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CRITERIA FOR RIGHT DISFOCALITY OF AN nTH ORDER LINEAR DIFFERENCE EQUATION

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1. Introduction. We consider the nth order linear difference equation

$$Pu(m) \equiv \sum_{i=0}^{n} \alpha_i(m)u(m+i) = 0, \qquad m \in I \tag{1.1}$$

where I is an integer interval $I = [a, b] = \{a, a+1, a+2, \dots, b\}$. We use the notation $I^k = [a, b+k]$ where k is an integer such that $k \geq a - b$, as used by Hartman [**5**]. We assume the coefficients $\alpha_i(m)$ are defined on I, with $\alpha_n(m) \equiv 1$ and that

$$(-1)^n \alpha_0(m) > 0 \qquad \text{for } m \in I. \tag{1.2}$$

In [**9**], necessary and sufficient conditions were given for right $(l, n-l)$-disconjugacy and left $(l, n-l)$-disconjugacy in terms of the coefficients $\alpha_i(m)$, $0 \leq i \leq n$, of equation (1.1), which lead to an improvement of a result of Hartman [**5**] for disconjugacy. Here we will give similar results for right ρ_l-disfocality. For the general nth order linear difference equation (1.1), we will give some necessary conditions in terms of the coefficients $\alpha_i(m)$ for right ρ_l-disfocality. In the special case $l = n-1$, we will give necessary and sufficient conditions in terms of the coefficients $\alpha_i(m)$ for right ρ_{n-1}-disfocality. For the second order linear difference equation these results lead to necessary and sufficient conditions in terms of the coefficients $\alpha_i(m)$ for right disfocality.

The concept of disfocality for linear differential operators was introduced by Nehari [**7**]. Nehari showed that a certain generalized linear differential equation is disfocal if and only if the principal minors of a Wronskian matrix are positive.

More recently, Eloe [**1**] brought over to linear difference equations many of the results given by Muldowney [**6**] and Eloe and Henderson [**3**]

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for disfocality of linear ordinary differential equations. Eloe [**1**] stated criteria for right disfocality of linear difference equations involving D-Markov, D-Fekete and D-Descartes system of solutions on I^n. Other recent papers concerned with disfocality criteria for linear difference equations are Eloe [**2**] and Eloe and Henderson [**4**].

We begin with some basic definitions and examples.

Definition 1.1. [**5**] We say u has a *generalized zero at* m provided either $u(m) = 0$ for $m \geq a$, or for $m > a$ there is a $k \in \{1, \dots, m-a\}$ such that $(-1)^k u(m-k)u(m) > 0$ and if $k > 1$ then $u(m-k+1) = \cdots = u(m-1) = 0$.

Hartman [**5**, Proposition 5.1] stated the following discrete analogue of Rolle's theorem which we will use frequently in this paper. We will be using the forward difference operator Δ defined by $\Delta u(m) = u(m+1) - u(m)$, and Δ^i is defined recursively by $\Delta^i u(m) = \Delta(\Delta^{i-1} u(m))$ with $\Delta^0 u(m) = u(m)$.

Proposition 1.1 [**5**] (Rolle's Theorem). *Suppose that the finite sequence* $u(1), \dots, u(j)$ *has* N_j *generalized zeros and that* $\Delta u(1), \dots, \Delta u(j-1)$ *has* M_j *generalized zeros. Then* $M_j \geq N_j - 1$.

Definition 1.2. Equation (1.1) is said to be *disfocal* on I^n provided there is no nontrivial solution u of equation (1.1) and points $m_0, \dots, m_{n-1} \in I^{n-i}$ such that $\Delta^i u(m)$ has a generalized zero at m_i for $0 \leq i \leq n-1$.

Definition 1.3. Equation (1.1) is said to be *right disfocal* on I^n provided there is no nontrivial solution u of equation (1.1) and points $a \leq m_0 \leq \cdots \leq m_{n-1} \leq b+1$ such that $\Delta^i u(m)$ has a generalized zero at m_i for $0 \leq i \leq n-1$.

Note that by using Rolle's theorem, it can be shown that if equation (1.1) is right disfocal on I^n, then equation (1.1) is also disconjugate on I^n. We say equation (1.1) is disconjugate on I^n provided there is no nontrivial solution of (1.1) that has n generalized zeros.

Definition 1.4. Fix $l \in \{1, \dots, n-1\}$. Equation (1.1) is said to be *right* ρ_l*-disfocal* on I^n provided there is no nontrivial solution u of equation (1.1) and points $a \le m_0 < m_1 \le \dots \le m_{n-k} \le b+1$ where $l \le k \le n$ such that $u(m_0) = \Delta u(m_0) = \dots = \Delta^{k-1} u(m_0) = 0$, and if $k < n$, $\Delta^{l+i} u(m)$ has a generalized zero at m_{i+1} for $0 \le i \le n-k-1$.

Note that, by using Rolle's theorem, it can be shown that if equation (1.1) is right ρ_l-disfocal on I^n, then equation (1.1) is also right $(l, n-l)$-disconjugate on I^n where equation (1.1) is said to be right $(l, n-l)$-disconjugate on I^n for fixed $l \in \{1, \dots, n-1\}$ provided there is no nontrivial solution u of (1.1) that has l consecutive zeros followed by consecutive $n-l-1$ zeros and a generalized zero.

It is clear from the definitions that disfocality implies right disfocality implies right ρ_l-disfocality on I^n. But, the converses are not true, as will be shown in later sections of this paper. Yet we will now present an example of an equation which is disconjugate, but is neither disfocal nor right disfocal on some interval I^n.

Example 1.1. Consider the following equation

$$2u(m) - 2u(m+1) + u(m+2) = 0, \qquad \text{for } m \in [0, 1]. \tag{1.3}$$

We can show equation (1.3) is disconjugate on $[0, 3]$ by the results in [**8**]. But there is a nontrivial solution u of equation (1.3) where $u(0) = 0$, $u(1) = 1$ and $u(2) = u(3) = 2$. That is, $u(0) = 0$ and $\Delta u(2) = 0$; hence, equation (1.3) is neither disfocal nor right disfocal on $[0, 3]$.

We will use the notation $D_k^l(m)$ to represent certain determinants of the coefficients of equation (1.1) as used in [**9, 10**].

$$D_k^l(m) = \begin{vmatrix} \alpha_l(m) & \alpha_{l+1}(m) & \cdots & \alpha_{l+k-1}(m) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{l-k+2}(m+k-2) & \alpha_{l-k+3}(m+k-2) & \cdots & \alpha_{l+1}(m+k-2) \\ \alpha_{l-k+1}(m+k-1) & \alpha_{l-k+2}(m+k-1) & \cdots & \alpha_l(m+k-1) \end{vmatrix} \tag{1.4}$$

where $\alpha_i(m) = 0$ for $i < 0$ or $i > n$, and $\alpha_n(m) = 1$.

We will define two more determinants $E_k^l(m)$ and $F_k^l(m)$ similar to $D_k^l(m)$. These two determinants will be defined in terms of equations equivalent to equation (1.1) in the same manner as $D_k^l(m)$ was defined for equation (1.1). The first of these equations we obtain using the formula

$$\begin{aligned}\Delta^{n-1}u(m+1) &= \sum_{j=0}^{n-1}(-1)^{n-j-1}\binom{n-1}{j}u(m+j+1)\\ &= \sum_{i=1}^{n}(-1)^{n-i}\binom{n-1}{i-1}u(m+i);\end{aligned}$$

namely,

$$Pu(m) \equiv \sum_{i=0}^{n-1}\beta_i(m)u(m+i) + \Delta^{n-1}u(m+1) = 0 \tag{1.5}$$

where

$$\begin{aligned}&\beta_0(m) = \alpha_0(m)\\ &\beta_i(m) = \alpha_i(m) + (-1)^{n-1-i}\binom{n-1}{i-1}, \qquad \text{for } 1 \le i \le n-1,\end{aligned}$$

where $\binom{j}{i}$ is the binomial coefficient $j!/[i!(j-i)!]$.

The following notation will be used for determinants involving the coefficients of (1.5). For fixed $l \in \{1, \dots, n-1\}$, we define $E_1^l(m) = \beta_l(m)$, and, for $k \ge 2$,

$$E_k^l(m) = \begin{vmatrix} \alpha_l(m) & \alpha_{l+1}(m) & \cdots & \alpha_{l+k-1}(m)\\ \vdots & \vdots & \ddots & \vdots\\ \alpha_{l-k+2}(m+k-2) & \alpha_{l-k+3}(m+k-2) & \cdots & \alpha_{l+1}(m+k-2)\\ \beta_{l-k+1}(m+k-1) & \beta_{l-k+2}(m+k-1) & \cdots & \beta_l(m+k-1)\end{vmatrix} \tag{1.6}$$

where $\alpha_i(m) = \beta_i(m) = 0$ for $i < 0$ or $i > n$, and $\alpha_n(m) = 1$.

Fix $j \in \{1, \dots, n-l-1\}$. By considering the binomial expansions of $\Delta^i u(m+n-j)$, for $1 \le i \le j$, we can write equation (1.1) in the form

$$Pu(m) \equiv \sum_{i=0}^{n-j}\gamma_i^j(m)u(m+i) + \sum_{i=1}^{j}\gamma_{n-j+i}^j(m)\Delta^i u(m+n-j) = 0 \tag{1.7}$$

where $\gamma_i^j(m) = \alpha_i(m)$ for $0 \le i \le n-j-1$ and $\gamma_n^j(m) = 1$ on I.

Similarly, we can write equation (1.1) in the form

(1.8)
$$Pu(m) \equiv \sum_{i=0}^{l} \gamma_i^{n-l}(m)u(m+i) + \sum_{i=1}^{n-l-1} \gamma_{l+i}^{n-l}(m)\Delta^i u(m+l) + \gamma_n^{n-l}(m)\Delta^{n-1}u(m+1) = 0$$

where $\gamma_i^{n-l}(m) = \alpha_i(m)$ for $0 \le i < l$ and $\gamma_n^{n-l}(m) = 1$ on I.

The following notation will be used for determinants involving the coefficients of equations (1.1), (1.7) and (1.8). We define $F_1^l(m) = \gamma_l^{n-l}(m)$, and, for $k \ge 2$,

(1.9)
$$F_k^l(m) = \begin{vmatrix} \gamma_l^{n-l-k+1}(m) & \gamma_{l+1}^{n-l-k+1}(m) & \cdots & \gamma_{l+k-1}^{n-l-k+1}(m) \\ \gamma_{l-1}^{n-l-k+2}(m+1) & \gamma_l^{n-l-k+2}(m+1) & \cdots & \gamma_{l+k-2}^{n-l-k+2}(m+1) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{l-k+1}^{n-l}(m+k-1) & \gamma_{l-k+2}^{n-l}(m+k-1) & \cdots & \gamma_l^{n-l}(m+k-1) \end{vmatrix}$$

where $\gamma_i^j(m) = 0$ for $i < 0$ or $i > n$. Also, $\gamma_n^j(m) = 1$, and $\gamma_i^j(m) = \alpha_i(m)$ for $j \le 0$, and $\gamma_i^j(m) = \alpha_i(m)$ for $0 \le i \le n-j-1$ where $j < n-l$. These are all elements of $F_k^l(m)$, except possibly the last row and last column are an $\alpha_i(m)$.

Note that for $l = n-1$, we have $E_k^l(m) = F_k^l(m)$.

2. Necessary conditions for right ρ_l-disfocality. In Theorem 2.1 we will give necessary conditions for right ρ_l-disfocality involving the signs of $E_k^l(m)$ and $F_k^l(m)$. But first we will prove several lemmas, which will be needed in the proof of Theorem 2.1.

Lemma 2.1. *Fix $l \in \{1, \dots, n-1\}$. Assume equation* (1.1) *is right ρ_l-disfocal on I^n. For $k \in \{1, \dots, \operatorname{card} I\}$ let $s \in I^{1-k}$. If u is a solution of* (1.1) *satisfying*

$$u(s+i) = 0, \qquad \text{for } 0 \le i \le l-1$$
$$u(s+l+k-1) = 1$$

and if

$$l < n-1, \qquad u(s+l+k+i) = 0, \qquad \text{for } 0 \le i \le n-l-2,$$

then

$$(-1)^{n-l-1}\Delta^{n-1}u(s+l+k-1) > 0.$$

Proof. The result is clear for $l = n-1$, thus assume $l \le n-2$. Note that $\Delta^{n-l-1}u(s+l+k-1) = (-1)^{n-l-1}$. We will show that $\Delta^i u(s+l+k-1)\Delta^{i+1}u(s+l+k-1) > 0$ for $n-l-1 \le i \le n-2$. Our result will then easily follow.

By $n-2$ applications of Rolle's theorem, there are points $s \le s_0 \le \dots \le s_{n-l-2} \le s+k$ such that $\Delta^i u(s) = 0$ for $0 \le i \le l-1$ and $\Delta^{l+i}u(m)$ has a generalized zero at s_i for $0 \le i \le n-l-2$. Let $i \in \{n-l-1, \dots, n-2\}$ and let $t = \max\{m : \Delta^i u(m)$ has a generalized zero at $m \le s+l+k-1\}$. We may assume $\Delta^i u(t) \ge 0$. We consider two cases $\Delta^i u(t) = 0$ and $\Delta^i u(t) > 0$.

For $\Delta^i u(t) = 0$, we may assume $\Delta^i u(t+1) > 0$. But then $\Delta^{i+1}u(t) > 0$. And since (1.1) is right ρ_l-disfocal on I^n, we must have $\Delta^i u(m) > 0$ for $m \in [t+1, s+l+k-1]$ and $\Delta^{i+1}u(m) > 0$ for $m \in [t, s+l+k-1]$, for otherwise we would have an extra generalized zero.

For $\Delta^i u(t) > 0$, by the way t was chosen we have that $\Delta^i u(t-1) < 0$. But then $\Delta^{i+1}u(t-1) > 0$. And since (1.1) is right ρ_l-disfocal on I^n, we must have $\Delta^i u(m) > 0$ and $\Delta^{i+1}u(m) > 0$ for $m \in [t, s+l+k-1]$. □

Lemma 2.2. *Assume $I = [a,b]$. If equation* (1.1) *is right ρ_l-disfocal on I^n for a fixed $l \in \{1, \dots, n-1\}$, then $E_k^l(m)$ is nonzero for $m \in I^{1-k}$ where $k = 1, \dots, \operatorname{card} I$.*

Proof. We will show the contrapositive. Assume there is an $s \in I^{1-k}$ such that $E_k^l(s) = 0$. We will show that equation (1.1) is not right ρ_l-disfocal on I^n. Since $E_k^l(s) = 0$, there are constants $A_1, \dots, A_k$ not all zero such that

$$
\begin{array}{ccccccc}
\alpha_l(s)A_1 & + & \cdots & + & \alpha_{l+k-1}(s)A_k & = & 0 \\
\alpha_{l-1}(s+1)A_1 & + & \cdots & + & \alpha_{l+k-2}(s+1)A_k & = & 0 \\
\vdots & & & & \vdots & & \\
\alpha_{l-k+2}(s+k-2)A_1 & + & \cdots & + & \alpha_{l+1}(s+k-2)A_k & = & 0 \\
\beta_{l-k+1}(s+k-1)A_1 & + & \cdots & + & \beta_l(s+k-1)A_k & = & 0.
\end{array} \tag{2.1}
$$

We will consider two cases $k \leq n-l$ and $k > n-l$.

First assume $k \leq n-l$. Let u be the solution of equation (1.1) satisfying the initial conditions

$$
\begin{aligned}
u(s+i) &= 0, && \text{for } 0 \leq i \leq l-1 \\
u(s+l+i) &= A_{i+1}, && \text{for } 0 \leq i \leq k-1
\end{aligned}
$$

and if $k < n-l$,

$$u(s+l+k+i) = 0, \qquad \text{for } 0 \leq i \leq n-l-k-1.$$

Note u is a nontrivial solution of (1.1) since at least one of $A_1, \dots, A_k$ is nonzero. Also note that $\Delta^i u(s) = 0$ for $0 \leq i \leq l-1$. Now consider equation (1.1) for $m = s+i$ where $0 \leq i \leq k-2$. For $m = s$,

$$\alpha_l(s)A_1 + \cdots + \alpha_{l+k-1}(s)A_k + \alpha_n(s)u(s+n) = 0.$$

By the first equation in (2.1), $u(s+n) = 0$. Thus, recursively for $1 \leq i \leq k-2$,

$$\alpha_{l-i}(s+i)A_1 + \cdots + \alpha_{l+k-1-i}(s+i)A_k + \alpha_n(s+i)u(s+n+i) = 0.$$

Then by (2.1), we have

$$u(s+n+i) = 0 \qquad \text{for } 1 \leq i \leq k-2.$$

Thus, u has $n-l-1$ consecutive zeros beginning at $s+l+k$. And since $u(s) = \cdots = u(s+l-1) = 0$, we have by $n-2$ applications of Rolle's theorem that $\Delta^i u(s) = 0$ for $0 \leq i \leq l-1$ and there are points $s \leq s_0 \leq \cdots \leq s_{n-l-2} \leq s+k$ such that $\Delta^{l+i}u(m)$ has a generalized zero at s_i for $0 \leq i \leq n-l-2$.

Now for $m = s + k - 1$ in equation (1.5),

$$\beta_{l-k+1}(s+k-1)A_1 + \cdots + \beta_l(s+k-1)A_k + \Delta^{n-1}u(s+k) = 0.$$

By the last equation in (2.1), $\Delta^{n-1}u(s+k) = 0$. Set $s_{n-l-1} = s+k$. Thus, we have $\Delta^i u(s) = 0$ for $0 \le i \le l-1$ and $\Delta^i u(m)$ has a generalized zero at s_i for $0 \le i \le n-l-1$. Hence, equation (1.1) is not right ρ_l-disfocal on I^n. This completes the proof for the case when $k \le n-1$.

Next consider the case where $k > n-l$. Let u be the solution of equation (1.1) satisfying the intial conditions

$$\begin{aligned} u(s+i) &= 0, && \text{for } 0 \le i \le l-1 \\ u(s+l+i) &= A_{i+1}, && \text{for } 0 \le i \le n-1. \end{aligned}$$

Note that since $k > n-l$, all the terms $u(s+l+i)$, for $0 \le i \le n-l-1$, are defined. Since $k+l > n$ and $\alpha_i(m) = 0$ for $i > n$, the system of equations (2.1) becomes

(2.2)
$$\begin{array}{cccc} \alpha_l(s)A_1 & +\cdots+ & \alpha_n(s)A_{n-l+1} & =0 \\ \alpha_{l-1}(s+1)A_1 & +\cdots+ & \alpha_n(s+1)A_{n-l+2} & =0 \\ \vdots & & \vdots & \\ \alpha_{n-k+1}(s+l-n+k-1)A_1 & +\cdots+ & \alpha_n(s+l-n+k-1)A_k & =0 \\ \alpha_{n-k}(s+l-n+k)A_1 & +\cdots+ & \alpha_{n-1}(s+l-n+k)A_k & =0 \\ \vdots & & \vdots & \\ \alpha_{l-k+2}(s+k-2)A_1 & +\cdots+ & \alpha_{l+1}(s+k-2)A_k & =0 \\ \beta_{l-k+1}(s+k-1)A_1 & +\cdots+ & \beta_l(s+k-1)A_k & =0. \end{array}$$

Consider equation (1.1) for $m = s+i$ where $0 \le i \le l-n+k-1$. Using the initial conditions, we get at $m = s$,

$$\alpha_l(s)A_1 + \cdots + \alpha_{n-1}(s)A_{n-l} + \alpha_n(s)u(s+n) = 0.$$

Solving for $u(s+n)$ and using the first equation in (2.2), we get

$$\begin{aligned} u(s+n) &= -\alpha_l(s)A_1 - \cdots - \alpha_{n-1}(s)A_{n-l} \\ &= A_{n-l+1}. \end{aligned}$$

Thus, recursively for $1 \le i \le l-n+k-1$ at $m = s+i$,

$$\alpha_{l-i}(s+i)A_1 + \cdots + \alpha_{n-1}(s+i)A_{n-l+i} + \alpha_n(s+i)u(s+n+i) = 0.$$

Solving for $u(s+n+i)$ and using (2.2), we get

$$\begin{aligned} u(s+n+i) &= -\alpha_{l-i}(s+i)A_1 - \cdots - \alpha_{n-1}(s+i)A_{n-l+i} \\ &= A_{n-l+i+1}. \end{aligned}$$

Hence, we have

$$u(s+n) = A_{n-l+1}, \ldots , u(s+l+k-1) = A_k.$$

And since $A_1, \ldots , A_k$ are not all zero, we now know that u is a nontrivial solution of equation (1.1).

Now consider equation (1.1) for $m = s+i$ where $l-n+k \le i \le k-2$. For $m = s+l-n+k$,

$$\begin{aligned} \alpha_{n-k}(s+l-n+k)A_1 + \cdots + \alpha_{n-1}(s+l-n+k)A_k& \\ + \alpha_n(s+l-n+k)u(s+l+k) &= 0. \end{aligned}$$

By (2.2), we have $u(s+l+k) = 0$. Thus, recursively for $l-n+k < i \le k-2$,

$$\alpha_{l-i}(s+i)A_1 + \cdots + \alpha_{l-i+k-1}(s+i)A_k + u(s+n+i) = 0$$

and by (2.2), $u(s+n+i) = 0$. That is,

$$u(s+l+k) = 0, \ldots , u(s+n+k-2) = 0.$$

And since $u(s) = \cdots = u(s+l-1) = 0$, we have by $n-2$ applications of Rolle's theorem that $\Delta^i u(s) = 0$ for $0 \le i \le l-1$ and there are points $s \le s_0 \le \cdots \le s_{n-l-2} \le s+k$ such that $\Delta^{l+i}u(m)$ has a generalized zero at s_i for $0 \le i \le n-l-2$.

Now for $m = s+k-1$ in equation (1.5),

$$\beta_{l-k+1}(s+k-1)A_1 + \cdots + \beta_l(s+k-1)A_k + \Delta^{n-1}u(s+k) = 0.$$

By the last equation in (2.2),

$$\Delta^{n-1}u(s+k) = 0.$$

Set $s_{n-l-1} = s + k$. Thus, we have $\Delta^i u(s) = 0$ for $0 \le i \le l-1$ and $\Delta^i u(m)$ has a generalized zero at s_i for $0 \le i \le n-l-1$. Hence, equation (1.1) is not right ρ_l-disfocal on I^n. $\square$

By arguing along the lines of the proof of Lemma 2, we can show a similar result for $F_k^l(m)$. That is, if equation (1.1) is right ρ_l-disfocal on I^n for a fixed $l \in \{1, \dots, n-1\}$, then $F_k^l(m)$ is nonzero for $m \in I^{1-k}$ where $k = 1, \dots, \operatorname{card} I$.

Theorem 2.1. *Fix $l \in \{1, \dots, n-1\}$. If equation (1.1) is right ρ_l-disfocal on I^n, then $(-1)^{k(n+l)} E_k^l(m) > 0$ and $(-1)^{k(n+l)} F_k^l(m) > 0$ for $m \in I^{1-k}$ where $k = 1, \dots, \operatorname{card} I$.*

Proof. Assume equation (1.1) is right ρ_l-disfocal on I^n. But then we have that equation (1.1) is right $(l, n-l)$-disconjugate on I^n. Hence, by [**10**, Theorem 3], we have $(-1)^{k(n+l)} D_k^l(m) > 0$ for $m \in I^{1-k}$ where $k = 1, \dots, \operatorname{card} I$.

We will first show that $(-1)^{k(n+l)} E_k^l(m) > 0$ for $m \in I^{1-k}$ where $k = 1, \dots, \operatorname{card} I$. For $k = 1$, let $s \in I$. Let u be the solution of equation (1.1) satisfying the initial conditions

$$\begin{aligned} u(s+i) &= 0, \qquad \text{for } 0 \le i \le l-1 \\ u(s+l) &= 1 \end{aligned}$$

and if $l < n-1$,

$$u(s+l+i) = 0, \qquad \text{for } 1 \le i \le n-l-1.$$

We have by $n-2$ applications of Rolle's theorem that $\Delta^i u(s) = 0$ for $0 \le i \le l-1$ and if $n-l > 1$ there are points $s \le s_0 \le \cdots \le s_{n-l-2} \le s+1$ such that $\Delta^{l+i} u(m)$ has a generalized zero at s_i for $0 \le i \le n-l-2$. Also, we have that

$$\Delta^{n-1} u(s) = (-1)^{n-l-1} \binom{n-1}{l}.$$

Since equation (1.1) is right ρ_l-disfocal on I^n, we have $(-1)^{n-l-1} \Delta^{n-1} u(s+1) > 0$.

Now consider equation (1.5) with $m = s$,

$$\beta_0(s)u(s) + \cdots + \beta_{n-1}(s)u(s+n-1) + \Delta^{n-1}u(s+1) = 0.$$

Using the initial conditions, we get

$$\beta_l(s) = -\Delta^{n-1}u(s+1).$$

Hence,

$$\begin{aligned}(-1)^{n+l}E_1^l(s) &= (-1)^{n+l}\beta_l(s)\\ &= (-1)^{n+l+1}\Delta^{n-1}u(s+1) > 0.\end{aligned}$$

Now assume $k > 1$. Let $s \in I^{1-k}$. Let u be the solution of equation (1.1) satisfying the boundary conditions

$$\begin{aligned} u(s+i) &= 0, \qquad \text{for } 0 \le i \le l-1 \\ u(s+l+k-1) &= 1 \end{aligned} \tag{2.3}$$

and if $l < n-1$,

$$u(s+l+k+i) = 0, \qquad \text{for } 0 \le i \le n-l-2.$$

We have by $n-2$ applications of Rolle's theorem that $\Delta^i u(s) = 0$ for $0 \le i \le l-1$ and if $n-l > 1$ there are points $s \le s_0 \le \cdots \le s_{n-l-2} \le s+k$ such that $\Delta^{l+i}u(m)$ has a generalized zero at s_i for $0 \le i \le n-l-2$. By Lemma 2.1, we have $(-1)^{n-l-1}\Delta^{n-1}u(s+l+k-1) > 0$. Hence, since equation (1.1) is right ρ_l-disfocal on I^n, we have $(-1)^{n-l-1}\Delta^{n-1}u(s+k) > 0$.

Consider the system of equations obtained from (1.1) for $m = s$, $s+1, \dots, s+k-2$ and equation (1.5) for $m = s+k-1$ with the boundary conditions (2.3)

$$\begin{aligned}
&\alpha_l(s)u(s+l) + \alpha_{l+1}(s)u(s+l+1) \\
&\qquad + \cdots + \alpha_{l+k-1}(s)u(s+l+k-1) = 0 \\
&\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&\alpha_{l-k+2}(s+k-2)u(s+l) + \alpha_{l-k+3}(s+k-2)u(s+l+1) \\
&\qquad + \cdots + \alpha_{l+1}(s+k-2)u(s+l+k-1) = 0 \\
&\beta_{l-k+1}(s+k-1)u(s+l) + \beta_{l-k+2}(s+k-1)u(s+l+1) \\
&\qquad + \cdots + \beta_l(s+k-1)u(s+l+k-1) = -\Delta^{n-1}u(s+k).
\end{aligned}$$

By Lemma 2.2, each $E_i^l(m)$ is nonzero; therefore, we can use Cramer's rule to solve for $u(s+l+k-1)$,

$$u(s+l+k-1) = \frac{[-\Delta^{n-1}u(s+k)][D_{k-1}^l(s)]}{E_k^l(s)}.$$

Since $u(s+l+k-1) = 1$ and $(-1)^{n+l-1}\Delta^{n-1}u(s+k) > 0$, we get

$$\begin{aligned}(-1)^{k(n+l)}E_k^l(s) &= [(-1)^{n+l-1}\Delta^{n-1}u(s+k)][(-1)^{(k-1)(n+l)}D_{k-1}^l(s)]\\ &> 0.\end{aligned}$$

Hence, we have $(-1)^{k(n+l)}E_k^l(m) > 0$ for $m \in I^{1-k}$ where $k = 1, \dots, \operatorname{card} I$.

By arguing in a similar manner, we can show that $(-1)^{k(n+l)}F_k^l(m) > 0$ for $m \in I^{1-k}$ where $k = 1, \dots, \operatorname{card} I$. □

3. Necessary and sufficient conditions. We will now show necessary and sufficient conditions for equation (1.1) to be right ρ_l-disfocal on I^n in the special case when $l = n-1$.

Note that by the way $E_k^{n-1}(m)$ was defined, we can express it as an $(n-1)$st order nonhomogeneous linear difference equation in terms of $D_k^{n-1}(m)$ depending on k with m fixed. That is,

$$E_k^{n-1}(m) = \sum_{i=0}^{n-1} \binom{n-1}{i} D_{k-i}^{n-1}(m), \tag{3.1}$$

where $k \geq 1$, $D_0^{n-1}(m) = 1$ and $D_i^{n-1}(m) = 0$ for $i < 0$. We can solve this as an initial value problem to get the following result.

Lemma 3.1. *Assume $(-1)^k E_k^{n-1}(m) > 0$ for $m \in I^{1-k}$ where $k = 1, \dots, \operatorname{card} I$. Then $(-1)^k D_k^{n-1}(m) > 0$ for $m \in I^{1-k}$.*

Proof. We first fix $m \in I$. We will show this lemma by solving the initial value problem (3.1) by induction on k where $k = 1, \dots, b+1-m$. We will show that the solution to the initial value problem (3.1) is

$$D_k^{n-1}(m) = \sum_{i=0}^{k} (-1)^i \binom{n+i-2}{i} E_{k-i}^{n-1}(m) \qquad \text{for } k \geq 1, \tag{3.2}$$

where $E_0^{n-1}(m) = 1$ and $E_i^{n-1}(m) = 0$ for $i < 0$.

If we assume that equation (3.2) is valid, we can easily see our lemma is true, since

$$\begin{aligned}(-1)^k D_k^{n-1}(m) &= (-1)^k \sum_{i=0}^{k} (-1)^i \binom{n-i-2}{i} E_{k-i}^{n-1}(m) \\ &= (-1)^k E_k^{n-1}(m) + (-1)^{k-1} \binom{n-1}{1} E_{k-1}^{n-1}(m) \\ &\quad + \cdots + \binom{n+k-2}{k} > 0.\end{aligned}$$

We now show equation (3.2) holds. For $k = 1$, we have from equation (3.1) and our initial conditions

$$E_1^{n-1}(m) = D_1^{n-1}(m) + \binom{n-1}{1}.$$

Transposing this equation, we get

$$\begin{aligned}D_1^{n-1}(m) &= E_1^{n-1}(m) - \binom{n-1}{1} \\ &= \sum_{i=0}^{1} (-1)^i \binom{n+i-2}{i} E_{1-i}^{n-1}(m).\end{aligned}$$

Hence, equation (3.2) is valid for $k = 1$.

For $k > 1$, assume equation (3.2) is valid for all values less than k. Solving equation (3.1) for $D_k^{n-1}(m)$, we get

$$\begin{aligned}D_k^{n-1}(m) &= E_k^{n-1}(m) - \sum_{j=1}^{n-1} \binom{n-1}{j} D_{k-j}^{n-1}(m) \\ &= E_k^{n-1}(m) - \sum_{j=1}^{n-1} \binom{n-1}{j} \sum_{i=0}^{k-j} (-1)^i \binom{n+i-2}{i} E_{k-j-i}^{n-1}(m) \\ &= E_k^{n-1}(m) - \sum_{j=1}^{n-1} \sum_{i=0}^{k-j} (-1)^i \binom{n-1}{j} \binom{n+i-2}{i} E_{k-j-i}^{n-1}(m).\end{aligned}$$

With a change of index, $i \to i-j$, we have

$$\begin{aligned}
D_k^{n-1}(m) &= E_k^{n-1}(m) - \sum_{j=1}^{n-1}\sum_{i=j}^{k}(-1)^{i-j}\binom{n-1}{j} \\
&\qquad\cdot \binom{n+i-j-2}{i-j} E_{k-i}^{n-1}(m) \\
&= E_k^{n-1}(m) - \sum_{i=1}^{n-1}\sum_{j=1}^{i}(-1)^{i-j}\binom{n-1}{j} \\
&\qquad\cdot \binom{n+i-j-2}{i-j} E_{k-i}^{n-1}(m) \\
&\quad - \sum_{i=n}^{k}\sum_{j=1}^{n-1}(-1)^{i-j}\binom{n-1}{j}\binom{n+i-j-2}{i-j} E_{k-i}^{n-1}(m) \\
&= E_k^{n-1}(m) \\
&\quad + \sum_{i=1}^{n-1}(-1)^i\Bigg[\sum_{j=1}^{i}(-1)^{j-1}\binom{n-1}{j} \\
&\qquad\cdot \binom{n+i-j-2}{n-2}\Bigg] E_{k-i}^{n-1}(m) \\
&\quad + \sum_{i=n}^{k}(-1)^i\Bigg[\sum_{j=1}^{n-1}(-1)^{j-1}\binom{n-1}{j} \\
&\qquad\cdot \binom{n+i-j-2}{n-2}\Bigg] E_{k-i}^{n-1}(m).
\end{aligned}$$

We now use two binomial identities found in [**13**, p. 619] to obtain

$$\begin{aligned}
D_k^{n-1}(m) = E_k^{n-1}(m) &+ \sum_{i=1}^{n-1}(-1)^i\binom{n+i-2}{n-2} E_{k-i}^{n-1}(m) \\
&+ \sum_{i=n}^{k}(-1)^i\binom{n+i-2}{n-2} E_{k-i}^{n-1}(m).
\end{aligned}$$

We then have

$$\begin{aligned} D_k^{n-1}(m) &= E_k^{n-1}(m) + \sum_{i=1}^{k} (-1)^i \binom{n+i-2}{i} E_{k-i}^{n-1}(m) \\ &= \sum_{i=0}^{k} (-1)^i \binom{n+i-2}{i} E_{k-i}^{n-1}(m). \end{aligned}$$

Hence the lemma follows from statements made at the beginning of the proof. □

Theorem 3.1. *A necessary and sufficient condition for equation (1.1) to be right ρ_{n-1}-disfocal on I^n is that $(-1)^k E_k^{n-1}(m) > 0$ for $m \in I^{1-k}$ where $k = 1, \dots, \operatorname{card} I$.*

Proof. Necessity follows immediately from Theorem 2.1. To show that the conditions are sufficient, assume $(-1)^k E_k^{n-1}(m) > 0$ for $m \in I^n$ where $k = 1, \dots, \operatorname{card} I$. By Lemma 3.1, we also have that $(-1)^k D_k^{n-1}(m) > 0$ for $m \in I^n$ where $k = 1, \dots, \operatorname{card} I$. But then by [**9**, Theorem 2] we have that equation (1.1) is right $(n-1,1)$-disconjugate on I^n.

Fix $s \in I$, then $k \in \{1, \dots, b-s+1\}$. Let u be the solution of equation (1.1) satisfying the initial conditions

$$\begin{aligned} u(s+i) &= 0, \qquad \text{for } 0 \le i \le n-2 \\ u(s+n-1) &= 1. \end{aligned}$$

We need to show that $\Delta^{n-1}u(m) > 0$ for all $m \in [s, b+1]$. We first consider the cases where $m = s,\ s+1$. By the initial conditions $\Delta^{n-1}u(s) = 1 > 0$. And by equation (1.5), we have

$$\Delta^{n-1}u(s+1) = -\beta_{n-1}(s) = -E_1^{n-1}(s) > 0.$$

Next, we consider the cases $m = s+k$ where $k > 1$. Consider the system of equations obtained from (1.1) for $m = s,\ s+1, \dots, s+k-2$ and equation (1.5) for $m = s+k-1$.

$\alpha_{n-1}(s)u(s+n-1) + \alpha_n(s)u(s+n) = 0$

$$\alpha_{n-2}(s+1)u(s+n-1)+\alpha_{n-1}(s+1)u(s+n)+\alpha_n(s+1)u(s+n+1)=0$$

$$\vdots \qquad \qquad \vdots$$

$$\alpha_{n-k+1}(s+k-2)u(s+n-1)+\alpha_{n-k+2}(s+k-2)u(s+n)$$
$$+\cdots+\alpha_n(s+k-2)u(s+n+k-2)=0$$

$$\beta_{n-k}(s+k-1)u(s+n-1)+\beta_{n-k+1}(s+k-1)u(s+n)$$
$$+\cdots+\beta_{n-1}(s+k-1)u(s+n+k-2)=-\Delta^{n-1}u(s+k).$$

Use Cramer's Rule to solve for $u(s+n+k-2)$ to get

$$u(s+n+k-2)=\frac{-\Delta^{n-1}u(s+k)D_{k-1}^{n-1}(s)}{E_k^{n-1}(s)}.$$

Solving for $\Delta^{n-1}u(s+k)$,

$$\Delta^{n-1}u(s+k)=u(s+n+k-2)\frac{(-1)^k E_k^{n-1}(s)}{(-1)^{k-1}D_{k-1}^{n-1}(s)}.$$

Now since equation (1.1) is right $(n-1,1)$-disconjugate on I^n and, by our initial conditions, we must have that $u(s+n+k-2)>0$. Hence, we have that $\Delta^{n-1}u(s+k)>0$. Thus, equation (1.1) is right ρ_{n-1}-disfocal on I^n. □

For the remainder of the paper, we will consider equation (1.1) for the special case when $n=2$. We can then write equation (1.1) as

$$\alpha_0(m)u(m)+\alpha_1(m)u(m+1)+u(m+2)=0 \tag{3.3}$$

where $\alpha_0(m)>0$ on I^2.

We now give an example of an equation which is right ρ_l-disfocal but is not right disfocal on some interval I^n.

Example 3.1. Consider the following equation

$$-u(m)+2u(m+1)-3u(m+2)+u(m+3)=0, \qquad \text{for } m\in[0,1]. \tag{3.4}$$

We can show that equation (3.4) is right ρ_2-disfocal on [0,4], by applying Theorem 3.1 which we will prove later in this paper. But there is a nontrivial solution u of equation (3.4) where $u(0) = 0$ and $u(1) = u(2) = u(3) = 1$. That is, $u(0) = 0$, $\Delta u(1) = 0$ and $\Delta^2 u(1) = 0$; hence, equation (3.4) is neither right disfocal nor right ρ_1-disfocal on $[0, 4]$.

We will show that right ρ_1-disfocality and right disfocality are equivalent for equation(3.3). Further, we will give better necessary and sufficient conditions for equation (3.3) to be right disfocal on I^n, which are similar to results for disconjugacy in [**8**].

Theorem 3.2. *Equation* (3.3) *is right disfocal on* I^2 *if and only if equation* (3.3) *is right* ρ_1*-disfocal on* I^2.

Proof. The one direction is clear from the definition of right disfocality. Thus, we assume that equation (3.3) is right ρ_1-disfocal on I^2. Then there is no solution u of equation (3.3) and points $a \le m_1 < m_2 \le b + 1$ such that $u(m_1) = 0$ and $\Delta u(m)$ has a generalized zero at m_2. But then equation (3.3) must be disconjugate on I^2.

Suppose, by way of contradiction, that equation (3.3) is not right disfocal on I^2. Then there is a nontrivial solution u of equation (3.3) and points $a \le m_1 \le m_2 \le b + 1$ such that u has a generalized zero at m_1 and $\Delta u(m)$ has a generalized zero at m_2. Let u be such a solution of equation (3.3) and let m_2 be the least such generalized zero for $\Delta u(m)$. We may assume that $m_1 > a$ and $u(m_1) > 0$ since, otherwise, our hypothesis would be contradicted. Then we would have $\Delta u(m_2 - 1) > 0$ and $\Delta u(m_2) \le 0$.

Let v be the solution of equation (3.3) satisfying the initial conditions $v(m_1 - 1) = 0$ and $v(m_1) = 1$. Since $\Delta v(m_1 - 1) = 1 > 0$, we have by the hypothesis that $\Delta v(m) > 0$ for $m \in [m_1 - 1, b + 1]$. We may assume that $u(m) < v(m)$ on $[m_1, b + 2]$ since a constant multiple of u is also a solution of equation (3.3).

Since $\Delta v(m_2) > 0$ and $\Delta u(m_2) \le 0$, we have that $v(m_2) < v(m_2 + 1)$ and $u(m_2) \ge u(m_2 + 1)$. Hence, we have $v(m_2) - u(m_2) < v(m_2 + 1) - u(m_2 + 1)$.

Let $\lambda = \min\{\hat{\lambda} : \hat{\lambda} u(m) = v(m)$ for some $m \in [m_1, m_2+1]\}$, and let m_0 be the point in $[m_1, m_2+1]$ where this occurs. Note that $m_0 \in [m_1, m_2]$. Let $w = v - \lambda u$. Then w is a solution of equation (3.3) where $w(m_0-1) > 0$, $w(m_0) = 0$ and $w(m_0+1) > 0$. But then w is a nontrivial solution of equation (3.3) with a zero followed by a generalized zero, which contradicts that equation (3.3) is disconjugate on I^2. □

Corollary 3.1. *Necessary and sufficient conditions for equation* (3.3) *to be right disfocal on* I^2 *is that* $(-1)^k E_k^1(m) > 0$ *for* $m \in I^{1-k}$ *where* $k = 1, \dots, b-a+1$.

The proof follows immediately from Theorems 3.1 and 3.2.

Theorem 3.3. *If there is a solution* u *of equation* (3.3) *such that* $u(a) = 0$ *and* $\Delta u(m) > 0$ *for* $m \in [a, b+1]$, *then equation* (3.3) *is right disfocal on* $[a, b+2]$.

Proof. Assume there is a solution u of equation (3.3) such that $u(a) = 0$ and $\Delta u(m) > 0$ for $m \in [a+1, b+1]$. Then we have that $u(m) > 0$ for $m \in [a+1, b+2]$. Hence, equation (3.3) is disconjugate on $[a, b+2]$.

We will show this by way of contradiction. Suppose equation (3.3) is not right disfocal on $[a, b+2]$. Then, by Theorem 3.2, equation (3.3) is not right ρ_1-disfocal on $[a, b+2]$. Hence, there is a nontrivial solution v of equation (3.3) and points $a < m_1 < m_2 \leq b+1$ such that $v(m_1) = 0$ and $\Delta v(m)$ has a generalized zero at m_2.

Without loss of generality, we may assume that $\Delta v(m_2 - 1) > 0$ and $\Delta v(m_2) \leq 0$. Further, we may assume that $0 < v(m) < u(m)$ on $[m_1+1, b+2]$. Since $\Delta v(m_2) \leq 0$ and $\Delta u(m_2) > 0$, we have $v(m_2) \geq v(m_2+1)$ and $u(m_2) < u(m_2+1)$. Hence, $0 < u(m_2) - v(m_2) < u(m_2+1) - v(m_2+1)$.

Let $\lambda = \min\{\hat{\lambda} : \hat{\lambda} v(m) = u(m)$ for some $m \in [m_1+1, m_2+1]\}$ and m_0 the point in $[m_1+1, m_2+1]$ where this occurs. Let $w = u - \lambda v$. Then w is a solution of equation (3.3) where $w(m_0-1) > 0$, $w(m_0) = 0$, and $w(m_0+1) > 0$. But then w is a nontrivial solution of equation

(3.3) with a zero followed by a generalized zero, which contradicts that equation (3.3) is disconjugate on $[a, b+2]$. □

Corollary 3.2. *Necessary and sufficient conditions for equation* (3.3) *to be right disfocal on* I^2 *is* $(-1)^k E_k^1(a) > 0$ *where* $k = 1, \dots, b-a+1$.

Proof. Necessity follows immediately from Corollary 3.1. To show that the conditions are sufficient, we assume that $(-1)^k E_k^1(a) > 0$ where $k = 1, \dots, b-a+1$. By Lemma 3.1, we have that $(-1)^k D_k^1(a) > 0$ where $k = 1, \dots, b-a+1$. But then, by [**8**, Theorem 2], we have that equation (3.3) is disconjugate on I^2.

Let u be a solution of equation (3.3) satisfying the initial conditions $u(a) = 0$ and $u(a+1) = 1$. Then $\Delta u(a) = 1 > 0$. We need to show that $\Delta u(m) > 0$ for $m \in I^1$, then the proof will follow from Theorem 3.3.

Using equation (1.5) at $m = a$, we have

$$\begin{aligned} \Delta u(a+1) &= -\beta_1(a) \\ &= -E_1^1(a) > 0. \end{aligned}$$

Next, we consider $m = a + k$ where $k > 1$. Consider the system of equations (3.3) for $m = a, a+1, \dots, a+k-2$ and equation (1.5) for $m = a + k - 1$,

$$\alpha_1(a)u(a+1) + u(a+2) = 0$$

$$\alpha_0(a+1)u(a+1) + \alpha_1(a+1)u(a+2) + u(a+3) = 0$$

$$\vdots \qquad\qquad \vdots$$

$$\alpha_0(a+k-2)u(a+k-2) + \alpha_1(a+k-2)u(a+k-1) + u(a+k) = 0$$

$$\beta_0(a+k-1)u(a+k-1) + \beta_1(a+k-1)u(a+k) = -\Delta u(a+k).$$

Use Cramer's Rule to solve for $u(a+k)$ to get

$$u(a+k) = \frac{-\Delta u(a+k) D_{k-1}^1(a)}{E_k^1(a)}.$$

Solve for $\Delta u(a+k)$,

$$\Delta u(a+k) = u(a+k)\frac{(-1)^k E_k^1(a)}{(-1)^{k-1}D_{k-1}^1(a)}. \tag{3.5}$$

Since equation (3.3) is disconjugate on I^2 and, by the initial conditions, we must have that $u(a+k) > 0$. Thus, since each term on the right side of (3.5) is positive, we have $\Delta u(a+k) > 0$. □

Now we give an example of an equation which is right disfocal but is not disfocal on some interval I^n.

Example 3.2. Consider the following equation

$$2.5u(m) - 3u(m+1) + u(m+2) = 0, \qquad \text{for } m \in [0,2]. \tag{3.6}$$

We can show that equation (3.6) is right disfocal on $[0,4]$ by using Corollary 3.2, which we will prove later in this paper. But there is a nontrivial solution u of equation (3.6) satisfying $u(0) = u(1) = 1$, $u(2) = .5$ and $u(3) = -1$. That is, $\Delta u(0) = 0$ and u has a generalized zero at $m = 3$. Hence, equation (3.6) is not disfocal on $[0,4]$.

Remark . Lloyd Jackson noted that the above result could be stated as an ordering of $D_k^1(a)$. That is, a necessary and sufficient condition for equation (3.3) to be right disfocal on I^2 is $(-1)^{k+1}D_{k+1}^1(a) > (-1)^k D_k^1(a) > 0$ for $k = 1, \dots, b-a$. This is easily seen from equation (3.1) and Corollary 2.

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