ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 3, Summer 1992

NEW PROOFS FOR TWO INFINITE PRODUCT IDENTITIES

RICHARD BLECKSMITH, JOHN BRILLHART AND IRVING GERST

Introduction. In the last few years, we have published a collection of identities in [**3**–**6**] which, in addition to being of interest in themselves, have partition interpretations. Recently, Bhargava, Adiga and Somashekara [**2**] have given alternate proofs of all but two of these identities using formulas in [**1**]. In this note we present new proofs for the remaining two identities (Theorems 2 and 3 below), using an idea found in the demonstrations in [**2**]. These proofs differ from our original proofs (as do those in [**2**]), in that no use of the quintuple product identity is made in them.

1. Preliminaries. We begin this section by reviewing some of the definitions and notations that will be used in this paper (cf. [**6**]).

Let $r_1, \dots, r_t$ be distinct residues modulo m, and let $S = \{n \in Z^+ : n \equiv r_1, \dots, r_t \pmod{m}\}$. We then denote the infinite products

$$\prod_{n\in S}(1-x^n) \quad \text{and} \quad \prod_{n\in S}(1+x^n)$$

by $(r_1, \dots, r_t)_m$ and $[r_1, \dots, r_t]_m$, respectively. For $\delta, \varepsilon \in \{0,1\}$, we define four T-functions:

$$\begin{aligned} T_{2\delta+\varepsilon}(k,l) &\doteq \sum_{-\infty}^{\infty}(-1)^{\delta\frac{n(n+1)}{2}-\varepsilon n}x^{kn^2+ln} \\ &= \prod_{n=1}^{\infty}(1-(-1)^{\delta n}x^{2kn})(1+(-1)^{n\delta+\varepsilon}x^{2kn-k+l}) \\ &\qquad \cdot (1+(-1)^{(n+1)\delta+\varepsilon}x^{2kn-k-l}). \end{aligned} \tag{1}$$

(The latter equality derives from the Jacobi triple product, see [**6**, p. 302].) For example, using the product notation above, we have

$$T_0(k,l) = \sum_{-\infty}^{\infty} x^{kn^2+ln} = (0)_{2k}[\pm(k-l)]_{2k}$$

Received by the editors on February 28, 1990.
Research supported in part by NSF Grant DMF8902258.

and

$$T_1(k,l) = \sum_{-\infty}^{\infty} (-1)^n x^{kn^2+ln} = (0, \pm(k-l))_{2k}.$$

Note that

$$T_\varepsilon(k,-l) = T_\varepsilon(k,l), \qquad \varepsilon = 0 \text{ or } 1. \tag{2}$$

We will also need the following

Lemma 1. *Let* $\varepsilon_1, \varepsilon_2 \in \{0,1\}$ *and* $(k,l_1),(k,l_2) \in \{(i/2, j/2) : (i,j) \in \mathbf{Z}^+ \times \mathbf{Z},\, i \equiv j \pmod 2\}$. *Then* $T_{\varepsilon_1}(k,l_1)T_{\varepsilon_2}(k,l_2) = T_\delta(2k, l_1 - l_2)T_\delta(2k, l_1+l_2) + (-1)^{\varepsilon_2} x^{k+l_2} T_\delta(2k, l_1-l_2-2k)T_\delta(2k, l_1+l_2+2k)$, *where* $\delta = [1-(-1)^{\varepsilon_1+\varepsilon_2}]/2$.

Proof. Put $m = 2$, $a = b = 1$, and $k = k_1 = k_2$ in the general expansion formula given in [**6**, Section 4]. □

We next prove a useful linear relationship between certain of the T's.

Theorem 1. *For* $\delta, \varepsilon \in \{0,1\}$ *and* $0 \le l \le k/2$,

$$T_\delta(2k, k-2l) + (-1)^\varepsilon x^l T_\delta(2k, k+2l) = T_{2\delta+\varepsilon}\left(\frac{k}{2}, \frac{k}{2} - l\right). \tag{3}$$

Proof. Using (1) and then (2) we have

$$\begin{aligned}
T_{2\delta+\varepsilon}\left(\frac{k}{2}, \frac{k}{2} - l\right) &= \sum_{-\infty}^{\infty} (-1)^{\delta \frac{n(n+1)}{2} + \varepsilon n} x^{\frac{k}{2}n^2 + (\frac{k}{2}-l)n} \\
&= \sum_{-\infty}^{\infty} (-1)^{\delta n} x^{\frac{k}{2}(2n)^2 + (\frac{k}{2}-l)2n} + \sum_{-\infty}^{\infty} (-1)^{\delta n + \varepsilon} x^{\frac{k}{2}(2n-1)^2 + (\frac{k}{2}-l)(2n-1)} \\
&= \sum_{-\infty}^{\infty} (-1)^{\delta n} x^{2kn^2 + (k-2l)n} + (-1)^\varepsilon x^l \sum_{-\infty}^{\infty} (-1)^{\delta n} x^{2kn^2 - (k+2l)n} \\
&= T_\delta(2k, k-2l) + (-1)^\varepsilon x^l T_\delta(2k, k+2l). \qquad \square
\end{aligned}$$

Corollary 1. *For* $\delta, \varepsilon_1, \varepsilon_2 \in \{0,1\}$ *and* $\delta \equiv \varepsilon_1 + \varepsilon_2 \pmod 2$,

$$T_{\varepsilon_1}(k,l)T_{\varepsilon_2}(k,k-l) = T_\delta(2k,k)T_{2\delta+\varepsilon_2}\left(\frac{k}{2},\frac{k}{2}-l\right). \tag{4}$$

Proof. If we put $l_1 = l$ and $l_2 = l-k$ in Lemma 1, then we find that

$$\begin{aligned} T_{\varepsilon_1}(k,l)T_{\varepsilon_2}(k,k-l) &= T_\delta(2k,k)T_\delta(2k,2l-k) \\ &\quad + (-1)^{\varepsilon_2}x^lT_\delta(2k,-k)T_\delta(2k,2l+k) \\ &= T_\delta(2k,k)[T_\delta(2k,k-2l)+(-1)^{\varepsilon_2}x^lT_\delta(2k,k+2l)] \\ &= T_\delta(2k,k)T_{2\delta+\varepsilon_2}\left(\frac{k}{2},\frac{k}{2}-l\right), \end{aligned}$$

using (2) and Theorem 1. □

Remark. An alternate proof of this corollary can be given using the product form for the T's.

2. Proofs of the identities.

Theorem 2. (cf. [**6**, Theorem 1])

$$\prod_{\substack{n=1 \\ n\equiv 0,\pm(5,7) \\ \pmod{24}}}^{\infty}(1-x^n)\prod_{\substack{n=1 \\ n\equiv\pm(1,4),6 \\ \pmod{12}}}^{\infty}(1+x^n) = \sum_{n=0}^{\infty}(x^{2n(n+1)}+x^{6n(n+1)+1}). \tag{5}$$

Proof. From the Gauss formula [**6**, (13)]

$$\sum_{n=0}^{\infty}x^{\frac{n(n+1)k}{2}} = \frac{(0)_{2k}}{(k)_{2k}}, \tag{6}$$

we can write the right-hand side of (5) as

$$\frac{(0)_8}{(4)_8}+x\frac{(0)_{24}}{(12)_{24}} = \frac{1}{(4)_8}\left[(0)_8+x\frac{(0)_{24}(4)_8}{(12)_{24}}\right].$$

But

$$\frac{(0)_{24}(4)_8}{(12)_{24}} = \frac{(0)_{24}(4,12,20)_{24}}{(12)_{24}} = (0,\pm 4)_{24} = T_1(12,8)$$

and

$$(0)_8 = (0,\pm 8)_{24} = T_1(12,4).$$

Thus, the right-hand side is equal to $[0]_4[T_1(12,4) + xT_1(12,8)]$, using the Euler result $[6,(18)]$ that

$$\frac{1}{(k)_{2k}} = [0]_k. \tag{7}$$

By Theorem 1, with $\delta = 1$, $\varepsilon = 0$, $k = 6$ and $l = 1$, this expression becomes

$$\begin{aligned}[0]_4T_2(3,2) &= [0]_4(0,\pm 5)_{12}[\pm 1,6]_{12} = [0]_{12}[\pm 4]_{12}(0)_{12}(\pm 5)_{12}[\pm 1,6]_{12}\\ &= (0)_{24}(\pm 5,\pm 7)_{24}[\pm 1,\pm 4,6]_{12}. \qquad \square\end{aligned}$$

Theorem 3. (cf. [**6**, Theorem 4])

$$\prod_{\substack{n=1\\ n\equiv 0,\pm 3\\ (\mathrm{mod}\ 10)}}^{\infty}(1-x^n)\prod_{\substack{n=1\\ n\equiv \pm(1,2,4)\\ (\mathrm{mod}\ 10)}}^{\infty}(1+x^n) = \sum_{n=0}^{\infty}(x^{n(n+1)} + x^{5n(n+1)+1}). \tag{8}$$

Proof. Using (6), we can write the right-hand side of (8) as

$$\frac{(0)_4}{(2)_4} + x\frac{(0)_{20}}{(10)_{20}} = \frac{(0)_4}{(2)_4}\left[1 + x\frac{(0)_{20}(2)_4}{(10)_{20}(0)_4}\right].$$

But

$$\frac{(0)_{20}(2)_4}{(10)_{20}(0)_4} = \frac{(0)_{20}^2(2,6,10,14,18)_{20}}{(0)_{20}(10)_{20}(0,4,8,12,16)_{20}} = \frac{T_1(10,8)T_1(10,4)}{T_1(10,6)T_1(10,2)},$$

so the right-hand side becomes

$$\frac{1}{(2)_4(0)_{20}}[T_1(10,6)T_1(10,2) + xT_1(10,8)T_1(10,4)].$$

But the expression in the brackets equals $T_1(5,2)T_0(5,4)$, which follows by applying Lemma 1 to the product $T_1(5,2)T_0(5,-4)$ and using (2). Thus, using (5), we obtain

$$\frac{[0]_2}{(0)_{20}}(0,\pm 3)_{10}(0)_{10}[\pm 1]_{10} = \frac{[0,2,4,6,8]_{10}(0)_{10}(0,\pm 3)_{10}[\pm 1]_{10}}{(0)_{10}[0]_{10}}$$
$$= (0,\pm 3)_{10}[\pm 1,\pm 2,\pm 4]_{10}. \qquad \square$$

REFERENCES

1. C. Adiga, B.C. Berndt, S. Bhargava and G.N. Watson, Chapter 16 of *Ramanujan's second notebook: Theta-functions and q-series*, Amer. Math. Soc. Memoirs **315**, Providence, 1985.

2. S. Bhargava, C. Adiga and D.D. Somashekara, *Parity results deducible from certain theta function identities*, found in Chapter 16 of *Ramanujan's second notebook*, The Mathematics Student **57** (1989), 89–100.

3. R. Blecksmith, J. Brillhart and I. Gerst, *Partition function parity theorems and theta function identity analogues*, Amer. Math. Soc. Abstracts **7** (1986), 225.

4. ———, *Parity results for certain partition functions and identities similar to theta function identities*, Math. Comp. **8** (1987), 29–38.

5. ———, *Some infinite product identities*, Amer. Math. Soc. Abstracts **8** (1987), 439.

6. ———, *Some infinite product identities*, Math. Comp. **51** (1988), 301–314.

Northern Illinois University, DeKalb, IL 60115

University of Arizona, Tucson, AZ 85721

SUNY at Stony Brook, Stony Brook, NY 11794