. Since we may in the above proof interchange Y and Y', we have the following complement of Lemma 2.1.

Remark 2.2. Any $\phi \in \mathfrak{L}^0_{Y'}(\mu)$ is weak* separating. If, in addition, $\phi \in \mathfrak{B}_{(Y',Y)}(\mu)$, then $\phi = \rho^*\phi$ a.e. (μ) for any lifting ρ of $\mathfrak{L}^\infty(\mu)$ and $||\rho^*\phi||_\infty < \infty$.

A Banach space Y is called weak* separating if for any complete probability space (Ω, Σ, μ) any weak* separating map $\phi \in \mathfrak{L}^0_{(Y',Y)}(\mu)$ is in $\mathfrak{L}^0_{Y'}(\mu)$. Let us call $\phi \in \mathfrak{L}^0_{(Y',Y)}(\mu)$ vector lifting invariant if $\phi = \lambda^*\phi$ a.e. (μ) for some vector lifting λ of $\mathfrak{L}^0(\mu)$. Clearly, any vector lifting invariant ϕ is weak* separating. The converse holds also since for a weak* separating ϕ there exists $\Omega_0 \in \Sigma$ such that $\mu(\Omega_0) = 1$ and $(y, \phi) = (z, \phi)$ a.e. (μ) for y, $z \in Y$ implies $(y, \phi(\omega)) = (z, \phi(\omega))$ for $\omega \in \Omega_0$, and for $\mathfrak{L} := \{(y, \phi) : y \in Y\}$. Lemma 1.2 gives a vector lifting λ of $\mathfrak{L}^0(\mu)$ such that $\lambda(y, \phi) = (y, \phi)$, i.e., $(\lambda^*\phi)(\omega) = \phi(\omega)$ for $\omega \in \Omega_0$.

Theorem 2.3. *For a Banach space Y the following conditions are all equivalent.*

(i) *Y has the weak* separation property.*

(ii) *For any complete probability space (Ω, Σ, μ) any weak* separating $\phi \in \mathfrak{B}_{(Y',Y)}(\mu)$ is in $\mathfrak{L}^\infty_{Y'}(\mu)$.*

(iii) *For any complete probability space* (Ω, Σ, μ) *any vector lifting invariant* $\phi \in \mathfrak{L}^0_{(Y',Y)}(\mu)$ *is in* $\mathfrak{L}^0_{Y'}(\mu)$.

Proof. The equivalence of (i) and (iii) follows from the remark preceding the theorem. The implication (i) $\Rightarrow$ (ii) is trivial since $||\rho^*\phi||_\infty < \infty$ for $\phi \in \mathfrak{B}_{(Y',Y)}(\mu)$ and any lifting ρ of $\mathfrak{L}^\infty(\mu)$.

(ii) $\Rightarrow$ (i). For $\phi \in \mathfrak{L}^0_{(Y',Y)}(\mu)$ we choose by Remark 1.1 $\phi_n \in \mathfrak{B}_{(Y',Y)}(\mu)$ such that $\phi = \sum_{n=1}^\infty \phi_n$, $S_n \in \Sigma$ with $\operatorname{supp}(\phi_n) \subseteq S_n$ and $S_n \cap S_m = \varnothing$ for $n \neq m$, $n, m \in \mathbf{N}$. Since $\{(y, \phi_n) \neq (z, \phi_n)\} \subseteq \{(y, \phi) \neq (z, \phi)\} \cap S_n$, we have ϕ_n weak* separating if ϕ is, and the result follows. □

A Banach space Y is called an Asplund space if Y' has the Radon-Nikodym property RNP (see [**6**, p. 61 and p. 213]). This definition is the most convenient one in our context. Equivalent conditions can be found in [**19, 20, 8, 9, 21** and **6**]. The equivalence of the conditions (ii) and (iii) of the following theorem is stated in [**21**, 3-4-1]. But I could find nowhere the equivalence of (i) and (ii) which is just what we need for Section 3 below and makes Asplund spaces useful for our purposes. We call $\phi \in \mathfrak{B}_{(Y',Y)}(\mu)$ weak* lifting invariant if we have $\rho^*\phi = \phi$ a.e. (μ) for some lifting ρ of $\mathfrak{L}^\infty(\mu)$.

Theorem 2.4. *For a Banach space* Y *the following conditions are all equivalent.*

(i) Y *is an Asplund space.*

(ii) *For any complete probability space* (Ω, Σ, μ) *any weak* lifting invariant* $\phi \in \mathfrak{B}_{(Y',Y)}(\mu)$ *is in* $\mathfrak{L}^\infty_{Y'}(\mu)$.

(iii) *For any complete probability space* (Ω, Σ, μ) *and any function* $\phi \in \mathfrak{B}_{(Y',Y)}(\mu)$, *there exists a function* $\psi \in \mathfrak{L}^\infty_{Y'}(\mu)$ *such that*

$$\phi \equiv \psi \sigma(Y', Y).$$

(iv) *For any complete probability space* (Ω, Σ, μ) *and any map* $\phi \in \mathfrak{L}^0_{(Y',Y)}(\mu)$, *there exists a map* $\psi \in \mathfrak{L}^0_{Y'}(\mu)$ *such that*

$$\phi \equiv \psi \sigma(Y', Y).$$

Proof. (i) $\Rightarrow$ (ii). Let Y be Asplund. For $\phi \in \mathfrak{B}_{(Y',Y)}(\mu)$, we define a continuous linear map u from $L^1(\mu)$ into Y' by means of

$$(y, u(\hat{g})) = \int (y, \phi) g \, d\mu \qquad \text{for} \quad y \in Y \text{ and } \hat{g} \in L^1(\mu).$$

By [**6**, Chapter III, Theorem 5] there exists a $\psi \in \mathfrak{L}^\infty_{Y'}(\mu)$ such that we have $(y, u(\hat{g})) = \int (y, \psi) g \, d\mu$ for $y \in Y$ and $\hat{g} \in L^1(\mu)$. This implies that $\phi \equiv \psi \sigma(Y', Y)$. By Remark 2.2, it holds that $\rho^* \psi = \psi$ a.e. (μ) for any lifting ρ of $\mathfrak{L}^\infty(\mu)$. If $\phi = \rho^* \phi$ a.e. (μ) for some lifting ρ of $\mathfrak{L}^\infty(\mu)$ then $\phi = \psi$ a.e. (μ) since $\rho^* \phi = \rho^* \psi$, i.e., $\phi \in \mathfrak{L}^\infty_{Y'}$.

(ii) $\Rightarrow$ (iii). For $\phi \in \mathfrak{B}_{(Y',Y)}(\mu)$ is $\psi := \rho^* \phi \in \mathfrak{B}_{(Y',Y)}(\mu)$ weak* lifting invariant and $\phi = \psi \sigma(Y', Y)$. By assumption, it holds that $\psi \in \mathfrak{L}^\infty_{Y'}(\mu)$.

(iii) $\Rightarrow$ (i). By [**6**, Chapter III, Theorem 5] it is sufficient to verify the Riesz representation property. For a continuous linear map u from $L^1(\mu)$ into Y', we find by [**10**, Chapter VII, Theorem 1, Corollary 1] a function $\phi \in \mathfrak{B}_{(Y',Y)}(\mu)$ such that $(y, u(\hat{g})) = \int (y, \phi) g \, d\mu$ for $y \in Y$ and $\hat{g} \in L^1(\mu)$. By assumption, there exists $\psi \in \mathfrak{L}^\infty_{Y'}(\mu)$ with $\phi = \psi \sigma(Y', Y)$. But then also $(y, u(\hat{g})) = \int (y, \psi) g \, d\mu$ for $y \in Y$ and $\hat{g} \in L^1(\mu)$.

(iv) $\Rightarrow$ (iii). For $\phi \in \mathfrak{B}_{(Y',Y)}(\mu)$, we have by assumption $\psi \in \mathfrak{L}^0_{Y'}(\mu)$ with $\phi = \psi \sigma(Y', Y)$. But then $\psi \in \mathfrak{B}_{(Y',Y)}(\mu)$, $||\rho^* \psi||_\infty < \infty$, and $\rho^* \phi = \phi$ a.e. (μ) by Remark 2.2. Then $\rho^* \psi \in \mathfrak{L}^\infty_{Y'}(\mu)$ and $\phi \equiv \rho^* \psi \sigma(Y', Y)$.

(iii) $\Rightarrow$ (iv). For $\phi \in \mathfrak{L}^0_{(Y',Y)}(\mu)$ we choose by Remark 1.1 sets $S_n \in \Sigma$ and $\phi_n \in \mathfrak{B}_{(Y',Y)}(\mu)$ with $\operatorname{supp}(\phi_n) \subseteq S_n$, $S_n \cap S_m = \varnothing$ for $n \neq m$, and $\phi = \sum_{n=1}^\infty \phi_n$ $(n, m \in \mathbf{N})$. By (iii), we find $\psi_n \in \mathfrak{L}^\infty_{Y'}(\mu)$ with $\phi_n \equiv \psi_n \sigma(Y', Y)$ for $n \in \mathbf{N}$, where we may assume $\operatorname{supp}(\psi_n) \subseteq S_n$ for $n \in \mathbf{N}$. Then $\psi := \sum_{n=1}^\infty \psi_n \in \mathfrak{L}^0_{Y'}(\mu)$ and $\phi = \psi \sigma(Y', Y)$. □

Corollary 2.5. *Any Banach space Y having the weak* separating property is an Asplund space.*

Proof. Clearly, condition (ii) of Theorem 2.3 implies condition (ii) of Theorem 2.4. □

Theorem 2.6. *A separable Banach space Y has the weak* separation property if and only if Y is an Asplund space, and then all conditions listed in Theorems* 2.3 *and* 2.4 *are all equivalent.*

Proof. Let Y be an Asplund space and let $\phi \in \mathfrak{B}_{(Y',Y)}(\mu)$ have the weak* separation property, i.e., there exists $\Omega_0 \in \Sigma$ with $\mu(\Omega_0) = 1$, and for $y, z \in Y$ holds $(y, \phi(\omega)) = (z, \phi(\omega))$ for $\omega \in \Omega_0$ if and only if $(y, \rho^*\phi) = (z, \rho^*\phi)$. Then kernel $(\phi(\omega))$ = kernel $(\rho^*\phi(\omega))$ for $\omega \in \Omega_0$, and therefore there exist $\alpha(\omega)$, $\beta(\omega) \in \mathbf{R}$ with $\phi(\omega) = \alpha(\omega)(\rho^*\phi)(\omega)$ and $(\rho^*\phi)(\omega) = \beta(\omega)\phi(\omega)$ for $\omega \in \Omega_0$. For $y \in Y$, there exists $\Omega_y \in \Sigma$ such that $\mu(\Omega_y) = 1$ and $(y, \phi(\omega)) = \rho(y, \phi)(\omega) = (y, \rho^*\phi(\omega)) = \beta(\omega)(y, \phi(\omega))$ for $\omega \in \Omega_y \cap \Omega_0$. This implies that $\beta(\omega) = 1$ for $\omega \in \Omega_y \cap \Omega_0 \cap \{(y, \phi) \neq 0\} =: B_y$. If $(y_n)_{n\in\mathbf{N}}$ is a dense sequence in Y, then we have

$$\{\phi \neq 0\} = \cup_{n=1}^{\infty}\{(y_n, \phi) \neq 0\} = \cup_{n=1}^{\infty} B_{y_n} =: B \text{ a.e.}(\mu).$$

Since $\beta(\omega) = 1$ for $\omega \in B$ we have $(\rho^*\phi)(\omega) = \phi(\omega)$ for all $\omega \in \Omega_0'$ if we define $\Omega_0' := B \cup \{\phi = 0\}$. But $\mu(\Omega_0') = 1$, i.e., ϕ is weak* lifting invariant, hence by Theorem 2.4 (ii), $\phi \in \mathfrak{L}^{\infty}_{Y'}(\mu)$. By Theorem 2.3, Y has the weak* separation property. □

By Theorem 2.6, the spaces c_0, l_p for $1 < p < \infty$ have the weak* separation property while l_∞ does not, by Corollary 2.5. A Banach space Y is called M^*-space if for any complete probability space (Ω, Σ, μ) and every $\eta \in \mathfrak{L}^0_{(Y',Y'')}(\mu)$ there exists a $\psi \in \mathfrak{L}^0_{Y'}(\mu)$ such that $\eta \equiv \psi\sigma(Y', Y)$, and we call Y an L^*-space if for any complete probability space (Ω, Σ, μ) any weak* lifting invariant $\phi \in \mathfrak{B}_{(Y',Y)}(\mu)$ is in $\mathfrak{L}^0_{(Y',Y'')}(\mu)$. A W^*-space is a Banach space Y such that for any complete probability space (Ω, Σ, μ) and any $\phi \in \mathfrak{L}^0_{(Y',Y)}(\mu)$ there exists an $\eta \in \mathfrak{L}^0_{(Y',Y'')}(\mu)$ such that $\phi \equiv \eta\sigma(Y', Y)$.

Lemma 2.7. *Any Asplund space is an L^*-space, and any L^*-space is a W^*-space.*

Proof. Y is a W^*-space by Remark 1.1 if and only if for any complete probability space (Ω, Σ, μ) and any $\phi \in \mathfrak{B}_{(Y',Y)}(\mu)$ there exists a map $\eta \in \mathfrak{L}^0_{(Y',Y'')}(\mu)$ with $\phi \equiv \eta\sigma(Y',Y)$. For an L^*-space Y and $\phi \in \mathfrak{B}_{Y',Y)}(\mu)$ there is $\eta := \rho^*\phi \in \mathfrak{B}_{(Y',Y)}(\mu)$, η is weak* lifting invariant, $\phi \equiv \eta\sigma(Y',Y)$, and hence η is in $\mathfrak{L}^0_{(Y',Y'')}(\mu)$ by assumption. □

Theorem 2.8. *For a Banach space Y the following conditions are all equivalent.*

(i) *Y is an Asplund space.*

(ii) *Y is an M^*-space and a W^*-space.*

(iii) *Y is an M^*-space and Y is an L^*-space.*

Proof. The implication (i) ⇒ (iii) is obvious from Theorem 2.4 since $\mathfrak{L}^0_{(Y',Y'')}(\mu) \subseteq \mathfrak{L}^0_{(Y',Y)}(\mu)$, and (iii) ⇒ (ii) is a consequence of Lemma 2.7. For the implication (ii) ⇒ (i), let $\phi \in \mathfrak{B}_{(Y',Y)}(\mu)$. Since Y is a W^*-space, we find a function $\eta \in \mathfrak{L}^0_{(Y',Y'')}(\mu)$ with $\phi \equiv \eta\sigma(Y',Y)$. But Y is an M^*-space. So we can choose a map $\psi \in \mathfrak{L}^0_{Y'}(\mu)$ such that $\eta \equiv \psi\sigma(Y',Y)$. Then $\phi \equiv \psi\sigma(Y',Y)$, i.e., Y is Asplund by Theorem 2.4. □

Remarks 2.9. A Banach space Y is a W^*-space if and only if Y' has the weak Radon Nikodym property WRNP (see [**17**] for definition) provided (i) Y is separable (see [**4**, 7.4.11]) or (ii) Y is separably complementable (see [**17**], Theorem 5] where [**17**, Theorem 5′] asserts the equivalence for arbitrary Banach spaces without proof). If a Banach space Y is not Asplund but a W^*-space, then Y is no M^*-space by Theorem 2.8. This situation is given e.g., for $Y = JT$, the separable James tree space by [**4**, p. 308] and the above remark (i). By [**8**, Proposition 5.4] any Banach space Y is an M^*-space if (Y', weak) is measure compact, the converse implication does not hold as witnessed by c^o_Γ for measure compact discrete space Γ (see end of Section 4). If we assume that the continuum τ is not measure compact, then Y is Asplund, hence an L^*-space and a W^*-space if (Y', weak) is measure compact by [**8**, Section 4 (3)].

If $Y = C[0,1]$, the separable space of all continuous functions from $[0,1]$ into $\mathbf{R}$, then we have (Y', weak) strongly lifting compact, hence measure compact by [**1**, 4.10 (iii)] if we assume τ measure compact.

Therefore, Y is an M^*-space. But Y does not possess the WRNP by [**17**, 161], hence there is no W^*-space by (i) above. By Lemma 2.7, Y is no L^*-space, and it does not have the weak* separation property by Corollary 2.5.

Theorem 2.10. *A Banach space Y is an M^*-space if and only if for any lifting ρ of $\mathfrak{L}^\infty(\mu)$ holds $\rho^*\eta \in \mathfrak{L}^\infty_{Y'}(\mu)$ for any $\eta \in \mathfrak{B}_{(Y',Y'')}(\mu)$.*

Proof. If $\eta \in \mathfrak{B}_{(Y',Y'')}(\mu)$ for an M^*-space Y, there exists a $\psi \in \mathfrak{L}^0_{Y'}(\mu)$ such that $\eta \equiv \psi\sigma(Y',Y)$. Then $\rho^*\eta = \rho^*\psi$ for any lifting ρ of $\mathfrak{L}^\infty(\mu)$. By Remark 2.2., $\rho^*\psi = \psi$ a.e. (μ) holds. Therefore, $\rho^*\eta \in \mathfrak{L}^\infty_{Y'}(\mu)$. For the converse, let us assume only that for any $\eta \in \mathfrak{B}_{(Y',Y'')}(\mu)$, there exists a lifting ρ of $\mathfrak{L}^\infty(\mu)$ with $\rho^*\eta \in \mathfrak{L}^\infty_{Y'}(\mu)$. For a given map $\eta \in \mathfrak{L}^0_{(Y',Y'')}(\mu)$, we choose by Remark 1.1 functions $\eta_n \in \mathfrak{B}_{(Y',Y'')}(\mu)$ and sets $S_n \in \Sigma$ with $\operatorname{supp}(\phi_n) \subseteq S_n$ for $n \in \mathbf{N}$, $S_n \cap S_m = \varnothing$ for $n \neq m$, $\cup_{n=1}^\infty S_n = \Omega$, $\sum_{n=1}^\infty \phi_n = \phi$. For any $n \in \mathbf{N}$ there exists a lifting ρ_n of $\mathfrak{L}^\infty(\mu)$ such that $\psi_n := \rho_n\eta_n \in \mathfrak{L}^\infty_{Y'}(\mu)$. Since $\operatorname{supp}(\psi_n) \subseteq \rho_n(S_n) := \rho_n(\chi_{S_n})$ the functions $\xi_n := \psi_n\chi_{S_n}$ have disjoint supports and satisfy $\xi_n \in \mathfrak{L}^\infty_{Y'}(\mu)$ and $\xi_n = \psi_n$ a.e. (μ) as well as $\eta_n \equiv \xi_n\sigma(Y',Y)$ since $\eta_n \equiv \psi_n\sigma(Y',Y)$ for $n \in \mathbf{N}$. This implies that $\eta \equiv \xi\sigma(Y',Y)$ and $\xi \in \mathfrak{L}^0_{Y'}(\mu)$. □

Remark . The second part of the above proof shows that a Banach space Y is an M^*-space if and only if for any $\eta \in \mathfrak{B}_{(Y',Y'')}(\mu)$ there exists a lifting ρ of $\mathfrak{L}^\infty(\mu)$ such that $\rho^*\eta \in \mathfrak{L}^\infty_{Y'}(\mu)$.

For $\phi \in \mathfrak{L}^0_{(Y',Y)}(\mu)$ we choose $\phi_n \in \mathfrak{B}_{(Y',Y)}(\mu)$ and S_n according to Lemma 1.1. If we apply for some lifting ρ of $\mathfrak{L}^\infty(\mu)$ Lemma 1.2 for $\mathfrak{L} := \{\sum_{n=1}^\infty \rho(y,\phi_n)\chi_{\rho(S_n)} : y \in Y\}$ and $\Omega_0 := \cup_{n=1}^\infty \rho(S_n)$ a vector lifting λ for $\mathfrak{L}^0(\mu)$ is obtained such that $(\lambda^*\phi)(\omega) = (\rho^*\phi_n)(\omega) \in Y'$ for $\omega \in \rho(S_n)$ and $n \in \mathbf{N}$. For $\phi \in \mathfrak{L}^0_{(Y,Y')}(\mu)$ and $\omega \in \Omega$ only $(\lambda'\phi)(\omega) \in Y'^*$ follows for arbitrary vector lifting λ of $\mathfrak{L}^0(\mu)$. We therefore call a Banach space Y linear lifting compact if for any complete probability space (Ω,Σ,μ) and any $\phi \in \mathfrak{L}^0_{(Y,Y')}(\mu)$ there exists a vector lifting λ for $\mathfrak{L}^0(\mu)$ and $\Omega_0 \in \Sigma$ with $\mu(\Omega_0) = 1$ such that $(\lambda'\phi)(\Omega_0) \subseteq Y$.

Theorem 2.11. *For a Banach space Y the following conditions are all equivalent.*

(i) *Y is linear lifting compact.*

(ii) *$(Y$, weak$)$ is measure compact.*

(iii) *$(Y$, weak$)$ is lifting compact.*

(iv) *For any complete probability space (Ω, Σ, μ) and any function $\phi \in \mathfrak{B}_{(Y,Y')}(\mu)$ there exists a lifting ρ for $\mathfrak{L}^\infty(\mu)$ and a set $\Omega_0 \in \Sigma$ with $\mu(\Omega_0) = 1$ such that $(\rho'\phi)(\Omega_0) \subseteq Y$.*

Proof. (i) $\Rightarrow$ (ii). If we choose λ and Ω_0 as in the definition of linear lifting compact spaces, then $\psi := (\lambda'\phi)\chi_{\Omega_0}$ has the separation property. Therefore, $\psi \in \mathfrak{L}^0_Y(\mu)$ by [**11**, Theorem 3] and clearly $\phi \equiv \psi\sigma(Y, Y')$ holds. From [**8**, 5.4], it follows that $(Y$, weak$)$ is measure compact. The equivalence of (ii) and (iii) is well known, see, e.g., [**3**, p. 252].

(ii) $\Rightarrow$ (iv). Let $\phi \in \mathfrak{B}_{(Y',Y)}(\mu)$. By [**8**, 5.4], there exists a function $\psi \in \mathfrak{L}^0_Y(\mu)$ with $\phi \equiv \psi\sigma(Y, Y')$. But then $\psi \in \mathfrak{B}_{(Y,Y')}(\mu)$ and by Lemma 2.1, $\psi = \rho'\psi = \rho'\phi$ a.e. (μ), i.e., there exists a set $\Omega_0 \in \Sigma$ with $\mu(\Omega_0) = 1$ such that $(\rho'\phi)(\Omega_0) = \psi(\Omega_0) \subseteq Y$.

(iv) $\Rightarrow$ (i). For $\phi \in \mathfrak{L}^0_{(Y,Y')}(\mu)$ we choose ϕ_n and S_n according to Remark 1.1 and then liftings ρ_n of $\mathfrak{L}^\infty(\mu)$ and $\Omega_n \in \Sigma$ with $\mu(\Omega_n) = \mu(S_n)$, $\Omega_n \subseteq S_n$ such that $(\rho'_n\phi_n)(\Omega_n) \subseteq Y$. If we apply Lemma 1.2 to the space $\mathfrak{L} := \{\sum_{n=1}^\infty \rho_n(\phi_n, y')\chi_{\Omega_n} : y' \in Y'\}$ and $\Omega_0 := \cup_{n=1}^\infty \Omega_n \in \Sigma$ we find a vector lifting λ for $\mathfrak{L}^0(\mu)$ with $\lambda(\phi, y') = \sum_{n=1}^\infty \rho_n(\phi_n, y')\chi_{\Omega_n} = \sum_{n=1}^\infty (\rho'_n\phi_n, y')\chi_{\Omega_n}$. This implies $(\lambda'\phi)(\omega) \in Y$ for $\omega \in \Omega_0$ and $\mu(\Omega_0) = 1$. □

The following characterization of strong lifting compactness in terms of vector valued lifted functions for Banach spaces under their weak topology should be compared with Theorem 3.1.

Theorem 2.12. *For a Banach space Y $(Y$, weak$)$, the Banach space Y under its weak topology $\sigma(Y, Y')$, is strongly lifting compact if and only if for any complete probability space (Ω, Σ, μ) and any lifting ρ of $\mathfrak{L}^\infty(\mu)$ holds $\phi = \rho'\phi$ a.e. (μ) for every $\phi \in \mathfrak{B}_{(Y,Y')}(\mu)$.*

Proof. If (Y, weak) is strongly lifting compact, then every $\phi \in \mathfrak{B}_{(Y,Y')}(\mu)$ is in $\mathfrak{L}^0_Y(\mu)$ by [**1**, Theorem 4.3], hence we have $\phi = \rho'\phi$ a.e. (μ) by Lemma 2.1 for arbitrary liftings ρ of $\mathfrak{L}^\infty(\mu)$. If for $\phi \in \mathfrak{B}_{(Y,Y')}(\mu)$ and $\Omega_0 \in \Sigma$ with $\mu(\Omega_0) = 1$ holds $\phi(\omega) = (\rho'\phi)(\omega)$ for $\omega \in \Omega_0$ for a lifting ρ of $\mathfrak{L}^\infty(\mu)$, then put $\psi(\omega) := \phi(\omega)$ for $\omega \in \Omega_0$, and $\psi(\omega) := 0$ otherwise. Then ψ is separating. By [**11**, Theorem 3] is ψ in $\mathfrak{L}^\infty_Y(\mu)$. Clearly $\phi = \psi$ a.e. (μ) holds. Therefore, (Y, weak) is strongly lifting compact by [**1**, Theorem 4.3]. □

3. The strong lifting compactness for the weak* topology. We call a Banach space Y weak* strongly lifting compact if Y' under its weak* topology $\sigma(Y', Y)$ is strongly lifting compact in the sense of [**1**, Section 3]. By Alaoglu's theorem, Y' is σ-compact under its weak* topology. Hence, it is always weak* strongly measure compact by [**16**, 3.43] and lifting compact by [**3**, Corollary 6.1].

Theorem 3.1. *A Banach space Y is weak* strongly lifting compact if and only if for any complete probability space (Ω, Σ, μ), any lifting ρ of $\mathfrak{L}^\infty(\mu)$ and any $\phi \in \mathfrak{B}_{(Y',Y)}(\mu)$ holds $\rho^*\phi = \phi$ a.e. (μ), i.e., ϕ is weak* lifting invariant for any $\phi \in \mathfrak{B}_{(Y,Y')}(\mu)$.*

Proof. First let Y be weak* strongly lifting compact and $\phi \in \mathfrak{B}_{(Y',Y)}(\mu)$. Then $|(y, \phi)| \leq M||y||$ a.e. (μ) for some constant $M < \infty$ and all $y \in Y$. For $y \in Y$, $r > 0$, let $h_{r,y} := \text{med}\,(-r||y||, y, r||y||)$, if as usual $\text{med}(a, b, c) := \max(\min(a, b), \min(a, c), \min(b, c))$ for $a, b, c \in \mathbf{R}$. Then $h_{r,y} \in C_b(Y')$, the space of all bounded, continuous functions on Y' under the weak* topology, $h_{r,y} \circ \phi = y \circ \phi$ a.e. (μ), and therefore $\rho(h_{r,y} \circ \phi) = \rho(y \circ \phi) = y \circ (\rho^*\phi)$ for $r > 0$, $y \in Y$. By assumption, there exists a set $\Omega_0 \in \Sigma$ such that $\mu(\Omega_0) = 1$, and for all $y \in Y$, $\omega \in \Omega_0$, $r > 0$ holds $\rho(h_{r,y} \circ \phi)(\omega) = (h_{r,y} \circ (\rho'(\phi))(\omega) = (h_{r,y} \circ \phi)(\omega)$ if $\rho'(\phi)$ is defined according to [**1**, p. 213]. This implies

$$(1) \qquad y \circ (\rho^*\phi)(\omega) = (h_{r,y} \circ \phi)(\omega) \quad \text{for} \quad y \in Y, \omega \in \Omega_0,\ r > 0.$$

For $\omega \in \Omega$, $(\rho^*\phi)(\omega) \in Y'$ holds since $||(\rho^*\phi)(\omega)|| \leq M$. For $y \in Y$ we have $|y \circ (\rho^*\phi)| \leq M||y||$ for $y \in Y$, and so we get for $r \geq M$, $\omega \in \Omega$

$$(2) \qquad h_{r,y}((\rho^*\phi)(\omega)) = y((\rho^*\phi)(\omega)).$$

Equations (1) and (2) yield

$$(3)\qquad h_{r,y}((\rho^*\phi)(\omega)) = h_{r,y}(\phi(\omega)) \quad \text{for} \quad y \in Y,\ r \geq M,\ \omega \in \Omega_0.$$

The family $H := \{h_{r,y} : y \in Y, r \geq M\}$ is separating on Y', and therefore (3) implies $(\rho^*\phi)(\omega) = \phi(\omega)$ for $\omega \in \Omega_0$.

For the converse, let ϕ be a Baire-measurable map from Ω into Y' under the weak* topology, i.e., $\phi \in \mathfrak{L}^0_{(Y',Y)}(\mu)$ by [**8**, 2.3]. According to Remark 1.1, we can choose functions $\phi_n \in \mathfrak{B}_{(Y',Y)}(\mu)$ and disjoint sets $S_n \in \Sigma$ with $\operatorname{supp}(\phi_n) \subseteq S_n$ for $n \in N$, $\cup_{n=1}^{\infty} S_n = \Omega$, and $\sum_{n=1}^{\infty} \phi_n = \phi$. By assumption, for any $n \in \mathbf{N}$ there exist sets $\Omega_n \in \Sigma$ with $\mu(\Omega_n) = 1$ and $(\rho^*\phi_n)(\omega) = \phi_n(\omega)$ for $\omega \in \Omega_n$. If we put $A_n := S_n \cap \rho(S_n) \cap \Omega_n$, we have $A_n \in \Sigma$, $A_n \cap A_m = \varnothing$ for $n \neq m$, and $(\rho^*\phi_n)(\omega) = \phi_n(\omega)$ for $\omega \in A_n$, $n \in \mathbf{N}$. Let $\Omega_0 := \cup_{n=1}^{\infty} A_n$. Then $\Omega_0 \in \Sigma$ and $\mu(\Omega_0) = 1$. For $n \in \mathbf{N}$ there exists $M_n < \infty$ with $||(\rho^*\phi_n)||_\infty \leq M_n$. The latter implies $||\phi_n(\omega)|| \leq M_n$ for $\omega \in \Omega_n$, $n \in \mathbf{N}$. If we put $h_{y,r_n} := \operatorname{med}(-r_n, y, r_n)$, then for $y \in Y$, $r_n \geq M_m$ holds $h_{y,r_n} \circ \phi = y \circ \phi_n$, $(h_{y,r_n} \circ \phi)(\omega) = (h_{y,r_n} \circ \phi_n)(\omega) = (y \circ \phi_n)(\omega)$ for $\omega \in S_n$, so we get $\rho(h_{y,r_n} \circ \phi)\chi_{\rho(S_n)} = \rho(y \circ \phi_n)\chi_{\rho(S_n)}$ for $n \in \mathbf{N}$. This yields $\rho(h_{y,r_n} \circ \phi)(\omega) = \rho(y, \phi_n)(\omega) = (y, \phi_n(\omega)) = (h_{y,r_n} \circ \phi_n)(\omega) = (h_{y,r_n} \circ \phi)(\omega)$ for $\omega \in A_n$, $n \in \mathbf{N}$, i.e., $h_{y,r_n}(\rho'(\phi)(\omega)) = h_{y,r_n}(\phi(\omega))$ for $n \in \mathbf{N}$, $y \in Y$, $\omega \in A_n$, $r_n \geq M_n$.

Again, $H := \{h_{y,r_n} : y \in Y, r_n \geq M_n, n \in \mathbf{N}\}$ is a separating family on Y'. Therefore, $(\rho\prime(\phi))(\omega) = \phi(\omega)$ for $\omega \in A_n$, $n \in \mathbf{N}$, and ϕ is strongly lifting compact with respect to the weak* topology. $\square$

Remarks . (1) Since, for $\phi \in \mathfrak{B}_{(Y',Y)}(\mu)$ we don't need the decomposition $\phi = \sum_{n=1}^{\infty} \phi_n$, the above proof shows in fact that a $\phi \in \mathfrak{B}_{(Y',Y)}(\mu)$ is strongly lifting compact with respect to the weak* topology on Y' if and only if $\rho^*\phi = \phi$ a.e. (μ) for any lifting ρ of $\mathfrak{L}^\infty(\mu)$.

(2) If Y is weak* strongly lifting compact, then the mapping ρ of $\mathfrak{B}_{(Y',Y)}(\mu)$ into $\mathfrak{L}^\infty_{Y'}(Y, \mu)$, the space of all functions $\phi \in \mathfrak{B}_{(Y',Y)}(\mu)$ with $||\phi||_\infty < \infty$, is bounded linear with $||\rho||_\infty \leq 1$ and the following holds

$$\rho^*\phi = \phi \text{ a.e. } (\mu),$$
$$\phi = \psi \text{ a.e. } (\mu) \quad \text{implies} \quad \rho^*\phi = \rho^*\psi$$

for $\phi, \psi \in \mathfrak{B}_{(Y',Y)}(\mu)$, i.e., ρ^* is a lifting for vector-valued functions in $\mathfrak{B}_{(Y',Y)}(\mu)$.

Corollary 3.2. *If Y is weak* strongly lifting compact and an M^*-space, then $(Y'$, weak$)$ is strongly lifting compact.*

Proof. Let $\phi \in \mathfrak{L}^0_{(Y',Y'')}(\mu)$ be given. We choose $\phi_n \in \mathfrak{B}_{(Y',Y'')}(\mu)$ and $S_n \in \Sigma$ according to Remark 1.1. We have $\phi_n = \rho^*\phi_n$ a.e. (μ) by Theorem 3.1 and $\rho^*\phi_n \in \mathfrak{L}^\infty_{Y'}(\mu)$ by Theorem 2.10. Since $\operatorname{supp}(\rho^*\phi_n) \subseteq \rho(S_n)$ we may put $\psi := \sum_{n=1}^\infty \rho^*\phi_n$. Then $\psi \in \mathfrak{L}^0_{Y'}(\mu)$ and $\phi = \psi$ a.e. (μ). By [**1**, Theorem 4.3] $(Y'$, weak$)$ is strongly lifting compact. □

Theorem 3.3. *Any separable Banach space Y is weak* strongly lifting compact and for any Baire and Borel measure μ on $(Y'$, weak$^*)$ every lifting of $\mathfrak{L}^\infty(\mu)$ is almost strong (here $(Y'$, weak$^*)$ denotes the space Y' under the weak* topology) and $(Y'$, weak$)$ is strongly lifting compact if and only if $(Y'$, weak$)$ is measure compact.*

Proof. For separable Y, $(Y'$, weak$^*)$ is metrizable, see, e.g., [**7**, V 5.1]. Therefore, Baire $(Y'$, weak$^*)$ = Borel $(Y'$, weak$^*)$, by [**1**; 2.2] $(Y'$, weak$^*)$ is strongly lifting compact, and every lifting of $\mathfrak{L}^\infty(\mu)$ is almost strong for any Baire measure μ on $(Y'$, weak$^*)$ by [**2**, Theorem 4]. Since $(Y'$, weak$)$ is submetrizable this space is strongly measure compact if and only if it is measure compact by [**1**, Theorem 4.7]. □

Example. The space $Y = C[0,1]$ is weak* strongly lifting compact by Theorem 3.3. If we assume the continuum τ is measure compact (i.e., it has no real valued measurable cardinal by [**8**, Section 4 (3)], a mild set theoretic assumption), then $(Y'$, weak$)$ is strongly lifting compact by [**1**, 4.10 (iii)], and Y is an M^*-space which is neither Asplund, nor an L^*-space nor a W^*-space as we have seen under Remark 2.9. But if we assume the continuum τ is not measure compact, then $(Y'$, weak$)$ is not strongly measure compact, not even measure compact. Hence, Y is no M^*-space by Corollary 3.2, and not a W^*-space by Remark 2.9, hence is not Asplund.

A Banach space Y is called SB^*-space if for any complete probability space (Ω, Σ, μ) and any $\phi \in \mathfrak{B}_{(Y',Y)}(\mu)$ there exists a function $\psi \in \mathfrak{L}^\infty_{Y'}(\mu)$ such that $\phi \equiv \psi$ a.e. (μ), and we call Y a strict W^*-space if for any complete probability space (Ω, Σ, μ) and any $\phi \in \mathfrak{B}_{(Y',Y)}(\mu)$ there exists an $\eta \in \mathfrak{L}^0_{(Y',Y'')}(\mu)$ such that $\phi = \eta$ a.e. (μ). Clearly, any strict W^*-space is a W^*-space by Remark 1.1. Every reflexive Banach space is a strict W^*-space.

Theorem 3.4. *For a Banach space Y the following conditions are all equivalent.*

(i) *Y is an* SB**-space.*

(ii) *For any complete probability space (Ω, Σ, μ) and any $\phi \in \mathfrak{L}^0_{(Y',Y)}(\mu)$, there exists a function $\psi \in \mathfrak{L}^0_{Y'}(\mu)$ with $\phi = \psi$ a.e. (μ).*

(iii) *$(Y'$, weak) is strongly lifting compact and Y is a strict W^*-space.*

(iv) *Y is weak* strongly lifting compact and an Asplund space.*

(v) *Y is weak* strongly lifting compact, an M^*-space, and a W^*-space.*

(vi) *Y is weak* strongly lifting compact, an M^*-space, and an L^*-space.*

(vii) *For every Baire probability measure μ on $(Y'$, weak$^*)$ is any weak Baire set μ-measurable, and every lifting of $\mathfrak{L}^\infty(\mu)$ is $C_b(Y'$, weak)-strong (respectively, there exists a $C_b(Y'$, weak)-strong lifting of $\mathfrak{L}^\infty(\mu)$).*

(viii) *$(Y'$, weak) is measure compact and for every Baire probability measure μ on $(Y'$, weak$^*)$ any weak Baire set is μ-measurable.*

(ix) *For any Baire probability measure μ on $(Y'$, weak$^*)$ any weak Baire set is μ-measurable and μ is supported by a μ-measurable closed linear subspace of Y' which is norm-separable.*

(x) *$(Y'$, weak) is measure compact and any norm Borel set in Y' is μ-measurable for any Baire probability measure μ on $(Y'$, weak$^*)$.*

(xi) *$(Y'$, weak) is measure compact and submetrizable, and for every Baire probability measure μ on $(Y'$, weak$^*)$ any weak Baire set is μ-measurable.*

(xii) *For any complete probability space (Ω, Σ, μ), any lifting ρ of $\mathfrak{L}^\infty(\mu)$, and any $\phi \in \mathfrak{B}_{(Y',Y'')}(\mu)$, $\phi = \rho'\phi$ holds a.e. (μ), and every weak Baire subset of Y' is μ-measurable.*

Proof. The implication (i) $\Rightarrow$ (ii) is immediate by Remark 1.1.

(ii) $\Rightarrow$ (iii). Since any $\phi \in \mathfrak{L}^0_{(Y',Y'')}(\mu)$ is in $\mathfrak{L}^0_{(Y',Y)}(\mu)$, there exists by assumption a $\psi \in \mathfrak{L}^0_{Y'}(\mu)$ such that $\phi = \psi$ a.e. (μ). Therefore, (Y', weak) is strongly lifting compact by [**1**, Theorem 4.3].

If $\phi \in \mathfrak{B}_{(Y',Y)}(\mu)$, then again by assumption, we find a $\psi \in \mathfrak{L}^0_{Y'}(\mu)$ with $\phi = \psi$ a.e. (μ). Clearly, $\psi \in \mathfrak{L}^0_{(Y',Y'')}(\mu)$, i.e., Y is a strict W^*-space.

(iii) $\Rightarrow$ (iv). Let $\phi \in \mathfrak{B}_{(Y',Y)}(\mu)$. Since Y is a strict W^*-space, there exists an $\eta \in \mathfrak{L}^0_{(Y',Y'')}(\mu)$ with $\eta = \phi$ a.e. (μ). Since (Y', weak) is strongly lifting compact, we find $\psi \in \mathfrak{L}^0_{Y'}(\mu)$ such that $\eta = \psi$ a.e. (μ) by [**1**, Theorem 4.3]. But then $\phi = \psi$ a.e. (μ), $\rho^*\psi \in \mathfrak{L}^\infty_{Y'}(\mu)$, and $\phi = \rho^*\psi$ a.e. (μ), i.e., Y is an Asplund space. By Remark 2.2 $\rho^*\psi = \psi$ a.e. (μ) holds true. Since $\rho^*\psi = \rho^*\phi$, we have $\phi = \rho^*\phi$ a.e. (μ), i.e., Y is weak* strongly lifting compact by Theorem 3.1.

Condition (iv) is equivalent with each of the conditions (v) and (vi) by Theorem 2.8.

(iv) $\Rightarrow$ (i). Let $\phi \in \mathfrak{B}_{(Y',Y)}(\mu)$. By Theorem 3.1 $\phi = \rho^*\phi$ a.e. (μ) holds since (Y', weak*) is strongly lifting compact. Since Y is Asplund, $\rho^*\phi \in \mathfrak{L}^\infty_{Y'}(\mu)$ follows by Theorem 2.4. Therefore, we put $\psi := \rho^*\phi$ and obtain $\phi = \psi$ a.e. (μ).

Let $\Omega := Y'$, $\Sigma :=$ (Baire(Y', weak),μ)$^\sim$ for a Baire probability measure on (Y', weak*). Then $\phi = id_{Y'} \in \mathfrak{L}^0_{Y',Y)}(\tilde{\mu})$ if $id_{Y'}$ denotes the identical map of Y'. If we now assume Y to be a strict W^*-space, then by Remark 1.1 we find a function $\eta \in \mathfrak{L}^0_{(Y',Y'')}(\tilde{\mu})$ with $\phi = \eta$ a.e. $(\tilde{\mu})$. This yields

$$\text{Baire}\,(Y', \text{weak}) \subseteq \Sigma. \tag{1}$$

If, conversely, we assume (1) and let ϕ be a Baire measurable map from Ω into (Y', weak*), then we define for a complete probability space (Ω, Σ, μ) a σ-algebra $B_\Sigma := \{A \subseteq Y' : \phi^{-1}(A) \in \Sigma\}$ and the image measure λ on B_Σ, $\lambda = \phi(\mu)$ of μ under ϕ by means of $\lambda(A) := \mu(\phi^{-1}(A))$ for $A \in B_\Sigma$. Since Baire (Y', weak) $\subseteq B_\Sigma$ we can define the restriction ν of λ to Baire (Y', weak*). But (Baire (Y', weak*)$, \nu)^\sim \subseteq B_\Sigma$ because B_Σ is complete. By (1) we find for a set $A \in$ Baire(Y', weak) sets B, $C \in B_\Sigma$ with $B \subseteq A \subseteq C$ and $\tilde{\nu}(C \backslash B) = \lambda(C \backslash B) = 0$. This implies $\phi^{-1}(B) \subseteq \phi^{-1}(A) \subseteq \phi^{-1}(C)$ and

$\mu(\phi^{-1}(C)\backslash\phi^{-1}(B)) = \lambda(C\backslash B) = 0$, so $\phi^{-1}(A) \in \Sigma$ since Σ is complete, i.e., $\phi \in \mathfrak{L}^0_{(Y',Y'')}(\mu)$, and we see that Y is a strict W^*-space. So we have the following remark.

(2) A Banach space Y is a strict W^*-space if and only if for any Baire probability measure μ on (Y', weak*),

$$\text{Baire }(Y',\ \text{weak}) \subseteq (\text{Baire }(Y',\ \text{weak}^*), \mu)^{\sim}$$

holds, i.e., any weak Baire set of Y' is μ-measurable.

For this reason we can extend any measure μ on Baire (Y', weak*), to a measure ν on Baire (Y', weak) and the completions with respect to μ and ν are identical.

If we replace in condition (iii) the term "Y is a strict W^*-space" by the equivalent condition (2) and the term "(Y', weak) is strongly lifting compact" by any one of the 13 equivalent conditions given in [**1**, Theorems 4.3 and 4.7, Corollaries 4.5 and 4.8], we get equivalent conditions for "Y is an SB*-space." We have noted above only some of these equivalent conditions. As indicated, condition (vii) follows from [**1**, Theorem 4.3 (iii)], (viii) from [**1**, Theorem 4.3 (iv)], (ix) from [**1**, Theorem 4.3 (v)], (xi) from [**1**, Theorem 4.3 (vi)], (xii) from [**1**, Theorem 4.3 (v)], and (xiii) from Theorem 2.12. □

As a corollary, we obtain a result of [**17**, p. 161], the remark before Theorem 3.

Corollary 3.5. *A separable Banach space is an* SB**-space if and only if it is Asplund. In that case the conditions of Theorems* 2.3, 2.4 *and* 3.4 *are all equivalent.*

According to [**17**, Theorem 3] a separable Banach space is a W^*-space if and only if it is a strict W^*-space or, equivalently, if Y does not contain any isomorphic copy of l_1.

Theorem 3.6. *If the Banach space Y is a strict W^*-space and an M^*-space, then Y is an Asplund space.*

Proof. If $\phi \in \mathfrak{B}_{(Y',Y)}(\mu)$ and Y is a strict W^*-space we find a function $\eta \in \mathfrak{L}^0_{(Y',Y'')}(\mu)$ with $\phi = \eta$ a.e. (μ). By Remark 1.1, we decompose η into functions $\eta_n \in \mathfrak{B}_{(Y',Y'')}(\mu)$ and find $S_n \in \Sigma$ with $\operatorname{supp}(\eta_n) \subseteq S_n$ for $n \in \mathbf{N}$. For a lifting ρ of $L^\infty(\mu)$ we have $\rho^*\eta_n \in \mathfrak{L}^\infty_{Y'}(\mu)$ for $n \in \mathbf{N}$ by Theorem 2.10. Since $\operatorname{supp}(\rho^*\eta_n) \subseteq \rho(S_n)$ for $n \in \mathbf{N}$ we can define the function $\psi := \sum_{n=1}^\infty \rho^*\eta_n$. Since $\psi \in \mathfrak{L}^0_{Y'}(\mu)$ and $\phi \equiv \psi\sigma(Y',Y)$ the space Y must be Asplund by Theorem 2.4. □

Theorem 3.7. *If the Banach space Y is a W^*-space and weak* strongly lifting compact, then Y is an L^*-space.*

Proof. Let $\phi \in \mathfrak{B}_{(Y',Y)}(\mu)$ be weak* lifting invariant, i.e., $\rho^*\phi = \phi$ a.e. (μ) for some lifting ρ of $\mathfrak{L}^\infty(\mu)$. Since Y is a W^*-space, we may choose a function $\eta \in \mathfrak{L}^0_{(Y',Y)}(\mu)$ with $\phi = \eta\sigma(Y',Y)$. This implies $\rho^*\phi = \rho^*\eta = \eta$ a.e. (μ), the latter since $\eta \in \mathfrak{B}_{(Y',Y)}(\mu)$, and Y is weak* strongly lifting compact. Therefore, $\phi = \eta$ a.e. (μ), and so $\phi \in \mathfrak{L}^0_{(Y',Y'')}(\mu)$. □

Examples 3.7. (i) The space $Y = C[0,1]$ is weak* strongly lifting compact but not an Asplund space as we have seen after 3.3 above. Hence, by Theorem 3.4, Y is not an SB*-space. Hence, by [**1**, 2.3], the condition (SL) of the introduction holds for (Y', weak*), i.e., for every Baire measure μ of (Y', weak*), every lifting of $\mathfrak{L}^\infty(\mu)$ is almost strong, but condition (SB) is not satisfied. By Theorem 3.4, Y is no strict W^*-space.

For completely regular Hausdorff space T imbeds T as well as βT, the Stone-Čech compactification of T, homeomorphically into $C_b(T)'$. Hence, for any weak* strongly lifting compact $C_b(T)$, βT as well as T must be necessarily strongly lifting compact by [**1**, 3.1 (i)]; therefore, T must be measure compact at least.

(ii) By Remarks 2.9, the James tree space $Y = JT$ is a W^*-space which is not Asplund, and therefore no SB*-space by Theorem 3.4, nor does Y have the weak* separation property by Corollary 2.5, but Y is an L^*-space by Theorem 3.7, since the separable space JT is weak* strongly lifting compact, and by [**17**, Theorem 3] and [**4**, 7.4.11] a strict W^*-space. By [**9**, 5.8] (Y', weak) is not measure compact, therefore not strongly lifting compact, so Y is no M^*-space by Theorem 3.6 in agreement with [**13**] (compare also [**17**, p. 163]). Since Y is a strict W^*-space, the completions of Baire measures with respect to the weak * as well as with respect to the weak topology are identical by Remark (2) in the proof of Theorem 3.4, hence so are the liftings of their spaces $\mathfrak{L}^\infty(\mu)$. Any such lifting ρ is almost strong with respect to the weak* topology by [**1**, Theorem 2.3] (applied to the identical map), but no such lifting is almost stronig with respect to the weak topology on Y' by [**1**, Theorem 4.3].

(iii) The reflexive space $Y = l^2_{[0,1]}$ is Asplund and a strict W^*-space, hence a W^*-space, an M^*-space, and an L^*-space by Theorem 2.8. By [**1**, 4.10 (ii)] (Y', weak) is not strongly lifting compact, therefore not weak* strongly lifting compact, but it is clearly measure compact. By Theorem 3.4, Y is no SB*-space.

(iv) The separable space $Y = L^1[0,1]$ is no W^*-space by [**17**, Theorem 3 (iv)], therefore no strict W^*-space. Hence, this space is neither Asplund nor an L^*-space by Theorem 2.8 and Lemma 2.7, and not an SB*-space by Theorem 3.4 but weak* strongly lifting compact by Theorem 3.3. By [**9**, Section 6, Example (3)] (Y', weak) is not measure compact, hence not strongly lifting compact. By Corollary 3.2, Y is no M^*-space, and by Corollary 2.3 not weak* separating. The function $\phi(\omega) := \chi_{[0,\omega]}$ of $[0,1]$ into Y' is by [**8**, p. 672] not strongly lifting compact with respect to the weak topology on Y'. Since for $\hat{g} \in L^1[0,1]$ the quantity $(g, \phi(\omega)) = \int_0^\omega g\, d\omega$ is a continuous function of ω and any lifting of $\mathfrak{L}^\infty[0,1]$ is almost strong, we have $\rho^*\phi = \phi$ a.e. (μ) for the Lebesgue measure on $[0,1]$. By Remark (1) following Theorem 3.1, the function ϕ is strongly lifting compact with respect to the weak* topology on Y'.

(v) The separable space l_1 is not a W^*-space by [**17**, Proposition 3] and Remarks 2.9, so it is neither a BS*-space nor an L^*-space. But Y is weak* strongly lifting compact by Theorem 3.3, not measure compact by [**9**, Table 1], hence not strongly lifting compact. So it is no M^*-space by Corollary 3.2, and no strict W^*-space by [**17**, Theorem 3].

4. Permanence properties. The first theorem is a complement to [**1**, Section 3]. The idea of using Michael's selection theorem traces back to [**8**, 6.2]. I am indebted to E. Michael for introducing me to and for commenting upon these selection theorems.

Theorem 4.1. *Let X be a Fréchet space and Y a closed subspace of X. Then (X, weak) is strongly lifting compact if (Y, weak) and (X/Y, weak) are both strongly lifting compact.*

Proof. Let (Y, weak) and (X/Y, weak) both be strongly lifting compact. If ϕ is a scalarly measurable map of Ω into X, then $\pi \circ \phi$ is scalarly measurable, therefore, Bochner measurable by [**1**, Theorem 4.3] since (X/Y, weak) is strongly lifting compact. Again, by [**14**] there exists a section σ from X/Y into X which is continuous with respect to the canonical metrics on X and X/Y for the canonical map π from X onto X/Y. Then $(\sigma \circ \pi \circ \phi)^{-1}(B) \in \Sigma$ for any open ball B in X and since $\pi \circ \phi$ is Bochner measurable, there exists $\Omega_0 \in \Sigma$ such that $\mu(\Omega_0) = 1$ and $(\pi \circ \phi)(\Omega_0)$ separable which implies $(\sigma \circ \pi \circ \phi)(\Omega_0)$ separable. Since X is metrizable $\sigma \circ \pi \circ \phi$ is Bochner measurable.

Again, by [**1**, Theorem 4.3] the scalarly measurable map $\psi = \phi - \sigma \circ \pi \circ \phi$ from Ω into Y is Bochner measurable since (Y, weak) is strongly lifting compact. Therefore, ϕ is Bochner measurable, too, i.e., (X, weak) is strongly lifting compact again by [**1**, Theorem 4.3]. If (X, weak) is strongly lifting compact for a Banach space X, then (Y, weak) is strongly lifting compact for any closed subspace Y by [**1**, 3.1 (i)]. If X is weak* strongly lifting compact, then dually X/Y is weak* strongly lifting compact. Indeed, recall that $(X/Y)'$ is isomorphic to $Y^\perp \subseteq X'$ (where, as usual, $Y^\perp := \{x' \in X' : x'|_Y = 0\}$) and the weak* topology of $(X/Y)'$ transforms under this isomorphis to the weak* topology of X' restricted to $Y^\perp$; therefore, the assertion follows again from [**1**, 3.1 (i)]. Since the quotient of an Asplund space is Asplund by [**18**] (see also [**4**, Theorem 5.8.1]) the same is true for SB*-spaces by Theorem 3.4 (iv). □

Theorem 4.2. *Let Y be a closed subspace of the Banach space X. If Y and X/Y are* SB**-spaces, then X is an* SB**-space.*

Proof. We verify condition (ii) of Theorem 3.4. For this let $\phi \in \mathfrak{L}^0_{(X',X)}(\mu)$. Y' is isomorphic to $X'/Y^\perp$, and the canonical surjection $\pi : X' \to X'/Y^\perp$ is weak*-weak* continuous. Then $\pi \circ \phi \in \mathfrak{L}^0_{(Y',Y)}(\mu)$ by [**8**, 2.3]. Since Y is weak* strongly lifting compact $\pi \circ \phi \in \mathfrak{L}^0_{Y'}(\mu)$ follows by Theorem 3.4. By [**14**] there exists a norm continuous section

σ for π. Then $\sigma \circ \pi \circ \phi \in \mathfrak{L}^0_{X'}(\mu)$ and $\psi = \phi - \sigma \circ \pi \circ \phi \in \mathfrak{L}^0_{(Y^\perp, X/Y)}(\mu)$ since $(X/Y)'$ is isomorphic to $Y^\perp$ by the remark preceding the theorem. Again, from Theorem 3.4 it follows that $\psi \in \mathfrak{L}^0_{Y^\perp}(\mu)$ since X/Y is weak* strongly lifting compact, therefore $\phi = \psi + \sigma \circ \pi \circ \phi \in \mathfrak{L}^0_{X'}(\mu)$. □

Theorem 4.3. *If for a Banach space Y the dual space Y' is weak strongly lifting compact or an* SB*-space, then $(Y$, weak$)$ is strongly lifting compact.*

Proof. The weak* topology of Y'' induces the weak topology on Y. If Y' is weak* strongly lifting compact the assertion follows from [**1**, Theorem 3.1].

Let ϕ be a Baire measurable map from Ω into Y, i.e., ϕ is scalarly measurable by [**8**, Theorem 2.3]. If $e : Y \to Y''$ is the evaluation map, then $e \circ \phi \in \mathfrak{L}^0_{(Y'', Y')}(\mu)$ and there exists a $\psi \in \mathfrak{L}^\infty_{Y''}(\mu)$ such that $\phi = \psi$ a.e. (μ), i.e., there exists a set $\Omega_0 \in \Sigma$ with $\mu(\Omega_0) = 1$ and $\psi(\Omega_0) = \phi(\Omega_0) \subseteq Y$. Put $\psi_0(\omega) := \psi(\omega)$ for $\omega \in \Omega_0$ and $\psi_0(\omega) = 0$ otherwise. Then $\psi_0 \in \mathfrak{L}^0_Y(\mu)$ and $\phi = \psi_0$ a.e. (μ). By [**1**, Theorem 4.3] the space $(Y$, weak$)$ is strongly lifting compact. □

By [**8**, 6.(4)] the space $D[0,1]$ of all right continuous functions with left limits from $[0,1]$ into $\mathbf{R}$ is not measure compact, so not strongly lifting compact. $(C[0,1]$, weak$)$ is strongly lifting compact since $C[0,1]$ is norm-separable, and $D[0,1]/C[0,1]$ is isomorphic with $c^0_{[0,1]}$. By Theorem 4.1 $(c^0_{[0,1]}$, weak$)$ is not strongly lifting compact, and by Theorem 4.3 $l^1_{[0,1]}$ is neither an SB*-space nor weak* strongly lifting compact.

For the c_0 respectively l_p product $(\oplus_{i \in I} Y_i)_{c_0}$ respectively $(\oplus_{i \in I} Y_i)_{l_p}$ of Banach spaces Y_i $(i \in I)$ see [**5**, p. 35]. Let $\phi_0(t) := (\delta_{ts})_{s \in [0,1]}$ for $t \in [0,1]$, where δ_{ts} is the Kronecker δ. For $1 \leq p < \infty$, $y = (y_t) \in l^p_{[0,1]}$ holds $(y, \phi_0)(t) = y_t \neq 0$ for at most countably many $t \in [0,1]$, and for any lifting ρ of $L^\infty(\mu)$, μ the Lebesgue measure, holds for all $t \in [0,1]$ always $\rho^* \phi_0(t) = 0 \neq \phi_0(t)$ for $t \in [0,1]$. This means that $l^p_{[0,1]} = (\oplus_{t \in [0,1]} \mathbf{R})_{l_p}$ is not weak* strongly lifting compact for $1 \leq p < \infty$ and not strongly lifting compact for the weak topology for $1 < p < \infty$. For a non measure compact discrete index set Γ, i.e., card (Γ) is a real valued measurable cardinal by [**8**, 4(3)], the space $(l^1_\Gamma$, weak) is not measure compact by [**8**, 6(1)], hence $l^1_\Gamma = (\oplus_{\gamma\in\Gamma}\mathbf{R})_{l^1}$ is not strongly lifting compact in the weak topology.

By Corollary 3.2 the space $c^0_\Gamma = (\oplus_{\gamma\in\Gamma}\mathbf{R})_{c_0}$ cannot be weak* strongly lifting compact, hence not an SB*-space. But products with countable index set have good stability properties.

Theorem 4.4. *For any $n \in \mathbf{N}$ let Y_n be weak* strongly lifting compact. Then the c_0 product $(\oplus_{n\in\mathbf{N}}Y_n)_{c_0}$ as well as the l_p product $(\oplus_{n\in\mathbf{N}}Y_n)_{l_p}$ are weak* strongly lifting compact for $1 \le p < \infty$. The theorem remains true if we replace "weak* strongly lifting compact" by "L^*-space."*

Proof. Let $Y = (\oplus_{n\in\mathbf{N}}Y_n)_{c_0}$ and $\phi \in \mathfrak{B}_{(Y',Y)}(\mu)$. Then $Y' = (\oplus_{n\in\mathbf{N}}Y'_n)_{l^1}$, $\phi = (\phi_n)_{n\in\mathbf{N}}$ with $\phi_n \in \mathfrak{B}_{(Y'_n,Y_n)}(\mu)$, hence for any lifting ρ of $\mathfrak{L}^\infty(\mu)$, $\rho^*\phi_n = \phi_n$ holds a.e. (μ) for $n \in \mathbf{N}$. Since $(y,\phi) = \sum_{n=1}^\infty (y_n,\phi_n)$ pointwise on Ω for $y = (y_n) \in Y$, we find by Egorov for any $m \in \mathbf{N}$ a set $\Omega_m \in \Sigma$ with $\mu(\Omega\backslash\Omega_m) \le 1/m$ and $\sum_{n=1}^\infty (y_n,\phi_n) = (y,\phi)$ uniformly on Ω_m, where we may assume $\Omega_m \subseteq \Omega_{m+1}$. Let $A_m := \Omega_m \cap \rho(\Omega_m)$. Then $A_m \subseteq A_{m+1}$, and $\sum_{n=1}^\infty (y_n,\phi_n)\chi_{A_m} = (y,\phi)\chi_{A_m}$ uniformly. Since ρ is continuous, this implies that $\sum_{n=1}^\infty \rho(y_n,\phi_n)\chi_{\rho(A_m)} = \rho(y,\phi)\chi_{\rho(A_m)}$. This yields $\rho(y,\phi)(\omega) = \sum_{n=1}^\infty \rho(y_n,\phi_n)(\omega)$ for $\omega \in \Omega_0 := \cup_{m=1}^\infty \rho(A_m)$, $\mu(\Omega_0) = 1$. By the weak* lifting invariance of ϕ_n we may choose $B_n \in \Sigma$ such that $\mu(B_n) = 1$ and $\rho(y_n,\phi_n)(\omega) = (y_n,\phi_n)(\omega)$ for $\omega \in B_n$, $n \in \mathbf{N}$. Put $A_0 := \Omega_0 \cap \cap_{n\in\mathbf{N}}B_n$. Then $\mu(A_0) = 1$, and for $\omega \in A_0$ we have $\rho(y,\phi)(\omega) = \sum_{n=1}^\infty \rho(y_n,\phi_n)(\omega) = \sum_{n=1}^\infty (y_n,\phi_n)(\omega) = (y,\phi)(\omega)$. From Theorem 3.1, it follows that Y is weak* strongly lifting compact. For l_p products the proof is similar and will be dropped. For L^*-spaces Y_n $(n \in \mathbf{N})$ the proof follows by a modification of the above proof. □

Corollary 4.5. *The c_0 product as well as the l_p product of a sequence of* SB**-spaces is itself an* SB**-space for $1 < p < \infty$.*

Proof. On the basis of Theorem 3.4 condition (iv) this follows from Theorem 4.4 and [**18**, Theorem 13]. □

Clearly, l_∞ is no SB*-space since it is not even Asplund (see, e.g., [**6**, p. 219]), so Corollary 4.5 cannot hold for the l_∞ product.

Theorem 4.6. *The c_0 product as well as the l_p product of a sequence of strongly lifting compact Banach spaces is itself strongly lifting compact for $1 \le p < \infty$.*

Proof. If we verify the condition given in Theorem 2.12 this amounts to interchanging Y and Y' in the proof of Theorem 4.4. □

Theorem 4.7. *The c_0 product as well as the l_p product of a sequence of strict W^*-spaces is itself a strict W^*-space.*

This follows immediately by definition. The space l_1 is not even a W^*-space.

Theorem 4.8. *Let Γ be a set with nonreal-valued measurable cardinal. For each $\gamma \in \Gamma$, let Y_γ be an M^*-space. Then the c_0 product $(\oplus_{\gamma\in\Gamma} Y_\gamma)_{c_0}$ as well as the l_p product $(\oplus_{\gamma\in\Gamma} Y_\gamma)_{l_p}$ are M^*-spaces for $1 < p < \infty$.*

Proof. Let $Y := (\oplus_{\gamma\in\Gamma} Y_\gamma)_{c_0}$ and $\eta = (\eta_\gamma) \in \mathfrak{L}_{(Y',Y)}(\mu)$ with $Y' = (\oplus_{\gamma\in\Gamma} Y'_\gamma)_{l_1}$. Interchanging $X = Y$ and $X' = Y'$ in the proof of [**9**, Theorem 3.4] we find that the set $\Gamma_0 := \{\gamma \in \Gamma : \eta_\gamma \not\equiv 0\sigma(Y'_\gamma, Y_\gamma)\}$ is countable. Let us put $\eta'_\gamma := \eta_\gamma$ for $\gamma \in \Gamma_0$, and $\eta'_\gamma := 0$ for $\gamma \in \Gamma\backslash\Gamma_0$. Since Γ has a nonreal-valued measurable cardinal, a reasoning similar to that in [**9**, Theorem 3.4] gives $\eta' = \eta\sigma(Y', Y)$. For $\gamma \in \Gamma_0$ we may choose $\psi_\gamma \in L^0_{Y'_\gamma}(\mu)$ with $\eta_\gamma \equiv \psi_\gamma\sigma(Y'_\gamma Y_\gamma)$. If we put $\psi_\gamma := 0$ for $\gamma \in \Gamma\backslash\Gamma_0$, then $\psi := (\psi_\gamma)_{\gamma\in\Gamma} \in L^0_{Y'}(\mu)$ and $\eta' = \psi\sigma(Y', Y)$ since Γ_0 is countable, i.e., $\eta = \psi\sigma(Y', Y)$. □

Again, the above theorem remains true if we replace "M^*-space" by "W^*-space."

REFERENCES

1. A.G.A.G. Babiker, G. Heller and W. Strauss, *On strong lifting compactness with applications to topological vector spaces*, J. Austral. Math. Soc. **41** (1986), 211–223.

2. A.G.A.G. Babiker and W. Strauss, *Almost strong liftings and τ-additivity*, Measure Theory, Oberwolfach 1979, Proceedings, Springer LNM **794** (1980).

3. A. Bellow, *Lifting compact spaces*, Measure Theory, Oberwolfach 1979, Proceedings, Springer LNM **794** (1980).

4. R.D. Bourgin, *Geometric aspects of convex sets with the Radon-Nikodym property*, Springer LNM **993** (1983).

5. M.M. Day, *Normed linear spaces*, Springer-Verlag, New York-Berlin-Heidelberg, 1973.

6. J. Diestel and J.J. Uhl, Jr., *Vector measures*, Amer. Math. Soc., 1977.

7. N. Dunford and J.T. Schwartz, *Linear operators* I: *General theory*, Pure Appl. Math. **7**, Interscience, New York, 1958.

8. G.A. Edgar, *Measurability in a Banach space*, Indiana Univ. Math. J. **26** (1977), 663–667.

9. ———, *Measurability in a Banach space* II, Indiana Univ. Math. J. **28** (1979), 559–579.

10. A. and C. Ionescu Tulcea, *Topics in the theory of lifting*, Springer-Verlag, New York-Berlin-Heidelberg, 1969.

11. A. Ionescu Tulcea, *On pointwise convergence, compactness, and equicontinuity* II, Advances in Math. **12** (1974), 171–177.

12. J.D. Knowles, *Measures on topological spaces*, Proc. London Math. Soc. **17** (1967), 139–156.

13. J. Lindenstrauss and C. Stegall, *Examples of separable spaces which do not contain l_1 and whose duals are non separable*, Studia Math. **54** (1975), 81–105.

14. E. Michael, *Continuous selections* I, Ann. of Math. **63** (1956), 361–382.

15. W. Moran, *Measures and mappings on topological spaces*, Proc. London Math. Soc. **19** (1969), 493–508.

16. S.E. Mosiman and R.F. Wheeler, *The strict topology in a completely regular setting*: *Relations to topological measure theory*, Canadian J. Math. **26** (1972), 873–890.

17. K. Musial, *The weak Radon-Nikodym property in Banach spaces*, Studia Math. **64** (1979), 151–173.

18. I. Namioka and R.R. Phelps, *Banach spaces which are Asplund spaces*, Duke Math. J. **42** (1975), 735–750.

19. C. Stegall, *The Radon-Nikodym property in conjugate Banach spaces*, Trans. Amer. Math. Soc. **206** (1975), 213–223.

20. ———, *The Radon-Nikodym property in conjugate Banach spaces* II, Trans. Amer. Math. Soc. **264** (1982), 507–519.

21. M. Talagrand, *Pettis integral and measure theory*, Memoirs Amer. Math. Soc. **307** (1984).

22. V.S. Varadarajan, *Measures on topological spaces*, Math. Sbornik **55** (1961), 33–100; English transl., Amer. Math. Soc. Transl. **48** (1965), 141–228.

Mathematisches Institut A, Universität Stuttgart, Pfaffenwaldring 57, D-7000 Stuttgart, Republic of Germany