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UNIFORM DENTABILITY, UNIFORM SMOOTHABILITY AND APPROXIMATIONS TO CONVEX SETS

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ABSTRACT. Uniform versions of dentability and smoothability are introduced, and shown to be dually related. It is shown that convex sets in spaces satisfying these properties can be approximated by the convex hull of a set of *uniformly sharp* corners in the former case and by the intersection of *uniformly flat* cones, generated by a point and the set, in the latter case.

Introduction. The notion of dentability in Banach spaces was introduced by Rieffel [**8**] and has been studied extensively since that time. One of the central results is that of Lindenstrauss (cf. [**7**]) which shows that if every bounded closed convex set in a Banach space X is dentable (the Radon-Nikodym property, abbreviated RNP), then every bounded closed convex set in X is the closed convex hull of its extreme points (the Krein-Milman property). This result was extended by Phelps [**7**] who showed that "extreme point" could be replaced by *strongly exposed point*. Loosely speaking, these results say that if X has the RNP, then a closed bounded convex set is the closed convex hull of its *corners*.

In [**5**], Finet defines a modulus of strong extremality for points of the unit ball which "measures how much a point is a strong extreme point of the unit ball." (See [**9**] or [**5**] for the definition of a strong extreme point.) It is then shown that the unit ball of a super-reflexive space can be approximated (arbitrarily close, using the Hausdorff metric) as the closed convex hull of a subset of the set of strong extreme points of the unit ball. The modulus of strong extremality for the points of this subset are uniformly bounded from below. (The bound depending on the closeness of the approximation.) Thus, thinking of the strong extreme points as corners, this says that the unit ball of a super-reflexive space can be approximated as the closed convex hull of a set of corners that are uniformly "sharp."

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In Section 2 of this paper we consider spaces that satisfy a uniform version of the RNP, called uniform dentability, and show that any closed bounded convex set in these spaces can be approximated as the closed convex hull of a set of *uniformly sharp corners* (with an altered notion of a corner).

The concept of smoothability was introduced by Edelstein [**3**], and later reformulated by Kemp [**6**]. These papers show that dentability and smoothability are, to a certain extent, dually related. In [**1**] a stronger form of smoothability, called strong smoothability, was introduced. It was shown that for a Banach space X to be strongly smoothable (i.e., every convex body is strongly smoothable) is equivalent to X being an Asplund space. Since X is an Asplund space if and only if X^* has the RNP, it follows that X is strongly smoothable if and only if X^* has the RNP.

With this in mind, it is natural to ask whether there is a uniform version of smoothability that is dual to uniform dentability. In Section 1 we define both uniform dentability and uniform smoothability, as well as weak* versions, and in Sections 3 and 4 we prove duality results.

While proving these duality results another approximation to certain convex sets is established, this time valid in uniformly smoothable spaces. This second approximation, not surprisingly, is in some dual to the previously mentioned approximation. Loosely speaking, it shows that a convex set with nonempty interior in a uniformly smoothable space can be approximated as the intersection of *quite flat* cones. (See the next section for definitions.) The distance from the vertex of each cone to the set is bounded from below. (Indeed, the bound only depends on the largest ball that fits inside the set and the norm of the space.) The measure of this second approximation is not, unfortunately, the Hausdorff metric. It is the radius of the largest ball that fits between the set and the approximating set.

The results in this paper concerning uniform dentability and uniform smoothability (in particular, Theorems 3.6, 3.7, 4.9 and 4.10) remain valid if the uniformity is dropped—indeed, the proofs would be shorter and less technical. However, we would then be considering the duality between strongly smoothable spaces and spaces with the RNP, which is investigated in [**1**].

1. Definitions and elementary properties. Throughout this paper, X denotes a Banach space with dual X^* and $K \subset X$ will be a closed convex set in X. For $x \in X$ and $r > 0$, $B(x, r)$ is the closed ball about x of radius r, $B(X)$ the closed unit ball of X and $S(X)$ the unit sphere of X. When no confusion arises, we use B instead of $B(X)$.

Let $f \in X^*$ and $\alpha > 0$ be given. The set $S(f, \alpha, K) = \{x \in K : f(x) \geq \sup f(K) - \alpha\}$ is called a *slice* of K. The *diameter* of $S(f, \alpha, K)$ is $\sup\{||x - y|| : x, y \in S(f, \alpha, K)\}$ and is denoted by diameter $S(f, \alpha, K)$. By the *depth* of $S(f, \alpha, K)$ we shall mean $\alpha/||f||$. Unfortunately, the depth depends not only on the point set $S(f, \alpha, K)$, but also on f and α. (It may be that $S(f_1, \alpha_1, K)$ and $S(f_2, \alpha_2, K)$ are the same set for different, indeed independent, f_1 and f_2.) If $S(f, \alpha, K) \neq K$, it is reasonable to define the depth by $\sup\{\alpha'/||f'|| : S(f', \alpha, K) = S(f, \alpha, K)\}$. This is clearly independent of f and α. (However, if $S(f, \alpha, K) = K$ this approach leads to an infinite depth.) Our results could be modified to use such a definition, but to avoid unnecessary complications, we do not do so.

The set $K \subset K$ is said to be *dentable* (weak* dentable) if for every $\varepsilon > 0$ there is $f \in X^*$ (f weak* continuous) and $\alpha > 0$ such that diameter $S(f, \alpha, K) \leq \varepsilon$. The space X is said to have the Radon-Nikodym property (abbreviated RNP, (weak* RNP)), or is said to be dentable (weak* dentable), if every closed (weak* closed) bounded convex set in X is dentable (weak* dentable). We say that X is *uniformly dentable* (weak* uniformly dentable) if there is a function $\Delta(\varepsilon)$, $0 < \varepsilon \leq 2$, such that $\Delta(\varepsilon) > 0$ and for every closed (weak* closed) convex set $K \subset B(X)$ and every ε, $0 < \varepsilon \leq 2$, there is a slice of K (weak* slice of K) with depth $\Delta(\varepsilon)$ and diameter at most ε. The function $\Delta(\varepsilon)$ is called a modulus of dentability (modulus of weak* dentability) for X. Note that the diameter of a slice $S(f, \alpha, K)$ for which $f^{-1}(\alpha) \cap K \neq \varnothing$ is at least as great as the depth, so $\Delta(\varepsilon) \leq \varepsilon$.

For $x \in X \sim K$ and $K \subset X$ let $k(x, K)$ be the closure of $\{\lambda(y - x) : y \in K, \lambda \geq 0\}$. The set $k(x, K)$ is called the *cone* of K with respect to x. If K is convex, then so is $k(x, K)$.

A set $K \subset X$ is said to be *strongly smoothable* (weak* strongly smoothable) (cf. [**1**]) if for every ε, $0 < \varepsilon < 1$ there is $f \in S(X^*)$ (f weak* continuous with norm 1) and $x \notin K$ such that $\{y \in B(X) : f(y) \geq \varepsilon\} \subset k(x, K)$. Note that $\{y \in B(X) : f(y) \geq \varepsilon\} = S(f, 1 -$

$\varepsilon, B(X))$. If every closed (weak* closed) convex set in X with nonempty interior is strongly smoothable (weak* strongly smoothable) we say that X is strongly smoothable (weak* strongly smoothable). A Banach space X is *uniformly smoothable* (weak* uniformly smoothable) if there is a positive function $\Sigma(\varepsilon)$, $0 < \varepsilon < 1$, such that for every closed (weak* closed) convex set $K \subset X$, $K \neq X$, with $B(X) \subset K$ there is $x \in X$ and $f \in S(X^*)$ (f weak* continuous with norm 1) with the interior of $B(x, \Sigma(\varepsilon))$ not intersecting K and $S(f, 1-\varepsilon, B(X)) \subset k(x, K)$. Letting $d(x, K) = \inf\{||x-y|| : y \in K\}$, the interior of $B(x, \Sigma(\varepsilon))$ not intersecting K is equivalent to $d(x, K) \geq \Sigma(\varepsilon)$. The function $\Sigma(\varepsilon)$ is called a modulus of smoothability (weak* modulus of smoothability) for X.

Remark 1.1. If $\Delta(\varepsilon)$ is a modulus of dentability for X and Δ' is a function with $0 < \Delta'(\varepsilon) \leq \Delta(\varepsilon)$, $0 < \varepsilon \leq 2$, then for any closed convex set $K \subset B(X)$, $S(f, \Delta'(\varepsilon), K) \subset S(f, \Delta(\varepsilon), K)$. Thus, diameter $S(f, \Delta'(\varepsilon), K) \leq$ diameter $S(f, \Delta(\varepsilon), K) \leq \varepsilon$, so $\Delta'(\varepsilon)$ is also a modulus of dentability for X. Also, if $\Sigma(\varepsilon)$ is a modulus of smoothability for X and $0 < \Sigma'(\varepsilon) \leq \Sigma(\varepsilon)$ for $0 < \varepsilon < 1$, then $\Sigma'(\varepsilon)$ is a modulus of smoothability for X.

Lemma 1.2. *If X has one of the properties: uniform dentability, weak* uniform dentability, uniform smoothability or weak* uniform smoothability, then any space isomorphic to X also has this property.*

The following three results concern uniformly convex spaces (cf. [**2**, page 145]) and uniform dentability. Similar results for uniformly smooth spaces (cf. [**2**]) and uniform smoothability could be obtained. However, since the dual of a uniformly convex space is uniformly smooth, the duality results that we establish later would make such results redundant. Since uniformly convex and uniformly smooth spaces are reflexive (cf. [**2**]) weak* versions are irrelevant.

Lemma 1.3. *Suppose that X is uniformly convex with modulus of convexity $\delta(\varepsilon)$. Then for any f, $||f|| = 1$, diameter $S(f, \delta(\varepsilon), B(X)) \leq \varepsilon$.*

Theorem 1.4. *Uniformly convex Banach spaces are uniformly dentable. Furthermore, if $\delta(\varepsilon)$ is the modulus of convexity, then $\Delta(\varepsilon)$ can be taken to be any function with $\Delta(\varepsilon) \le \delta(\varepsilon)$.*

Proof. Let X be a uniformly convex Banach space with modulus of convexity $\delta(\varepsilon)$. Let $K \subset B = B(X)$ be closed and convex, $r \in S(X)$ be arbitrary, but fixed, and $\lambda_0 = \sup\{\lambda : K + \lambda r \subset B\}$. Note that $K + \lambda_0 r \subset B$.

Let $\mu > 0$ be arbitrary. Then for some $y \in K$, $y + (\lambda_0 + \mu)r \notin B$. Let $z = y + \lambda_0 r$ and choose $f \in S(X^*)$ to support $B(X)$ at $z/||z||$. Consider $S(f, \delta(\varepsilon), B) \cap (K + \lambda_0 r)$. By Lemma 1.3 this has diameter no greater than ε. Now $S(f, \delta(\varepsilon), B) \cap (K + \lambda_0 r) = S(f, \delta(\varepsilon) - (1 - \sup f(K+\lambda_0 r)), K+\lambda_0 r)$ and $1 - \sup f(K+\lambda_0 r) \le 1 - f(z) = 1 - ||z|| \le ||z+\mu r|| - ||z|| \le ||\mu r|| = \mu$. Thus, $S(f, \delta(\varepsilon), B) \cap (K+\lambda_0 r) \supseteq S(f, \delta(\varepsilon) - \mu, K + \lambda_0 r) = S(f, \delta(\varepsilon) - \mu, K) + \lambda_0 r$ so diameter $S(f, \delta(\varepsilon) - \mu, K) \le \varepsilon$. Since this is true for all $\mu > 0$, the continuity of f and the convexity of K show that diameter $S(f, \delta(\varepsilon), K) \le \varepsilon$. Thus, if $\Delta(\varepsilon) \le \delta(\varepsilon)$, then diameter $S(f, \Delta(\varepsilon), K) \le$ diameter $S(f, \delta(\varepsilon), K) \le \varepsilon$. □

Corollary 1.5. *Super-reflexive Banach spaces are uniformly dentable.*

It is natural to ask whether or not super-reflexivity is equivalent to uniform dentability (or uniform smoothability). We are unable to answer this question. Of course, if so, the duality results that follow would be obvious.

Theorem 1.6. *Let X be a Banach space.*

(1) *If X^* is weak* uniformly dentable, then X^* is uniformly dentable.*

(2) *If X^* is uniformly smoothable, then X^* is weak* uniformly smoothable.*

Proof. (1) Let X^* have weak* modulus of dentability $\Delta(\varepsilon)$, let $K \subset B(X^*)$ be closed and convex, and let ε, $0 < \varepsilon \le 2$, be fixed. If $\overline{K}^*$ is the weak* closure of K, there is a weak* continuous functional f on X^*, $f \in S(X^*)$, such that diameter $S(f, \delta(\varepsilon), \overline{K}^*) \le \varepsilon$. Since $\sup f(\overline{K}^*) = \sup f(K)$, $S(f, \Delta(\varepsilon), \overline{K}^*) \supset S(f, \Delta(\varepsilon), K)$ and hence diameter $S(f, \Delta(\varepsilon), K) \leq \varepsilon$. Thus, X^* is uniformly dentable with modulus $\Delta(\varepsilon)$.

(2) Let X^* have modulus of smoothability $\Sigma(\varepsilon)$, let $K \subset X^*$ be an arbitrary weak* closed convex set with $B(X) \subset K$ and let $\varepsilon > 0$ be given. Then there is $g \in S(X^{**})$ and $x \in X^*$ with $S(g, 1-\varepsilon/3, B(X^*)) \subset k(x, K)$ and $d(x, K) \geq \Sigma(\varepsilon/3)$. Using the separation principle, let $f \in S(X^{**})$ be weak* continuous and such that $\inf f(K) \geq f(x)$. By (the proof of) [**1**], Lemma 2.7, $S(f, 1-\varepsilon, B(X^*)) \subset k(x, K)$, showing that X^* is weak* uniformly smoothable with modulus $\Sigma(\varepsilon/3)$. □

Before concluding this section, we introduce some notation that will be used later. Let $K \subset X$ and $q \notin K$. Define

$$l(q, K) = \mathrm{cl}\,\{(1-\lambda)q + \lambda x : \lambda \geq 1 \text{ and } x \in K\}.$$

The set $l(q, K)$ is a subset of $k(q, K)+q$: those points of $k(q, K)+q$ that have a point $k \in K$ between them and q. For the sake of consistency it would have been nice to either use $l(q, K) - q$ instead of $l(q, k)$ or $k(q, K) + q$ instead of $k(q, K)$; however, to be consistent with [**1**] and to not always be writing $l(q, K) + q$, we prefer this notation.

Lemma 1.7. *Let $K \subset X$ be closed and convex with $B(X) \subset K$ and let $q \notin K$. Then*

(a) *$l(q, K)$ is convex and $B(X) \subset l(q, K)$.*

(b) *If $x \in l(q, K)$, then $x - \lambda q \in l(q, K)$ for any $\lambda \geq 0$.*

Equivalently, if $x \notin l(q, K)$, then $x + \lambda q \notin l(q, K)$ for $\lambda \geq 0$.

(c) *If $x \in (k(q, K) + q) \sim l(q, K)$, then $k(x, K) = k(x, l(q, K))$ and $d(x, K) = d(x, l(q, K))$.*

(To prove parts b and c, consider the two-dimensional plane containing 0, x and q.)

Section 2. In this section (in Theorem 2.2) we approximate convex sets in uniformly dentable spaces as the closed convex hull of uniformly "sharp" corners. By a corner of K we mean a slice of K, or we could just as well mean the slice intersected with the boundary of K. By the sharpness of the corner we mean the ratio of the depth of the slice to the diameter. We actually show more than this, as the "sharpness" of the corners used to obtain a given closeness for the approximation depends only on the space X (and the norm) and the radius of the set to be approximated. Furthermore, the corners chosen are disjoint.

The closeness of the approximation will be given by the Hausdorff metric (cf. [**4**]). For two sets A and K, we let $H(A, K)$ denote the Hausdorff distance between the sets. If $A \subset K$, the situation here, $H(A, K) < \varepsilon$ is equivalent to $d(x, A) < \varepsilon$ for all $x \in K$.

The following lemma is a uniform version of that of Phelps ([**7**, Lemma 4]).

Lemma 2.1. *Let X be a uniformly dentable Banach space with modulus of dentability $\Delta(\varepsilon)$. Let $\varepsilon > 0$ be given and define*

$$\delta(\varepsilon) = \Delta\left(\frac{\varepsilon}{2}\left(1 + \frac{4}{\varepsilon}\right)^{-2}\right).$$

Let $g \in S(X^*)$, $K \subset B(X)$ and α, $|\alpha| < 1$ be given. Define $C = \{x \in K : g(x) \geq \alpha\}$. If $\sup g(K) > \alpha + \varepsilon$, then there is $f \in S(X^*)$ such that $S(f, \delta(\varepsilon), K) \subset C$ and $\operatorname{diam} S(f, \delta(\varepsilon), K) \leq \varepsilon$.

Proof. Let $t \in B(X)$ satisfy $g(t) = \alpha$. Let $K_0 = 1/2(K - t)$ and let $C_0 = 1/2(C - t)$. Then $K_0 \subset B(X)$, $C_0 = \{x \in K_0 : g(x) \geq 0\}$, and $\sup g(C_0) = \sup g(K_0) > \varepsilon/2$. To prove the lemma, it is sufficient to find $f \in S(X^*)$ such that $S(f, \delta(\varepsilon)/2, K_0) \subset C_0$ and has diameter at most $\varepsilon/2$.

Choose $z \in C_0$ such that $g(z) > \varepsilon/2$. For each $p \in D \overset{(\mathrm{def})}{=} K_0 \cap g^{-1}(0)$, define $T_p : X \to X$ by $T_p(y) = y - 2(g(y)/g(z))(z - p)$. ($T_p$ is reflection along $z - p$ through the hyperplane $g^{-1}(0)$.) Then $||T_p|| \leq 1 + (2/g(z))||z - p|| \leq 1 + 4/\varepsilon$, $T_p^{-1} = T_p$, $T_p z = 2p - z$, and for each $p \in D$, $p = z/2 + T_p(z)/2$.

Let $C_1 = \overline{\text{co}}(\cup_{p \in D} T_p(C_0) \cup C_0)$, where $\overline{\text{co}}$ means the closed convex hull. Suppose that $x \in \cup_{p \in D} T_p(C_0) \cup C_0$. Then, either $x = T_p(y)$ for some $p \in D$ and $y \in C_0$, or $x \in C_0$. If $x \in C_0$, then $||x|| \leq 1$, since $C_0 \subset B(X)$. If $x = T_p(y)$, then $||x|| \leq ||T_p||\,||y|| \leq 1 + 4/\varepsilon$. Thus, $C_1 \subset (1 + 4/\varepsilon)B(X)$.

Since X is uniformly dentable, with modulus $\Delta(\varepsilon)$, there is a slice $S(f_1, \delta(\varepsilon), C_1/(1 + (4/\varepsilon)))$ of diameter at most $\varepsilon(1 + 4/\varepsilon)^{-2}/2$, where $f_1 \in S(X^*)$.

Letting $\mu = (1 + 4/\varepsilon)\delta(\varepsilon)$ it follows that diameter $S(f_1, \mu, C_1) \leq (\varepsilon/2)(1 + 4/\varepsilon)$. Choose $y \in \cup_{p\in D} T_p(C_0) \cup C_0$ such that $\sup f_1(C_1) < f_1(y) + \mu/2$.

Case I. $y \in C_0$. Let $f = f_1$. It is straightforward to show that $S(f, \mu, C_1) \supset S(f, \mu/2, C_0)$. We wish to show that $S(f, \mu/2, K_0) = S(f, \mu/2, C_0)$. If not, there is $x \in K_0 \sim C_0$ (so $g(x) < 0$) with $f(x) \geq \sup f(C_0) - \mu/2$. There is also $w \in C_0$ (so $g(w) \geq 0$) with $f(w) \geq \sup f(C_0) - \mu/2$. Thus, there is q, between x and w, with

$$q \in D \cap S(f, \mu/2, C_0) \subset S(f, \mu, C_1).$$

However, $q = z/2 + T_q(z)/2$ so, since both z and $T_q(z)$ are in C_1, either z or $T_q(z)$ is in $S(f, \mu, C_1)$. Now $||z - q|| \geq |g(z - q)| > \varepsilon/2$ and $||T_q(z) - q|| = ||2q - z - q|| > \varepsilon/2$ so either of q or $T_q(z)$ being in $S(f, \mu, C_1)$ contradicts diameter $S(f, \mu, C_1) \leq \varepsilon(1 + 4/\varepsilon)^{-1}/2 < \varepsilon/2$. Thus, $S(f, \mu/2, K_0) = S(f, \mu/2, C_0)$. Noting that $\delta(\varepsilon) < \mu$ and $\varepsilon(1 + 4/\varepsilon)^{-1}/2 < \varepsilon/2$, the lemma is complete.

Case II. $y \notin C_0$. Then $y \in T_p(C_0)$ for some $p \in D$ and as above, $S(f_1, \mu, C_1) \supset S(f_1, \mu/2, T_p(C_0))$ so the latter slice has diameter at most $\varepsilon(1+4/\varepsilon)^{-1}/2$. As in Case I (for $T_p(C_0)$ and $T_p(K_0)$ rather that C_0 and K_0) it can be shown that $S(f_1, \mu/2, T_p(C_0)) = S(f_1, \mu/2, T_p(K_0))$. Thus, by [**7**, Lemma 3],

$$S\left(T_p^* f_1, \frac{\mu}{2}, K_0\right) = T_p^{-1}\left(S\left(f_1, \frac{\mu}{2}, T_p(K_0)\right)\right) = S\left(T_p^* f_1, \frac{\mu}{2}, C_0\right) \subset C_0$$

and has diameter at most $||T_p^{-1}||$ diameter $S(f_1, \mu/2, T_p(C_0))$ since $T_p^{-1} = T_p$ and $||T_p|| < 1 + 4/\varepsilon$, diameter $S(T_p^* f_1, \mu/2, K_0) \leq \varepsilon/2$. Let $f = (T_p^* f_1/||T_p^* f_1||)$. Then, since $||T_p^*|| \leq 1 + 4/\varepsilon$,

$$S\left(T_p^* f_1, \frac{\mu}{2}, K_0\right) = S\left(f, \frac{\mu}{2||T_p^* f_1||}, K_0\right) \supset S\left(f, \frac{\delta(\varepsilon)}{2}, K_0\right).$$

Thus, $S(f, \delta(\varepsilon)/2, K_0) \subset C_0$ and has diameter at most $\varepsilon/2$, completing the lemma. $\square$

Theorem 2.2. *Let X be a uniformly dentable Banach space. Given $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that for every closed convex set $K \subset B(X)$ there are functionals $\{f_\alpha\} \subset S(X^*)$ such that the slices $S(f_\alpha, \delta(\varepsilon), K)$ are pairwise disjoint and*

$$H(\overline{\text{co}}(\cup S(f_{\alpha'}\delta(\varepsilon), K))) \leq \varepsilon.$$

Proof. Let $\delta(\varepsilon)$ be defined from a modulus of dentability for X as in Lemma 2.1. We now fix ε, let $\delta = \delta(\varepsilon)$ and fix a closed convex set $K \subset B(X)$. Consider the collection $\mathcal{F}$ of all sets $F \subset S(X^*)$ such that $\{S(h, \delta(\varepsilon), K) : h \in F\}$ are pairwise disjoint and each has diameter at most ε. Order the collection $\mathcal{F}$ by inclusion. It is easily seen that $\cup F_\alpha$ is an upperbound for any chain $\{F_\alpha\}$ in $\mathcal{F}$. Thus, by Zorn's lemma, there is a maximal member $F \in \mathcal{F}$. Let $K' = \overline{\text{co}}(\cup_{h\in F} S(h, \delta, K))$ and suppose that $H(K', K) > \varepsilon$. Then there is $x \in K$ with $d(x, K') > \varepsilon$ and so there is $g \in S(X^*)$ that strictly separates $B(x, \varepsilon)$ and K'. We may assume that $\sup g(K') < \inf g(B(x, \varepsilon))$. If α is between $\sup g(K')$ and $\inf g(B(x, \varepsilon))$, then $|\alpha| < 1$ and $\sup g(K) \geq g(x) > \alpha + \varepsilon$. Thus, we may apply Lemma 2.1 to find $f \in S(X^*)$ such that $S(f, \delta, K) \subset \{x \in K : g(x) \geq \alpha\}$ and hence $S(f, \delta, K)$ is disjoint from $S(h, \delta, K)$ for any $h \in F$. Then $F \cup \{h\}$ is in $\mathcal{F}$, contradicting the maximality of F.

Section 3. In this section we prove that if X is uniformly dentable, then X^* is weak* uniformly smoothable. To prove that if X^* is weak* uniformly dentable, then X is uniformly smoothable is similar, so the proof will be omitted.

The following lemma is one of the crucial steps in our argument. Notice that it, as well as results 3.2 and 3.3, do not involve any notion of dentability or smoothability, but simply concern certain sets and their duals.

Lemma 3.1. *Let $K \subset X$ be a closed convex nonempty set and let $f \in X^*$ satisfy* $\sup f(K) > 1$. *Then* $l(f, K^0) = S(f, \sup f(K) - 1, K)^0$.

Proof. Let $L = l(f, K^0)$, $S = S(f, \sup f(K) - 1, K)$ and suppose that $g \in L$. Then $g = (1 - \lambda)f + \lambda h$ for some $\lambda \geq 1$ and $h \in K^0$. Let $x \in S$.

Then $f(x) \geq 1$ and $h(x) \leq 1$. Thus, since $1 - \lambda \leq 0$,

$$g(x) = (1 - \lambda)f(x) + \lambda h(x) \leq 1 - \lambda + \lambda = 1$$

showing that $g \in S^0$ and, since $g \in L$ was arbitrary, that $L \subset S^0$.

Now suppose that $g \in S^0$. To show that $g \in L$, we need to show that for some $h \in K^0$ and $\lambda \geq 1$, $g = (1 - \lambda)f + \lambda h$. This is equivalent to showing that for some α, $0 < \alpha \leq 1$, $\alpha g + (1 - \alpha)f \in K^0$, which is how we proceed. Note that $K^0 \subset L$.

The first case we consider is that of $g = cf$ for some c. If $c \geq 0$, we have $\sup g(K) = c \sup f(K) = c \sup f(S) = \sup g(S) \leq 1$, so $g \in K^0 \subset L$ and if $c < 0$, then $1/(1-c) < 1$ and $(1/(1-c))g + (1 - 1/(1-c))f = 0 \in K^0$, so once more $g \in L$.

We now assume that f and g are independent and $g \notin K^0$. Let $Y = f^{-1}(1) \cap g^{-1}(1)$. If $x \in K$ either $f(x) < 1$ ($x \notin S$) or $g(x) \leq 1$ (if $x \in S$). Thus, it follows that Y does not intersect the interior of K. A consequence of the Hahn Banach Theorem, ([**2**, Theorem I, 6.2]), shows that there is a functional $h \in K^0$ such that $Y \subset h^{-1}(1)$.

Fix $y_0 \in Y$. Since the kernels of f, g, and h all contain $Y - y_0$, a subspace of X having codimension 2, and since f and g are independent, it follows that $h = \alpha g + (1 - \alpha)f$ for some $\alpha \in \mathbf{R}$. Choosing $s \in S$ with $f(s) > 1$ and noting that $g(s)$ and $h(s)$ are at most 1, we see that $\alpha > 0$. Since $g \in S^0 \backslash K^0$, there is $x \in K$ with $g(x) > 1$ and $f(x) < 1$ from which it follows that $\alpha < 1$. Hence $g \in L$, completing the proof. □

Corollary 3.2. *With f and K as in Lemma* 3.1

$$k(f, K^0) = k(f, S(f, \sup f(K) - 1, K)^0).$$

Lemma 3.3 relates the size of a slice from a set K to the smoothness of K^0.

Lemma 3.3. *Suppose that $\varnothing \neq K \subset B(X)$ and $f \in X^*$ are such that $\sup f(K) > 1$ and diameter $S(f, \mu, K) \leq \varepsilon$ where $\mu = \sup f(K) - 1$.*

Then $||f-h|| \geq \mu$ *for all* $h \in S(f,\mu,K)^0 \supset K^0$. *Also, for any* $x \in S(f,\mu,K)$,

$$S(-x, ||x|| - (\varepsilon+\mu), B(X^*)) \subset k(f, K^0).$$

Proof. The proof of the first part is straightforward. Thus, fix $x \in S = S(f,\mu,K)$ and let $h \in S(-x, ||x|| - (\varepsilon+\mu), B(X^*))$. Then $h(x) \leq -(\varepsilon+\mu)$. For any $y \in S$,

$$\begin{aligned}(f+h)(y) &= f(y) + h(x) + h(y-x)\\ &\leq 1 + \mu - (\varepsilon+\mu) + ||y-x||\\ &\leq 1.\end{aligned}$$

Thus, $f+h \in S^0$ so $h \in S^0 - f \subset k(f, S^0) = k(f, K^0)$ by Corollary 3.2, completing the lemma. □

Lemma 3.4. *Let* X *be a uniformly dentable Banach space with modulus of dentability* $\Delta(\varepsilon)$, *and let* $\delta(\varepsilon)$ *be defined as in Lemma* 2.1. *Let* $K \subset B(X)$ *be closed and convex with* $\sup ||K|| = 1$. *Then, given* ε, $0 < \varepsilon < 1/3$, *there is* $f \in S(X^*)$ *such that* $\sup f(K) > (\delta(\varepsilon)/\varepsilon)(1-3\varepsilon)$, *diameter* $S(f, \delta(\varepsilon), K) \leq \varepsilon$ *and* $\inf ||S(f,\delta(\varepsilon),K)|| > 1-3\varepsilon$.

Proof. Let $\delta = \delta(\varepsilon)$ and $A = \overline{\text{co}}(K \cup -K)$. By Theorem 2.2 there are functionals $\{f_\alpha\} \subset S(X^*)$ such that $H(\overline{\text{co}}(\cup S(f_\alpha,\delta,A)), A) \leq \varepsilon$ and diameter $S(f_\alpha,\delta,A) \leq \varepsilon$ for each α. If $S(f_\alpha,\delta,A) \subset (1-2\varepsilon)B$ for all α then $\overline{\text{co}}(\cup S(f_\alpha,\delta,A) \subset (1-2\varepsilon)B$ and so $A \subset (1-\varepsilon)B$, contradicting $\sup ||K|| = 1$. Thus, choose α such that $\sup ||S(f_\alpha,\delta,A)|| > 1-2\varepsilon$ and let $S = S(f_\alpha,\delta,A)$.

Now diameter $S \leq \varepsilon$, so clearly $\inf ||S|| > 1-3\varepsilon$. Since $\varepsilon < 1/3$, this shows that $0 \notin S$ and since $0 \in A$, it follows that $\sup f_\alpha(A) > \delta$ and hence $f_\alpha(x) > 0$ for all x in S.

Let $x \in S$ be arbitrary. Since $0 \in A$ convexity shows that $((\sup f_\alpha(A) - \delta)/f_\alpha(x)) \cdot x \in S$. Thus, as diameter $S \leq \varepsilon$,

$$\varepsilon \geq \left\| x - \frac{\sup f_\alpha(A) - \delta}{f_\alpha(x)} \cdot x \right\| = \left(\frac{f_\alpha(x) - \sup f_\alpha(A) + \delta}{f_\alpha(x)} \right) ||x||.$$

Since this is true for all $x \in S$, $\inf ||S|| > 1 - 3\varepsilon$, and $\sup f(A) = \sup f(S)$, then $\varepsilon > (\delta / \sup f_\alpha(A))(1-3\varepsilon)$, so $\sup f_\alpha(A) > (\delta/\varepsilon)(1-3\varepsilon)$.

Now $\sup f_\alpha(A)$ is the same as either $\sup f_\alpha(K)$ or $\sup f_\alpha(-K)$. Let $f = f_\alpha$ for the former case and $f = -f_\alpha$ for the latter. Then either $S \supset S(f, \delta, K)$ or $-S \supset S(f, \delta, K)$ and the lemma follows easily. □

Lemma 3.5. *Let X be uniformly dentable. Then for every ε, $0 < \varepsilon < 1$, there is $\Sigma(\varepsilon) > 0$ such that if K is closed and convex, $\varnothing \neq K \subset B(X)$ and $K \neq \{0\}$, then there is $f \in X^*$ such that $d(f, K^0) \geq \Sigma(\varepsilon)$ and for each $x \in S(f, \Sigma(\varepsilon), K)$, $S(-x, 1-\varepsilon, B(X^*)) \subset k(f, K^0)$.*

Proof. Define $\Sigma(\varepsilon) = \delta(\varepsilon/7)$ where δ is as in Lemma 2.1. Fix $\varepsilon > 0$ and let $\lambda = \sup ||K||$. Note that $0 < \lambda \leq 1$.

Lemma 3.4 shows that there is an $f_1 \in S(X^*)$ such that diameter $S(f_1, \Sigma(\varepsilon), K/\lambda) \leq \varepsilon/7$, $\sup f_1(K/\lambda) > 7(\Sigma(\varepsilon)/\varepsilon)(1 - 3\varepsilon/7)$ and $\inf\ ||S(f_1, \Sigma(\varepsilon), K/\lambda)|| > 1 - 3\varepsilon/7$.

Define $f_2 = f_1/(\sup f_1(K/\lambda) - \Sigma(\varepsilon))$. Then $S(f_1, \Sigma(\varepsilon), K/\lambda) = S(f_2, \sup f_2(K/\lambda) - 1, K/\lambda)$ so Lemma 3.3 shows that $||f_2 - h|| \geq \sup f_2(K/\lambda) - 1 > \Sigma(\varepsilon)$ for all $h \in (K/\lambda)^0 = \lambda K^0$. Thus, $d(f_2/\lambda, K^0) \geq \Sigma(\varepsilon)/\lambda$. Since $0 < \lambda \leq 1$, letting $f = f_2/\lambda$ gives $d(f, K^0) \geq \Sigma(\varepsilon)$.

Lemma 3.3 also shows that, for any $x \in S(f_1, \Sigma(\varepsilon), K/\lambda)$, $S(-x, ||x|| - (\varepsilon/7 + \sup f_2(K/\lambda) - 1), B(X^*)) \subset k(f_2, \lambda K^0) = k(\lambda f, \lambda K^0) = k(f, K^0)$.

Now

$$\begin{aligned}
\sup f_2\left(\frac{K}{\lambda}\right) - 1 &= \frac{\sup f_1(\frac{K}{\lambda})}{\sup f_1(\frac{K}{\lambda}) - \Sigma(\varepsilon)} - 1 \\
&= \frac{\Sigma(\varepsilon)}{\sup f_1(\frac{K}{\lambda}) - \Sigma(\varepsilon)} \\
&< \frac{\Sigma(\varepsilon)}{7\frac{\Sigma(\varepsilon)}{\varepsilon}(1 - \frac{3\varepsilon}{7}) - \Sigma(\varepsilon)} \\
&= \frac{\varepsilon}{7 - 4\varepsilon} < \frac{\varepsilon}{3}
\end{aligned}$$

since $0 < \varepsilon < 1$.

Thus $S(-x, ||x|| - \varepsilon/3 - \varepsilon/7, B(X^*)) \subset k(f, K^0)$.

Since $\inf ||S(f_1, \Sigma(\varepsilon), K/\lambda)|| > 1 - 3\varepsilon/7$, it follows that

$$S(-x, 1-\varepsilon, B(X^*)) \subset S(-x, 1-(19/21)\varepsilon, B(X^*)) \subset k(f, K^0)$$

for all $x \in S(f_1, \Sigma(\varepsilon), K/\lambda)$, completing the lemma. □

Theorem 3.6. *If X is uniformly dentable, then X^* is weak* uniformly smoothable.*

Proof. Every weak* closed convex set $K \subset X^*$ with $B(X^*) \subset K$ satisfies $(K_0)^0 = K$ (where K_0 is the prepolar of K) so the proposition follows directly from Lemma 3.5. (The modulus of smoothability can be taken to be any function strictly less than the function $\Sigma(\varepsilon)$ given by Lemma 3.5.) □

Theorem 3.7. *If X^* is weak* uniformly dentable, then X is uniformly smoothable.*

The proof of this follows exactly the same course as the proof of Theorem 3.6; hence, we do not repeat it. More precisely, weak* and prepolar versions of Results 2.1, 2.2 and 3.1–3.6 are needed.

Section 4. In this section we show that if X is uniformly smoothable (X^* is weak* uniformly smoothable) then X^* is weak* uniformly dentable (X is uniformly dentable). In the course of doing this, we establish an approximation to closed convex bodies in uniformly smoothable spaces (Theorem 4.10).

The section proceeds by first introducing some notation, then presenting a series of technical lemmas that lead up to the main results. Much of the argument is, in some sense, dual to Sections 2 and 3.

Let X be a Banach space, and let $q \in X$ and $0 < r < ||q||$ be given. Let $Q = k(0, B(q, r))$. Note that $Q^0 = \{f \in X^* : \sup f(Q) = 0\}$. For $h \in Q^0$, define $T_h : X \to X$ by $T_h(x) = x - (2h(x)/h(q)) \cdot q$. ($T_h$ is reflection along q through $h^{-1}(0)$.) It is easy to verify that $T_h = T_h^{-1}$ and that $||T_h|| \le M$ where $M = 1 + 2(||q||/r)$. Since $T_h = T_h^{-1}$, $1/M \le ||T_h|| \le M$ and $1/M \le ||T_h^*|| \le M$.

Let $x \in X \sim Q$ be fixed, but arbitrary, and let π be the plane spanned by x and q. Then $Q \cap \pi$ is bounded by two rays. At least one of these rays, call it R, is in the same half plane determined by the line through q and the origin as x. If $||x|| = 1$, by the separation theorem, let $h_x \in Q^0$ satisfy $h_x(R) = 0$. For $x = \lambda y$, $\lambda > 0$ and $||y|| = 1$, let $h_x = h_y$. Thus, h_x is defined for all $x \in X \sim Q$ in such a way that $h_{\lambda x} \equiv h_x$ for $\lambda > 0$, h_x separates Q and x, and $h_x(R) = 0$. Note that $T_{h_x}(x) \in Q$ for all $x \in X \sim Q$.

It is not hard to see that $\{T_h(x) : h \in Q^0\}$ is a closed segment parallel to the vector q, and that the endpoint of this segment, at the end pointing in the same direction as q, is $T_{h_x}(x)$. (If x lies on the line through q and 0, the segment is the single point $-x$.)

Lemma 4.1. *Let q, r and Q be as above and let K be closed and convex with $0 \in K$ and $q \notin K$. Let $L = l(q, K)$. If $x \in X \sim Q$ satisfies either $x \notin L$ or $T_h(x) \notin L$ for some $h \in Q^0$, then $T_{h_x}(x) \notin L$.*

Proof. If $x \notin L$, since $T_{h_x}(x) = x - 2h_x(x)q/h_x(q)$ and $h_x(x)/h_x(q) < 0$, then $T_{h_x}(x) \notin L$ follows from Lemma 1.7(b). If $T_h(x) \notin L$, $h \in Q^0$, then from the above discussion, $T_{h_x}(x) = T_h(x) + \lambda q$, $\lambda > 0$, so Lemma 1.7(b) once more shows that $T_{h_x}(x) \notin L$. □

Lemma 4.2. *Let K be closed and convex with $B \subset K \subset X$. Suppose that q and r are such that $d(q, K) > r$. Define Q, T_h, h_x and M as above. Let $L = l(q, K)$ and define $L_1 = \cap_{h \in Q^0} T_h(L) \cap L$. Then $L_1 \subset M(||q|| + r)B$.*

Proof. Let $x \in L_1$. If $x \in Q$, then $x \in L \cap Q$, so $||x|| < ||q|| + r$. If $x \notin Q$, then $T_{h_x}(x) \in Q$. Since $x \in L_1$, $x \in T_{h_x}(L)$ and hence $T_{h_x}(x) \in L$. Thus, $T_{h_x}(x) \in L \cap Q$, and so $||T_{h_x}(x)|| < ||q|| + r$. Since $||T_{h_x}^{-1}|| \leq M$, $||x|| < M(||q|| + r)$. Noting that $M \geq 1$, the lemma is complete. □

Remark . The set L_1 is to play a role similar to that of the set C_1 in Lemma 2.1.

Lemma 4.3. *Let K be closed and convex with $B(X) \subset K \subset X$. Let $q \notin K$ and $r > 0$ be given with $r < ||q||$. Let $Q = k(0, B(q, r))$. If $x \in Q$ and $||x|| < (||q|| - r)/(1 + r)$, then $x \in k(q, K) + q$.*

Proof. It is easy to see that $Q = k(0, B(q/(1 + r), r/(1 + r)))$ so $x = \lambda y$ for some $y \in B(q/(1 + r), r/(1 + r))$ and $\lambda \geq 0$. Since $||x|| < (||q|| - r)/(1 + r)$, $0 \leq \lambda < 1$. Now $k(q, K) + q \supset k(q, B) + q = k(q, B((q/1 + r), r/(1 + r))) + q \supset B(q/(1 + r), r/(1 + r))$. Thus, $y \in k(q, K) + q$ and since $0 \in k(q, K) + q$, convexity shows that $x = \lambda y \in k(q, K) + q$.

Lemma 4.4. *Let $K \subset X$ be closed and convex with $B(X) \subset K$. If $0 < \lambda < 1$ and λx is not in the interior of K, $d(x, K) \geq (1 - \lambda)/\lambda$. If $\lambda x \notin K$, then $d(x, K) > (1 - \lambda)/\lambda$.*

Lemma 4.5. *Let $R > 1$ be given, and let α be the positive solution to $(1 + x)^2 + x = R$. Let $K \subset X$ be closed and convex with $inf|| \sim K|| = 1$. Then there is $u \in \sim K$ such that $d(u, K) > \alpha$, $1 + \alpha < ||u|| < (1 + \alpha)^2$ and if $Q = k(0, B(u, \alpha))$ and $x \in Q$ satisfies $||x|| > R$, then $d(x, K) > (||x||/R) - 1$.*

Notation. The following notation will be used for the remainder of this section. Let α be the positive solution to $(1 + x)^2 + x = 1.25$, $r = 2\alpha/((1 + \alpha)^2 - 3\alpha) = 2\alpha/(1.25 - 4\alpha)$, $\beta = 2(1 + \alpha)^2/(1 - 2\alpha)$ and $M = 1 + 2\beta/r$ ($\alpha \approx .081, r \approx .175, \beta \approx 2.79, M \approx 32.8$).

We will be working in a uniformly smoothable Banach space, with modulus of smoothability $\Sigma(\varepsilon)$. Thus, for $0 < \varepsilon < 1$, define

$$\sigma_1(\varepsilon) = \frac{\Sigma(\frac{\varepsilon}{M^2})}{2M\left(M(\beta + r) + \frac{\Sigma(\frac{\varepsilon}{M^2})}{M}\right)}$$

and $\sigma(\varepsilon) = \min\{\sigma_1(\varepsilon), .2\}$. (Notice that $\Sigma(\varepsilon/M^2)$ is defined for $0 < \varepsilon < 1$ and $\sigma(\varepsilon) < \Sigma(\varepsilon/M^2)$).

Lemma 4.6. *Let $K \subset X$ satisfy* $\inf|| \sim K|| = 1$. *Then there is $q \in \sim K$ with $||q|| < \beta$ and $d(q, K) > r$ such that if $Q = k(0, B(q, r))$*

and $x \in Q$ *has* $||x|| < 2$ *then* $x \in k(q, K) + q$. *Furthermore, for any* $x \in Q$ *with* $||x|| > 1.5$, $d(x, K) > .2$.

Proof. By Lemma 4.5 (with $R = 1.25$) there is $u \in\sim K$ such that $d(u, K) > \alpha$, $1 + \alpha < ||u|| < (1 + \alpha)^2$ and if $Q_1 = k(0, B(u, \alpha))$ and $x \in Q_1$ satisfies $||x|| > 1.25$, then $d(x, K) > (||x||/1.25) - 1$. Thus, if $||x|| > 1.5$, then $d(x, K) > .2$.

Let $\lambda = 2/(||u|| - 3\alpha)$ and $q = \lambda u$. Then $||q|| < \beta$ and since $\lambda > 1$ (indeed $\lambda > 2$) and $0 \in K$,

$$d(q, K) \geq d(q, \lambda K) = \lambda d(u, K) > \lambda\alpha > r.$$

Now $Q_1 = k(0, B(q, \lambda\alpha))$ so Lemma 4.3 shows that if $x \in Q_1$ satisfies $||x|| < (||q|| - \lambda\alpha)/(1 + \lambda\alpha) = 2$, then $x \in k(q, K) + q$. Since $Q \subset Q_1$, the lemma is complete. □

The following lemma contains the crucial steps in proving the main results in this section—Theorems 4.9 and 4.12. The proof is somewhat dual to Lemma 2.1, and the result is similar in that it gives us some control over where K is smoothable. Lemma 4.11 is another such result.

Lemma 4.7. *Let* X *be a uniformly smoothable Banach space with modulus* $\Sigma(\varepsilon)$. *Define* $\sigma(\varepsilon)$ *as in the notation before Lemma* 4.6. *Then, for every closed convex set* K *with* $\inf || \sim K|| = 1$ *and for every* ε, $0 < \varepsilon < 1$, *there is* $p \notin K$ *and* $f \in S(X^*)$ *with* $||p|| < 2$, $\sigma(\varepsilon) < d(p, K) < 2\sigma(\varepsilon)$, *and* $S(f, 1 - \varepsilon, B) \supset k(p, K)$.

Proof. Let $\alpha, r, \beta, M, \sigma_1$, as well as σ be defined as in the notation before Lemma 4.6. Let q and Q be as in Lemma 4.6 and define $L = l(q, K)$. Let T_h and h_x be defined as in the discussion at the beginning of this section (using the above q and r). Let $\varepsilon, 0 < \varepsilon < 1$, be fixed but arbitrary. We caution the reader that, since $T_h = T_h^{-1}$, T_h appears where one might expect T_h^{-1}.

As in Lemma 4.2, define $L_1 = \cap_{h \in Q^0} T_h(L) \cap L$. Since $||T_h^{-1}|| \leq M$, $(1/M)B \subset T_h(B)$ for all $h \in Q^0$, and hence $(1/M)B \subset L_1$. Since X is uniformly smoothable with modulus $\Sigma(\varepsilon)$, there is $y_1 \in X$ and $g \in S(X^*)$ such that $d(y_1, ML_1) \geq \Sigma(\varepsilon/M^2)$ and $S(g, 1-(\varepsilon/M^2), B) \subset k(y_1, ML_1)$. (Since $M > 1$, $\Sigma(\varepsilon/M^2)$ is defined.)

Letting $y_2 = (y_1/M)$ it is not hard to show that $k(y_1, ML_1) = k(y_2, L_1)$ and $d(y_2, L_1) \geq \Sigma(\varepsilon/M^2)/M$. Now, since $0 \in L_1$, there is γ, $0 < \gamma \leq 1$, such that if $y = \gamma y_2$, then $d(y, L_1) = \Sigma(\varepsilon/M^2)/M$. Note that $k(y, L_1) \supset k(y_2, L_1) \supset S(g, 1-(\varepsilon/M^2), B)$. Furthermore, $||y|| \geq 1/M + \Sigma(\varepsilon/M^2)/M$.

Let $\lambda = 1 - \Sigma(\varepsilon/M^2)/(2M||y||)$. Then $0 < \lambda < 1$ and $||y - \lambda y|| = \Sigma(\varepsilon/M^2)/2M < d(y, L_1)$ so $\lambda y \notin L_1$. We now have two cases to consider: that of $\lambda y \in Q$ and that of $\lambda y \notin Q$. If $\lambda y \in Q$, it follows that $\lambda y \notin L$, for otherwise Lemma 1.7 (b) would show that $T_h(\lambda y) \in L$ for all $h \in Q^0$, so for $h \in Q^0$ $\lambda y = T_h T_h(\lambda y) \in T_h(L)$ and hence λy would be in L_1. We will not pursue this case further as from this point it is similar to the case $\lambda y \notin Q$. (Simply use I instead of T below. The inequalities will not be sharp but will be satisfied.)

Suppose that $\lambda y \notin Q$. Since $\lambda y \notin L_1$, Lemma 4.1 shows that $T_{h_y}(\lambda y) \notin L$. (Recall that, by definition, $h_\lambda = h_{\lambda y}$.) Let $T = T_{h_y}$. Lemma 4.4 shows that $d(T(y), L) > (1-\lambda)/\lambda > 1 - \lambda = \Sigma(\varepsilon/M^2)/(2M||y||)$. Lemma 4.2 shows that $L_1 \subset M(||q|| + r)B$ so, since $d(y, L_1) = \Sigma(\varepsilon/M^2)/M$, $||y|| < M(\beta + r) + \Sigma(\varepsilon/M^2)/M$, hence $d(T(y), L) > \sigma_1(\varepsilon)$.

Let $p_1 = T(y)$ and $f = T^*g/||T^*g||$. Since $||g|| = 1$, $||T^*g|| \geq 1/M$ and since $y \notin Q$, $T(y) = p_1 \in Q$ (cf. the discussion at the beginning of this section). Now

$$S\left(g, 1 - \frac{\varepsilon}{M^2}, B\right) \subset k(y, L_1)$$

so

$$\begin{aligned} T\left(S\left(g, 1 - \frac{\varepsilon}{M^2}, B\right)\right) &\subset T(k(y, L_1)) \\ &= k(p_1, TL_1) \\ &\subset k(p_1, L). \end{aligned}$$

Also,

$$\begin{aligned} T\left(S\left(g, 1 - \frac{\varepsilon}{M^2}, B\right)\right) &= T\left\{x \in B : g(x) > \frac{\varepsilon}{M^2}\right\} \\ &\supset \left\{x \in TB : f(x) > \frac{\varepsilon}{M}\right\} \end{aligned}$$

$$\supset \left\{ x \in \frac{1}{M}B : f(x) > \frac{\varepsilon}{M} \right\}$$
$$= \frac{1}{M}S(f, 1-\varepsilon, B).$$

Thus, $S(f,1-\varepsilon,B) \subset Mk(p_1,L) = k(p_1,L)$. Define

$$p_2 = \begin{cases} p_1 & \text{if } ||p_1|| < 1.75 \\ 1.75\frac{p_1}{||p_1||} & \text{otherwise.} \end{cases}$$

Note that $p_2 \in Q$ and $k(p_1,L) \subset k(p_2,L)$. Lemma 4.6 shows that $p_2 \in k(q,K) + q$, hence Lemma 1.7 (c) shows that $S(f,1-\varepsilon,B) \subset k(p_2,L) = k(p_2,L) = k(p_2,K)$. Now

$$d(p_2,L) = \begin{cases} d(p_1,L) & \text{if } ||p_1|| < 1.75 \\ d\left(1.75\frac{p_1}{||p_1||}, L\right) & \text{otherwise.} \end{cases}$$

However, Lemma 4.6 shows that $d(1.75(p_1/||p_1||),L) > .2$. Since $d(p_1,L) > \sigma_1(\varepsilon)$, it follows that

$$d(p_2,L) > \min\{\sigma_1(\varepsilon), .2\} = \sigma(\varepsilon).$$

Lemma 1.7 (c) shows that $d(p_2,K) = d(p_2,L) > \sigma(\varepsilon)$. For any λ, $0 < \lambda \le 1$, such that $\lambda p_2 \notin K$, $k(\lambda p_2,K) \supset k(p_2,K) \supset S(f,1-\varepsilon,B)$. Thus, letting $p = \lambda p_2$, for a suitable choice of λ, will give $\sigma(\varepsilon) < d(p,L) < 2\sigma(\varepsilon)$, $||p|| \le ||p_2|| \le 1.75 < 2$, and $S(f,1-\varepsilon,B) \subset k(p,K)$. □

Lemma 4.8. *Let $K \subset X$, $f \in X^*$ and $\mu > 0$ be given. Suppose that $0 < \lambda \le 1$ and diameter $S(f,\mu,(1/\lambda)K) \le \varepsilon$. Then diameter $S(f,\mu,K) \le \varepsilon$.*

Proof. Let $m > 0$ be arbitrary, but fixed. Let $x, y \in S(f,\mu - (1/m),K)$ be arbitrary. Choose $z \in (1/\lambda)K$ such that $f(z) \ge (1/\lambda)\sup f(K) - (1/m)$ and let $x' = x + (1-\lambda)z$ and $y' = y + (1-\lambda)z$. Then x' and y' are in $S(f,\mu,(1/\lambda)K)$ so $||x' - y'|| \le \varepsilon$. However, $||x' - y'|| = ||x - y||$ so $||x - y|| \le \varepsilon$. Hence, diameter

$S(f, \mu - (1/m), K) \leq \varepsilon$. Since this is true for all $m > 0$, diameter $S(f, \mu, K) \leq \varepsilon$. □

We are now prepared for the main results of this section.

Theorem 4.9. *If X is uniformly smoothable, then X^* is weak* uniformly dentable.*

Proof. Suppose that X has modulus of uniform smoothability $\Sigma(\varepsilon)$, $0 < \varepsilon < 1$. Define $\Sigma'(\varepsilon) = \min\{\Sigma(\varepsilon), M\varepsilon\}$. Then $\Sigma'(\varepsilon) \leq \Sigma(\varepsilon)$, so it too is a modulus of smoothability for X. Define $\sigma(\varepsilon)$ as in the notation before Lemma 4.6, using $\Sigma'(\varepsilon)$ rather than $\Sigma(\varepsilon)$. Then $\sigma(\varepsilon) < \Sigma'(\varepsilon/M) \leq \varepsilon$.

Let $K \subset B(X^*)$ be weak* closed and convex. We first consider the case that $\sup ||K|| = 1$.

Fix ε, $0 < \varepsilon < 1$. Let A be the convex hull of $K \cup (-K)$ and note that A is weak* closed. Then $A \subset B(X^*)$, $(A_0)^0 = A$, $A_0 \supset B(X)$, $\sup ||A|| = 1$, and $\inf || \sim A_0|| = 1$. Lemma 4.7 shows that there is $p \in X \sim A_0$ and $f \in S(X^*)$ with $||p|| < 2$, $\sigma(\varepsilon) < d(p, A_0) < 2\sigma(\varepsilon)$, and $S(f, 1 - \varepsilon, B(X)) \subset k(p, A_0)$.

Since $d(p, A_0) > \sigma(\varepsilon)$, there is a $g \in X^*$ that strictly separates $B(p, \sigma(\varepsilon))$ and A_0. Since $0 \in A_0$, it may be assumed that $\sup g(A_0) \leq 1 \leq \inf g(B(p, \sigma(\varepsilon)))$ so $g \in (A_0)^0 = A$.

Consider $S = \{h \in X^* : \sup h(A_0) \leq 1 \leq h(p)\} = S(p, \sup p(A) - 1, A)$. Then $g \in S$ and $\sup p(A) - 1 \geq p(g) - 1 \geq 1 + ||g||\sigma(\varepsilon) - 1 = ||g||\sigma(\varepsilon)$. Now $||p|| < 2$ and $g(p) > 1$ so $||g|| > 1/2$. Hence, $\sup p(A) - 1 > \sigma(\varepsilon)/2$. The depth of the slice S is $(1/||p||)(\sup p(A) - 1)$, which is more than $\sigma(\varepsilon)/4$.

Next we wish to find an upperbound for the diameter of S. Since $S((f/\varepsilon), (1/\varepsilon) - 1, B) = S(f, 1 - \varepsilon, B) \subset k(p, A_0)$ and $||f|| = 1$, Lemma 3.1 shows that $k(p, A_0)^0 \subset l((f/\varepsilon), B(X^*))$. Also, [**1**, Lemma 2.4] shows that $k(0, S) = k(p, A_0)^0$, so $S \subset k(0, S) \subset l((f/\varepsilon), B(X^*)) \subset k((f/\varepsilon), B(X^*)) + (f/\varepsilon) = k(f, \varepsilon B(X^*)) + (f/\varepsilon)$. From $k(0, S) \subset k(f, \varepsilon B(X^*)) + (f/\varepsilon)$, it follows that $x - (f/\lambda\varepsilon) \in (1/\lambda)k(f, \varepsilon B(X^*)) = k(f, \varepsilon B(X^*))$ for all $\lambda > 0$ and $x \in S$. Letting $\lambda \to \infty$ shows that $x \in k(f, \varepsilon B(X^*))$ and hence $S \subset k(f, \varepsilon B(X^*))$.

Lemma 2.6 of [**1**] shows that

$$\text{diameter}\, S \leq 8\varepsilon||p|| + 2(\sup p(A) - 1) < 16\varepsilon + 2(\sup p(A) - 1).$$

Since $d(p, A_0) < 2\sigma(\varepsilon)$, there is $q \in A_0$ with $||q - p|| < 2\sigma(\varepsilon)$. Thus, since $A \subset B(X^*)$, $\sup p(A) \leq \sup(p - q)A + \sup q(A) \leq ||p - q|| + 1$. Thus, $\sup p(A) - 1 < 2\sigma(\varepsilon)$. Hence, diameter $S < 16\varepsilon + 4\sigma(\varepsilon)$. Recalling from the beginning of this proof that $\sigma(\varepsilon) < \varepsilon$, we have diameter $S < 20\varepsilon$.

Now A is the convex hull of $K \cup (-K)$. Thus, $\sup p(A)$ is either $\sup p(K)$ or $\sup p(-K)$. Assume the former, the latter case being similar. (Simply replace p by $-p$.) Since $\sup p(K) = \sup p(A)$, $S(p, \sup p(K) - 1, K)$ has the same depth as S—more than $\sigma(\varepsilon)/4$ and, since $K \subset A$, diameter $S(p, \sup p(K) - 1, K) \leq$ diameter $S < 20\varepsilon$.

Define $\Delta(\varepsilon) = (1/4)\sigma(\varepsilon/20)$. Then it follows from the above that for every set $K \subset B(X^*)$ with $\sup || \sim K|| = 1$, and every ε, $0 < \varepsilon < 2$, there is $p \in S(X)$ (obtained by normalizing the above choice for p) such that diameter $S(p, \Delta(\varepsilon), K) \leq \varepsilon$.

To complete the theorem, we need to consider the case that $\sup ||K|| < 1$. Let ε, $0 < \varepsilon < 2$ be given and let $\lambda = \sup ||K||$. Then, by the above, there is $p \in S(X)$ such that diameter $S(p, \Delta(\varepsilon), (1/\lambda)K) \leq \varepsilon$ and Lemma 4.8 shows then that diamter $S(p, \Delta(\varepsilon), K) \leq \varepsilon$. □

Theorem 4.10. *If X^* is weak* uniformly smoothable, then X is uniformly dentable.*

Proof. The proof of this is much the same as the proof of Theorem 4.9, so it shall not be repeated, but we shall comment on the one major change that is necessary. In the introduction to this section, the transformations T_h are defined for $h \in Q^0$. To prove Theorem 4.10 we would, of course, use only those h in Q_0, the prepolar of Q. The set Q is $k(0, B(q, r))$ for some $q \in X$, $0 < r < ||q||$, and this is a weak* closed set. For $x \notin Q$, we considered the plane π spanned by q and x and let R be a ray bounding $Q \cap \pi$ that is in the same half plane of π determined by q and the origin as x. Then h_x was defined to be a function in Q^0 such that $h_x(R) = 0$. Unfortunately, we cannot necessarily choose h_x to be in Q_0.

The properties of h_x that are needed for Theorem 4.9 appear only in Lemmas 4.1, 4.2 and Proposition 4.7. The functional h_x is such that $T_{h_x}(x) \in Q$ and for K a closed convex set with $B(X) \subset K$, $q \notin K$ and $L = l(q, K)$, if $x \notin L$ or $T_h(x) \notin L$ for some $h \in Q^0$, then $T_{h_x}(x) \notin L$. It was not necessary that h_x be chosen independent of K. Using the separation principle it is possible to choose $h_x \in Q_0$ with these properties, although the choice may depend on the set K. □

Lemma 4.11. *Let X be uniformly smoothable with modulus $\Sigma(\varepsilon)$ and let $K \subset X$ satisfy $B \subset K$. Let $q \notin K$ and $\varepsilon > 0$, $0 < \varepsilon < 1$, satisfy $d(q, K) > \varepsilon$ (or equivalently $B(q, \varepsilon) \cap K = \varnothing$). Then there is $p \notin K$ and $f \in S(X^*)$ such that $(\varepsilon/4)\sigma(\varepsilon) < d(p, K) < (1/2)\sigma(\varepsilon)$ (where $\sigma(\varepsilon)$ is defined in the notation before Lemma 4.6), $S(f, 1-\varepsilon, B) \subset k(p, K)$ and $B(q, \varepsilon) \not\subset k(p', K) + p'$ for any $p' \in X \sim K$ with $||p - p'|| \leq (\varepsilon/2)$.*

Proof. Let $\lambda = \sup\{\lambda' \in (0,1) : B(\lambda' q, (1/4)(1 - \lambda' + \varepsilon\lambda')) \subset K\}$ and let $s = (1/4)(1 - \lambda + \varepsilon\lambda)$. Note that $0 < \lambda < 1$, $\varepsilon/4 < s < 1/4$ and, since K is closed, $B(\lambda q, s) \subset K$. By the choice of λ, $B(\lambda q, s') \not\subset K$ for $s' > s$. Hence, $\inf || \sim ((1/s)(K - \lambda q))|| = 1$. Applying Lemma 4.7 to the set $(1/3)(K - \lambda q)$ shows that there is $p \notin K$ and $f \in S(X^*)$ with $||p - \lambda q|| < 2s$, $(\varepsilon/4)\sigma(\varepsilon) < s\sigma(\varepsilon) < d(p, K) < 2s\sigma(\varepsilon) < (1/2)\sigma(\varepsilon)$, and $s(f, 1-\varepsilon, B) \subset k(p, K)$.

Now $p = \lambda((\varepsilon/4s)(p - \lambda q) + q) + (1 - \lambda)(p - \lambda q)/4s$, $(\varepsilon/4s)(p - \lambda q) + q$ is in the interior of $B(q, \varepsilon/2)$, and $(p - \lambda q)/4s$ is in the interior of $B/2$. Thus, p is in the interior of $\lambda B(q, \varepsilon/2) + (1 - \lambda)(B/2) = B(\lambda q, 2s)$. Suppose that $p' \in X \sim K$ has $||p' - p|| \leq \varepsilon/2 < 2s$. Then p' is in the interior of $B(\lambda q, 4s)$. If $B(q, \varepsilon) \subset k(p', K) + p'$, then, since $k(p', K) + p'$ is convex, $B \subset K \subset k(p', K) + p'$ and $\lambda B(q, \varepsilon) + (1 - \lambda)B = B(\lambda q, 4s)$, p' would be an interior point of $k(p', K) + p'$, a contradiction. □

The following theorem is the second approximation to convex sets that was promised (Theorem 2.2 being the first).

Theorem 4.12. *Let X be a uniformly smoothable Banach space with modulus $\Sigma(\varepsilon)$ and define $\sigma(\varepsilon)$ as earlier. Let ε, $0 < \varepsilon < 1$, be given. Then for every closed convex set $K \subset X$, with $B \subset K$ there are pairs*

(p_i, f_i), $p_i \in X \sim K$, $f_i \in S(X^*)$ *and* i *in an index set* I *such that*

$$\frac{\varepsilon}{4}\sigma(\varepsilon) < d(p_i, K) < \frac{1}{2}\sigma(\varepsilon),$$
$$||p_i - p_j|| > \frac{\varepsilon}{2}, S(f_i, 1-\varepsilon, K) \subset k(p_i, K)$$

for all $i, j \in I$ *and* $(\cap_i(k(p_i, K) + p_i)) \sim K$ *contains no balls of radius* ε.

Proof. Consider the collection $\mathcal{F}$ of all sets $F = \{(p_i, f_i) :$ i in an index set$\}$ where $(\varepsilon/4)\sigma(\varepsilon) < d(p_i, K) < (1/2)\sigma(\varepsilon)$, $||p_i - p_j|| > \varepsilon/2$ $(i \neq j)$ and $S(f_i, 1-\varepsilon, K) \subset k(p_i, K)$, ordered by inclusion. The union of any chain is easily seen to be an upper bound for that chain so, by Zorn's lemma, let F be a maximal element for $\mathcal{F}$. We claim that the pairs $(p_i, f_i) \in F$ are as desired. It remains only to show that $(\cap_{p_i \in F} k(p_i, K) + p_i) \sim K$ contains no balls of radius ε.

To do this, suppose that $B(q, \varepsilon) \subset (\cap k(p_i, K) + p_i) \sim K$. By Lemma 4.11, there are $p \in X$ and $f \in S(X^*)$ with $(\varepsilon/4)\sigma(\varepsilon) < d(p, K) < (1/2)\sigma(\varepsilon)$, $S(f, 1-\varepsilon, K) \subset k(p, K)$ and $B(q, \varepsilon) \not\subset k(p', K) + p'$ for any p' with $||p - p'|| \leq (\varepsilon/2)$. But then $F \cup \{p\}$ is in $\mathcal{F}$ and $F \cup \{p\} \geq F$, so this contradicts the maximality of F. □

Corollary 4.13. *With* K *and* (p_i, f_i), $i \in I$ *as above,* $\cap l(p_i, K) \sim K$ *contains no balls of radius* ε.

Proof. This follows immediately since

$$K \subset l(p_i, K) \subset k(p_i, K) + p_i. \qquad \square$$

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