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A CONGRUENCE FOR $c\phi_{h,k}(n)$

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ABSTRACT. This paper is a sequel to a recent paper [**2**] on congruences for generalized Frobenius partitions. With the aid of some congruence properties for compositions, we will derive a congruence, modulo h^2, for $c\phi_{h,k}(n)$, the number of generalized Frobenius partitions of n with h colors and at most k repetitions, provided $(h, k+1) = 1$.

Introduction. Let $c\phi_{h,k}(n)$ be the number of generalized Frobenius partitions, F-partitions for short, of n with h colors and (at most) k repetitions as introduced in [**3**]. These combinatorial objects are an extension of two classes of F-partitions introduced by Andrews [**1**]. In two recent papers [**3, 4**] the generating functions and the Hardy-Ramanujan-Rademacher expansions for $c\phi_{h,k}(n)$ were derived. In this paper we will prove two congruences for $c\phi_{h,k}(n)$ which are similar to congruences for two other classes of F-partitions.

It has been shown [**2**] that $\sum_{d|(h,n)} \mu(d) c\phi_{h/d}(n/d) \equiv 0 (\operatorname{mod} h^2)$ and $\sum_{d|(h,n)} \mu(d) k\phi_{h/d}(n/d) \equiv 0 (\operatorname{mod} h^2)$ where $c\phi_h(n)(k\phi_h(n))$ are the number of F-partitions of n with h colors without (with unrestricted) repetitions. In this paper we will prove the following.

Theorem 1.

$$\sum_{d|(h,n)} \mu(d) c\phi_{h/d,k}\left(\frac{n}{d}\right) \equiv 0 (\operatorname{mod} h)$$

Theorem 2.

$$\sum_{d|(h,n)} \mu(d) c\phi_{h/d,k}\left(\frac{n}{d}\right) \equiv 0 (\operatorname{mod} hH)$$

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where H is the product of all the prime power factors of h which are relatively prime to $k+1$.

As an immediate corollary, we have

Corollary.

$$\sum_{d|(h,n)} \mu(d) \mathrm{c}\phi_{h/d,k}\left(\frac{n}{d}\right) \equiv 0 (\operatorname{mod} h^2)$$

provided $(h, k+1) = 1$.

We begin by introducing the idea of a color chart associated with a colored F-partition. Let

$$\begin{matrix} \lambda_1 \dots \lambda_1 & \dots & \lambda_s \dots \lambda_s \\ \beta_1 \dots \beta_1 & \dots & \beta_t \dots \beta_t \end{matrix}$$

be an arbitrary F-partition using h colors where in each row if the colors are ignored the λ_i's and the β_j's represent distinct nonnegative integers with $0 \le \lambda_s < \dots < \lambda_1$ and $0 \le \beta_t < \dots < \beta_1$. The color chart associated with this colored F-partition is an h-column, $(s+t)$-rowed array where the ith row, $1 \le i \le s$, gives the color distribution for λ_i and the $(s+j)$th row, $1 \le j \le t$, gives the color distribution for β_j.

For example, the color chart associated with

$$\begin{matrix} 2_4 & 2_2 & 2_1 & 1_3 & 0_3 & 0_1 \\ 3_1 & 3_1 & 1_3 & 1_3 & 1_2 & 1_2 \end{matrix}$$

an F-partition of 23 using 4 colors (the colors are designated by subscripts) is

$$\begin{matrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0. \end{matrix}$$

Proof of Theorem 1. The proof of Theorem 1 is easy since if $d|(h,n)$ then the F-partitions enumerated by $\mathrm{c}\phi_{h/d,k}(n/d)$ can be viewed as

F-partitions of n with h colors and k repetitions in the following way: repeat each entry d times and increment the color by h/d each time. This amounts to just repeating the color chart of an F-partition enumerated by $c\phi_{h/d,k}(n/d)$ d times to form a color chart having h columns. Thus the F-partitions of n with h colors and k repetitions enumerated by $c\phi_{h/d,k}(n/d)$ have order dividing h/d under cyclic permutation of the columns of its color chart.

A simple inclusion/exclusion argument shows that $\sum_{d|(h,n)} \mu(d) \cdot c\phi_{h/d,k}(n/d)$ enumerates the F-partitions of n with h colors and k repetitions whose order is h under cyclic permutation of the columns of its color chart. Therefore, this sum is congruent to zero modulo h. □

Before we begin our proof of Theorem 2, we need to introduce some intermediate results concerning the number of compositions of a positive integer into positive parts.

Intermediate results. Let $c(r,s;n)$ be the number of compositions of n into exactly r positive parts each less than or equal to s. Define $b(r,s,t;n)$ to be the number of compositions of n into exactly r positive parts each less than or equal to s whose order under cyclic permutation is t. When $t=r$, we will simply write $b(r,s;n)$.

Properties of $b(r,s;n)$.

(1) For d a positive integer $b(r,s;n) = b(dr,s,r;dn)$.

(2) The number of compositions of n into exactly r nonnegative parts each less than or equal to s whose order under cyclic permutation is r is $b(r,s+1;n+r)$.

(3) $b(r,s;n) = \sum_{d|(r,n)} \mu(d) c(r/d,s;n/d)$.

(4) For $D|r$ with $(D,s)=1$,

$$b(r,s;n) \equiv 0 \begin{cases} \bmod \frac{r(D,n)}{(2,D)} & n \equiv r-2 \equiv 0 \pmod 4 \\ \bmod r(D,n) & \text{otherwise} \end{cases}$$

(5) If $n \equiv r-2 \equiv 0 \pmod 4$ and $(2,s)=1$,

$$\frac{1}{r} b(r,s;n) + \frac{2}{r} b\left(\frac{r}{2}, s; \frac{n}{2}\right) \equiv 0 \pmod 2.$$

The first and second properties for $b(r,s;n)$ are obvious and the third follows by observing that $c(r/d,s;n/d)$ is the number of compositions of n into exactly r parts each less than or equal to s whose order under cyclic permutation divides r/d.

In order to prove the fourth property, it is sufficient to show that if $(p,s)=1$, $c(R,s;N)\equiv c(R/p,s;N/p) \pmod{p^{t+j+c}}$ where p^t exactly divides R, p^j divides (R,N) and $c=0$ unless $p=2$, $t=1$, and 4 divides N in which case $c=-1$. The proof of this will be based on the following four theorems in which $\sum_i$ represents a sum of terms q^j with $(p^t,j)=p^i$.

Theorem 1. *If* $(m,p)=1$, *then* $(1+q)^{mp^t}=(1+q^p)^{mp^{t-1}}+p^t\sum_0+p^{t+1}\sum_1+\cdots+p^{2t}\sum_t$ *for all odd primes* p, $t\geq 1$ *and for* $p=2$, $t=1$.

Theorem 2. *If* $(m,p)=1$, *then* $(1-q)^{mp^t}=(1-q^p)^{mp^{t-1}}+p^t\sum_0+p^{t+1}\sum_1+\cdots+p^{2t}\sum_t$ *for all odd primes* p, $t\geq 1$ *and for* $p=2$, $t\geq 2$.

Theorem 3. *If* $(m,p)=1$, *then* $(q+q^2+\dots)^{mp^t}=(q^p+q^{2p}+\dots)^{mp^{t-1}}+p^t\sum_0+p^{t+1}\sum_1+\cdots+p^{2t}\sum_t$ *for all odd primes* p, $t\geq 1$ *and for* $p=2$, $t\geq 2$.

Theorem 4. *If* $(m,2)=1$, *then* $(q+q^2+\dots)^{2m}=(q^2-q^4+q^6-\dots)^m+2\sum_0+4\sum_1$.

The generating function for $c(r,s;n)$ is given by $(q+q^2+\cdots+q^s)^r$. Thus we have $\sum[c(R,s;N)-c(R/p,s;N/p)]q^N=(q+\cdots+q^s)^R-(q^p+\cdots+q^{ps})^{R/p}=(1-q^s)^R(q+q^2+\dots)^R-(1-q^{ps})^{R/p}(q^p+q^{2p}+\dots)^{R/p}=[(1-q^s)^R-(1-q^{ps})^{R/p}](q+q^2+\dots)^R+(1-q^{ps})^{R/p}[(q+q^2+\dots)^R-(q^p+q^{2p}+\dots)^{R/p}]=(p^t\sum_0+\cdots+p^{2t}\sum_t)\sum\binom{j-1}{R-1}q^j+\sum(-1)^{j/p}\binom{R/p}{j/p}q^{sj}(p^t\sum_0+\cdots+p^{2t}\sum_t)=p^t\sum_0+\cdots+p^{2t}\sum_t$ provided p is an odd prime, $t\geq 1$ or $p=2$, $t\geq 2$ and $(p,s)=1$.

If 2 exactly divides R and $(2,s)=1$, we have $\sum[c(R,s;N)+(-1)^{N/2}c(R/2,s;N/2)]q^N=(q+\cdots+q^s)^R+(-q^2+q^4-\cdots-q^{2s})^{R/2}=$

$(1-q^s)^R(q+q^2+\dots)^R-(1+q^{2s})^{R/2}(q^2-q^4+\dots)^{R/2}=[(1-q^s)^R-(1+q^{2s})^{R/2}](q+q^2+\dots)^R+(1+q^{2s})^{R/2}[(q+q^2+q^3+\dots)^R-(q^2-q^4+\dots)^{R/2}]=(2\sum_0+4\sum_1)\sum\binom{j-1}{R-1}q^j+\sum\binom{R/2}{j/2}q^{sj}(2\sum_0+4\sum_1)=2\sum_0+4\sum_1$.

From the above results, the congruence for $c(R,s;N)$ follows immediately and hence property four is verified.

The fifth property follows by observing that

$$\begin{aligned}
&\frac{1}{r}\sum_{d|(r,n)} db\left(\frac{r}{d},s;\frac{n}{d}\right)\\
&\quad=\frac{1}{r}\sum_{f|(r,n)} d\sum_{e|(r/d,n/d)}\mu(e)c\left(\frac{r}{de},s;\frac{n}{de}\right)\\
&\quad=\frac{1}{r}\sum_{f|(r,n)} c\left(\frac{r}{f},s;\frac{n}{f}\right)\sum_{d|f} d\mu\left(\frac{f}{d}\right)\\
&\quad=\frac{1}{r}\sum_{f|(r,n)}\phi(f)c\left(\frac{r}{f},s;\frac{n}{f}\right)\\
&\quad=\frac{1}{r}\sum_{f|(r/2,n/2)}\phi(f)\left[c\left(\frac{r}{f},s;\frac{n}{f}\right)+c\left(\frac{r}{2f},s;\frac{n}{2f}\right)\right].
\end{aligned}$$

From the last result, we see that the expression in brackets is divisible by 4 when 2 exactly divides r, 4 divides n and $(2,s)=1$ since in this situation f will be odd. This sum will therefore be even.

Thus,

$$\frac{1}{r}\sum_{d|(r,n)} db\left(\frac{r}{d},s;\frac{n}{d}\right)=\sum_{d|(r/2,n/2)}\left[\frac{d}{r}\,b\left(\frac{r}{d},s;\frac{n}{d}\right)+\frac{2d}{r}\,b\left(\frac{r}{2d},s;\frac{n}{2d}\right)\right]$$

is even and since d is odd in this last sum we see that 2 exactly divides r/d and 4 divides n/d. Hence, we can proceed by induction on the size of $R=r/d$ and $N=n/d$ so that $(1/R)b(R,s;N)+(2/R)b(R/2,s;N/2)$ will be even for $d>1$ and since the whole sum is even we must have $(1/r)b(r,s;n)+(2/r)b(r/2,s;n/2)$ is even.

Proof of Theorem 2. To extend the divisibility to hH, we begin by considering an arbitrary uncolored F-partition of n

$$\begin{matrix} \lambda_1 \dots \lambda_1 & \dots & \lambda_s \dots \lambda_s \\ \beta_1 \dots \beta_1 & \dots & \beta_t \dots \beta_t \end{matrix}$$

where λ_i appears $f_i > 0$ times and β_j appears $f_{s+j} > 0$ times. We note that $f_1 + \dots + f_s = f_{s+1} + \dots + f_{s+t}$ since the lines of the array are of equal length. We will now count the number of ways of coloring this F-partition using h colors and (at most) k repetitions so that its order is h under cyclic permutation of the h colors.

It is easy to see that the number of ways of coloring this F-partition using h colors and k repetitions so that h is the order of its color chart under cyclic permutation of the columns is $\sum \prod_{i=1}^{s+t} b(h, k+1, h/d_i; h + f_i)$ where the sum is over all sets of positive integers $\{d_1, \dots, d_{s+t}\}$ such that d_i divides both h and f_i and $\operatorname{lcm}(h/d_1, h/d_2, \dots, h/d_{s+t}) = h$. By property 1 we immediately see that

$$\sum \prod_{i=1}^{s+t} b\left(h, k+1, \frac{h}{d_i}; h + f_i\right) = \sum \prod_{i=1}^{s+t} b\left(\frac{h}{d_i}, k+1; \frac{h+f_i}{d_i}\right).$$

Now we fix $\{d_1, \dots, d_{s+t}\}$ and consider the prime factorization of h. Let p^e be the highest power of a prime dividing H. Since $\operatorname{lcm}(h/d_1, \dots, h/d_{s+t}) = h$, there exists d_j such that p does not divide d_j. Let p^a, $a \geq 0$, be the highest power of p which divides $(d_1, \dots, d_{j-1}, d_{j+1}, \dots, d_{s+t})$. Then p^a divides f_j since p^a divides all f_i, $i \neq j$, and $f_1 + \dots + f_s = f_{s+1} + \dots + f_{s+t}$. Furthermore, p^a divides f_j/d_j since p does not divide d_j. Also, p^{e-a} divides some h/d_i, $i \neq j$. Unless $p = 2$, $e = 1$, $a = 1$ and 2 exactly divides f_j, we have p^{e+a} divides $b(h/d_j, k+1; (h+f_j)/d_j)$ and p^{e-a} divides $\prod_{i \neq j} b(h/d_i, k+1; (h+f_i)/d_i)$. Hence, p^{2e} divides $\prod_{i=1}^{s+t} b(h/d_i, k+1; (h+f_i)/d_i)$.

In the exceptional case if 4 divides $b(h/d_j, k+1; (h+f_j)/d_j)$ we are done; so we assume $b(h/d_j, k+1; (h+f_j)/d_j)/2$ is odd. Now let $\{f_{i_1}, \dots, f_{i_{2u}}\}$, $i_1 < i_2 < \dots < i_{2u}$, be the set of f_i's such that 2 exactly divides f_i. The cardinality of this set is even since $f_1 + \dots + f_s = f_{s+1} + \dots + f_{s+t}$ and in the situation we are considering all of the f_i's are even. Suppose $f_j = f_{i_r}$. Consider $b(h/d_{i_{r'}}, k+1; (h + f_{i_{r'}})/d_{i_{r'}})$

where $r' = r+1$ if r is odd and $r' = r-1$ if r is even. If $b(h/d_{i_{r'}}, k+1; (h+f_{i_{r'}})/d_{i_{r'}})$ is even we have the needed factor of 2; so we assume $b(h/d_{i_{r'}}, k+1; (h+f_{i_{r'}})/d_{i_{r'}})$ is odd.

The set $\{d'_1, \dots, d'_{s+t}\}$ where $d'_i = d_i$, $i \neq j$, $i_{r'}$, $d'_j = 2d_j$, and $d'_{i_{r'}} = d_{i_{r'}}/2$ is among the set of d_i's over which we are summing and we have $b(h/d'_j, k+1; (h+f_j)/d'_j)$ and $b(h/d'_{i_{r'}}, k+1; (h+f_{i_{r'}})/d'_{i_{r'}})/2$ are odd by property 4 for $b(r,s;n)$. Hence, $\prod_{i=1}^{s+t} b(h/d_i, k+1; (h+f_i)/d_i) + \prod_{i=1}^{s+t} b(h/d'_i, k+1; (h+f_i)/d'_i)$ is divisible by 4.

Noting that the pairing of sets as described above is a well-defined operation, we can therefore conclude that $\sum \prod_{i=1}^{s+t} b(h/d_i, k+1; (h+f_i)/d_i)$ where the sum extends over all sets of positive integers $\{d_1, \dots, d_{s+t}\}$ such that d_i divides both h and f_i and $\operatorname{lcm}(h/d_1, \dots, h/d_{s+t}) = h$ is congruent to zero modulo hH. This is sufficient to prove Theorem 2. □

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