ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 4, Fall 1992

ON THE EXTENSION OF DERIVATIONS TO SOME CLOSURES

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Introduction. Let A be a noetherian ring with integral closure $\overline{A}$ in its total quotient ring $k(A)$. When A is an integral domain, it is well known that any differentiation $\underline{D} = (1, D_1, \dots, D_i, \dots)$ of A extends to $\overline{A}$ [**3**, Section 2], but, generally, $\underline{D}$ doesn't extend to a ring lying between A and $\overline{A}$ (see [**4** ex. 1.1 or ex. 2.6]). Now let $A \subset B$ be noetherian integral domains. In Section 1, we consider some *closures* of A in B with respect to a given property. We prove that a differentiation $\underline{D}$ of A which extends to B also extends to the *u-closure* and to the *F-closure* of A in B; moreover, $\underline{D}$ extends to the *t-closure* of A in B whenever B is finite as an A-module. As regards (n-root)-*closure*, we show $\underline{D}$ can be extended under particular assumptions but not as a general rule (Section 1, Remark 1.9); we note that the (2,3)-*closure* has been already studied in [**5**].

The above problem can be considered from another point of view. If A is any noetherian ring, each ring between A and $\overline{A}$ can be seen as a suitable *closure* of A in $k(A)$ called Δ-*closure* and denoted with A^Δ (where Δ is a set of ideals of A, according to [**7**]). Since generally neither a differentiation of A nor an integrable derivation extends to A^Δ, we wonder when a derivation of A can be extended to A^Δ (Section 2). For any A^Δ, we give a sufficient (but not necessary) condition in order that a derivation D of A can be extended to A^Δ (Proposition 2.5), whereas, under suitable assumptions, we show that the extension of D to A^Δ can be characterized by certain properties of the conductor β of A in A^Δ. In particular, we prove the following. If A has (S_1)-property, D extends to $\overline{A}$ if and only if β is D-differential (Corollary 2.8).

Finally, in Section 3 we consider some classes of Δ-closures A^Δ and of derivations D of A satisfying the sufficient condition of Proposition 2.5, so that $D(A^\Delta) \subset A^\Delta$.

Received by the editors on August 7, 1990.

In this paper, all rings are assumed to be commutative with a unit element.

Section 1. Let A, B be noetherian rings such that $A \subset B$. The ring A is called $*$-*closed in* B if each $b \in B$ satisfying a property $(*; A)$ belongs to A. The smallest subring of B containing A that is $*$-closed is called the $*$-*closure* of A in B; we shall denote it generically with ${}^{*}{}_{B}A$.

In this section we are concerned with the following question: if $\underline{D}$ (respectively, D) is a differentiation (respectively, an integrable derivation) of A which extends to B, then does $\underline{D}$ (respectively, D) extend to ${}^{*}{}_{B}A$? In particular, $*$-*closure* will mean, respectively, *t-closure*, *u-closure*, *F-closure* and (n-root)-*closure* (as regards (2,3)-*closure*, see [**5**]).

First, we recall the basic definitions; generally, they concern integral domains, so that in this section we shall consider noetherian integral domains.

Definition 1.1. Let A, B be noetherian integral domains such that $A \subset B$. If $b \in B$, then:

1) b satisfies $(u; A)$ if: $b^2 - b \in A$, $b^3 - b^2 \in A$;

2) b satisfies $(t; A)$ if: $b^2 - rb \in A$, $b^3 - rb^2 \in A$ for some $r \in A$;

3) b satisfies $(F; A)$ if: b^2, b^3, $nb \in A$ for some positive integer n;

4) b satisfies (n-root;A) if: $b^n \in A$ for some positive integer n.

(See, e.g., [**9, 10**]). If $*$ means one of the properties 1) through 4), we denote ${}^{*}{}_{B}A$ respectively with: ${}^{u}{}_{B}A$, ${}^{t}{}_{B}A$, ${}^{F}{}_{B}A$, ${}^{n\text{-root}}{}_{B}A$.

Remark 1.2. a) Each property $(*; A)$ in Definition 1.1 is expressed by conditions like $P_i^*(b) \in A$, where, for each i, $P_i^*(b)$ belongs to $A[b]$.

b) If b satisfies one of the properties 1) through 4) of Definition 1.1, then b is integral over A.

c) If $*$ is one of the properties 1) through 4) of Definition 1.1 and b satisfies $(*; A)$, then b satisfies also $(*; C)$ for each ring C such that $A \subset C \subset B$.

Proposition 1.3. *Let A, B be as in Definition* 1.1. *For each $b \in {}^*{}_B A$, one has $b \in A[x_1, \dots, x_k] \subset B$, where x_j satisfies $(*; A[x_1, \dots, x_{j-1}])$ for $j = 1, \dots, k$.*

Proof. We call *$*$-finite extension of* A in B a ring like $A_k = A[x_1, \dots, x_k] \subset B$ where x_j satisfies $(*; A[x_1, \dots, x_{j-1}])$ for $j = 1, \dots, k$. Then put $X = \{A_k | A_k \text{ is } *\text{-finite extension of } A \text{ in } B\}$ and

$$C = \bigcup_k A_k, \text{ where } A_k \in X. \tag{1}$$

First we note the union (1) is filtered, according to Remark 1.2c). Then we prove ${}^*{}_B A = C$. It is obvious that $C \subset {}^*{}_B A$; so, let us prove the opposite inclusion. If ${}^*{}_B A \not\subset C$, then $C \neq {}^*{}_B C$ (otherwise, $C = {}^*{}_B C \supset {}^*{}_B A$, since $A \subset C \subset B$), so that we can find $b \in B \backslash C$ such that b satisfies $(*; C)$. Then, according to Remark 1.2a), there are some *polynomials* $P_i^*(b)$ in $C[b]$ such that $P_i^*(b) \in C$. Therefore, since the union (1) is filtered, and according to the definition of C, one can find a suitable $A_k \in X$ such that, for each i, $P_i^*(b)$ and its coefficients belong to A_k. The above statement implies b satisfies $(*; A_k)$, then $A_k[b]$ is a *$*$-finite extension of* A in B, i.e., $A_k[b] = A_h$ for some $A_h \in X$. So, $b \in C$, a contradiction. Then ${}^*{}_B A = C$. $\square$

In many cases, the *$*$-closures* with respect to the properties $*$ considered in Definition 1.1 are stable under passage to formal series, as we recall.

Proposition 1.4. *Let A, B be noetherian integral domains such that $A \subset B$. Then*:

1) *If A is u-closed in B, so is $A[[X]]$ in $B[[X]]$.*

2) *If A is t-closed in B and B is finite as an A-module, then $A[[X]]$ is t-closed in $B[[X]]$.*

3) *If A is F-closed in B, so is $A[[X]]$ in $B[[X]]$.*

4) *If* char$(A) = p$ *and* A *is* (p*-root*)*-closed in* B*, then* $A[[X]]$ *is* (p*-root*)*-closed in* $B[[X]]$*.*

Proof. 1) See [**6**, n. 4, Proposition 7].

2) See [**6**, n. 4., Proposition 8, (8.2)].

3) Let us suppose $A = {}^{F}{}_{B}A$, and let $f(X) = \sum a_i X^i \in B[[X]]$ be such that $f(X)$ satisfies $(F; A[[X]])$. One immediately has $a_0 \in A$; so, in order to prove $f(X) \in A[[X]]$, it is enough to show that $(f(X)-a_0)^2$, $(f(X) - a_0)^3$, $n(f(X) - a_0)$ belong to $A[[X]]$ for some positive integer n (for, then we replace $f(X)$ by $f(X) - a_0 = X\ (a_1 + a_2X + \dots)$ and iterate the procedure with respect to $a_1 + a_2X + \dots$). Now, expanding out, and according to the assumptions over A, it is enough to show that $a_0 f(X) \in A[[X]]$, i.e., $a_0 a_i \in A$ for each $i \geq 0$. It can be shown as in [**1**, Theorem (1), page 283].

4) Let us suppose that $A = {}^{p-\mathrm{root}}{}_{B}A$, and let $f(X) = \sum a_i X^i \in B[[X]]$ be such that $[f(X)]^p \in A[[X]]$. Since char$(A) = p$, one has $[f(X)]^p = \sum a_i^p X^{ip}$, so that $a_i^p \in A$ for each i; then $a_i \in A$, according to the assumptions on A. □

Now let $\underline{D} = (1, D_1, \dots , D_i, \dots)$ be a differentiation (of Hasse-Schmidt) of A, i.e., a sequence of additive endomorphisms $D_i : A \to A$ such that $D_n(ab) = \sum_{i,j}(D_i(a).D_j(b))$ (where $i+j = n$) for all $a, b \in A$ (so that D_1 is a derivation of A). A derivation D of A is called *integrable* if there exists a differentiation $\underline{D} = (1, D_1, \dots , D_i, \dots)$ of A such that $D_1 = D$. We denote with $HS(A)$ the group of the differentiations of A, and with Der(A) (respectively with IDer(A)) the A-module of all the derivations of A (respectively of the integrable derivations of A).

It is known that, if A is a ring, to each $\underline{D} = (1, D_1, \dots , D_i, \dots) \in HS(A)$ one can associate a ring homomorphism $E : A \to A[[X]]$ defined by $E(a) = \sum D_i(a)X^i$ for each $a \in A$ (see [**3**, Section 1]. Then, according to Proposition 1.4, we wonder when a $\underline{D} \in HS(A)$, such that $\underline{D}$ extends to B, can extend to ${}^{*}{}_{B}A$. We can prove:

Theorem 1.5. *Let* A, B *be as in Proposition* 1.4*, and let* $\underline{D} = (1, D_1, \dots , D_i, \dots)$ *in* $HS(A)$ *be such that* $\underline{D}$ *extends to* B*. Then*:

1) $\underline{D}$ *extends to* ${}^{u}{}_{B}A$.

2) *If* B *is finite as an* A*-module, then* $\underline{D}$ *extends to* ${}^{t}{}_{B}A$.

3) $\underline{D}$ *extends to* ${}^{F}{}_{B}A$.

4) *If* char $(A) = p$, $\underline{D}$ *extends to* ${}^{p-root}{}_{B}A$.

Proof. Let $E : A \to A[[X]]$ be the ring homomorphism associated to $\underline{D}$; let us denote with E also the ring homomorphism of B in $B[[X]]$ associated to the extension of $\underline{D}$ to B. So, for each closure ${}^{*}{}_{B}A$ considered in 1) through 4), we have to prove: $E({}^{*}{}_{B}A) \subset ({}^{*}{}_{B}A)[[X]]$. Now, according to Proposition 1.3, it is enough to show that $E(A[x_1, \dots , x_k]) \subset ({}^{*}{}_{B}A)[[X]]$ for each $A[x_1, \dots , x_k]$ contained in B such that x_j satisfies $(*; A[x_1, \dots , x_{j-1}])$ for $j = 1, \dots , k$; then we have to prove that $E(x_j) \in ({}^{*}_{B}A)[[X]]$ for each x_j as above, $1 \le j \le k$. We show it by induction on k. Since x_1 satisfies $(*; A)$, there is a finite set of *polynomials* $\{P_i^*(x_1)\}$ such that, for each i, $P_i^*(x_1) \in A[x_1]$, say $P_i^*(x_1) = \sum_h (a_{ih} x_1^h)$ (see Definition 1.1 and Remark 1.2a)). Then, for each i, in $B[[X]]$, one has $E(P_i^*(x_1)) = \sum_h E(a_{ih})[E(x_1)]^h$, so that $E(x_1)$ satisfies $(*; A[[X]])$, i.e., $E(x_1) \in *_{B[[X]]}A[[X]]$. Then $E(x_1) \in ({}^{*}{}_{B}A)[[X]]$, according to Proposition 1.4. Now let us suppose that $E(x_1), \dots , E(x_{j-1}) \in ({}^{*}{}_{B}A)[[X]]$. The same argument as before shows that $E(x_j)$ satisfies $(*; ({}^{*}{}_{B}A)[[X]])$, so $E(x_j) \in ({}^{*}_{B}A)[[X]]$, according to Proposition 1.4. □

Corollary 1.6. *Let* A *be a noetherian integral domain,* $\overline{A}$ *its integral closure,* ${}^{*}A$ *be the* $*$-closure *of* A *in* $\overline{A}$, $\underline{D} \in HS(A)$. *Then*:

a) $\underline{D}$ *extends to* ${}^{u}A$, ${}^{F}A$,

b) *if* $\overline{A}$ *is finite as an* A*-module,* $\underline{D}$ *extends to* ${}^{t}A$,

c) *if* char $(A) = p$, $\underline{D}$ *extends to* ${}^{p\text{-}root}A$.

Proof. Since $\underline{D}$ extends to $\overline{A}$ (see [**3**, Theorem 3]), it follows immediately from Theorem 1.5. □

Corollary 1.7. *The results of Theorem* 1.5 *and Corollary* 1.6 *hold by considering* $D \in I\mathrm{Der}\,(A)$ *instead of* $\underline{D} \in HS(A)$.

Corollary 1.8. *Let R be an integral domain such that* $\mathrm{char}\,(R) = 0$, *$k(R)$ be the quotient field of R, and $A = R[[X]]$, $B = (k(R))[[X]]$. If $\underline{D}$ is the differentiation of $HS(B)$ defined by $\underline{D} = (1, D = \partial/\partial X, \dots, D_i = D^{(i)}/i!, \dots)$, then*:

a) $\underline{D} \in HS(A)$

b) *$\underline{D}$ extends to ${}^{u}{}_{B}A$, ${}^{F}{}_{B}A$.*

Proof. a) We have $D_j(x^i) = 0$ if $i < j$, and, if $i \geq j$, $D_j(X^i) = \binom{i}{j} X^{i-j} \in A$. Then $D_j(A) \subset A$.

b) It follows immediately from a) and Theorem 1.5, 1, 3). □

Remark 1.9. If A, B, D are as in Theorem 1.5, generally $\underline{D}$ doesn't extend to ${}^{n\text{-root}}{}_{B}A$, as the following example shows. Let $A = \mathbf{Z}[[X]]$, $B = \mathbf{Q}[[X]]$, $\underline{D} = (1, D = \partial/\partial X, \dots, D_i = D^{(i)}/i!, \dots) \in HS(A) \cap HS(B)$ (Corollary 1.8 a)). Let $f(X) = 1 + (1/2)X - (1/8)X^2 + (1/16)X^3 - (5/128)X^4 + \cdots \in B$, considered in [**10**, Example 1]. As proved in [**10**], one has $[f(X)]^2 \in \mathbf{Z}[[X]] = A$, so that $f(X) \in {}^{2\text{-root}}{}_{B}A$. On the other hand, $D(f(X)) = (1/2) - (1/4)X + (3/16)X^2 + \dots$, so $[D(f(X))]^2 = (1/4) - (1/4)X + \dots$. Then, $D(f(X)) \notin {}^{2\text{-root}}{}_{B}A$. In fact, according to Proposition 1.3, a series $g(X) = \sum g_i X^i \in B$ belonging to ${}^{2\text{-root}}{}_{B}A$ is such that $g(X) \in A[x_1, \dots, x_k]$, where, for $j = 1, \dots, k$, $x_j = \sum_i (h_{ij} X^i) \in B = \mathbf{Q}[[X]]$ and $x_j^2 \in A[x_1, \dots, x_{j-1}]$; then, $h_{0j}^2 \in \mathbf{Z}$, i.e., $h_{oj} \in \mathbf{Z}$ (for $j = 1, \dots, k$, as one can easily see), which implies that $g_0 \in \mathbf{Z}$. So, D (then, even more so, D), doesn't extend to ${}^{2\text{-root}}{}_{B}A$.

Section 2. Let A be a noetherian ring, $k(A)$ be the total quotient ring of A and $\overline{A}$ be the integral closure of A in $k(A)$. Each ring B such that $A \subset B \subset \overline{A}$ can be seen as a suitable *closure* of A in $k(A)$, as we now recall. We refer to [**7**]. If Δ is a multiplicatively closed set of nonzero ideals of A, for each ideal I of A we denote with I_Δ the set $\bigcup\{IK : K | K \in \Delta\}$. Now, a Δ-*extension* of A

in $k(A)$ is a subring B of $k(A)$ containing A such that $aB \cap A \subset (aA)_\Delta$ for all regular nonunits a in A. We call Δ-*closure of A in $k(A)$* the largest Δ-extension of A that is contained in $k(A)$, and we denote it with A^Δ (see [**7**, 2.1, 6.1] and following remark). We recall some basic properties of A^Δ.

Proposition 2.1. *Let A be a noetherian ring, $k(A)$ the total quotient ring of A, and Δ a multiplicatively closed set of nonzero ideals of A. Then $A^\Delta = A[S] = S$, where*:

$$S = \{b/a \in k(A) | a \text{ is a regular nonunit in } A \text{ and } b \in (aA)_\Delta\}.$$

(See [**7**, Theorem (6.2)].)

Further, we have

Proposition 2.2. *Let A, $k(A)$, and Δ be as in Proposition* 2.1, *and let B be a ring such that $A \subset B \subset \overline{A}$. Then $B = A^\Delta$, where*:

$$\Delta = \{\text{finite products of the ideals } aB \cap A | a \text{ is a regular nonunit in } A\}.$$

In particular, $\overline{A} = A^\Delta$ where Δ is the set of nonzero ideals of A that are not contained in any minimal prime ideal.

(See [**7**, (8.1) and the following remarks, and (6.3)].)

So, according to Proposition 2.2, any subring of $\overline{A}$ containing A can be seen as a suitable Δ-closure of A in $k(A)$.

If $D \in I\mathrm{Der}\,(A)$, generally D doesn't extend to A^Δ (see [**4**, ex. 1.1 or ex. 2.6]). So, in this section we wonder when a derivation $D \in \mathrm{Der}\,(A)$ can be extended to A^Δ. We recall that an ideal I of A is called D-*differential* if $D(I) \subset I$. According to Proposition 2.1, each element of A^Δ looks like $s = b/a$, where:

(2) $$a \text{ is a regular nonunit in } A$$

(3) $$b \in (aA)_\Delta, \text{ i.e., } bK \subset aK \text{ for some ideal } K \in \Delta.$$

We have

Proposition 2.3. *Let* A, Δ *be as in Proposition* 2.1, $s = b/a \in A^{\Delta}$, K *be an ideal of* Δ *satisfying* (3), *and* $D \in \mathrm{Der}\,(A)$. *If* K *is* D*-differential, then* $D(s) \in A^{\Delta}$.

Proof. Since $D \in \mathrm{Der}\,(A)$, one has $D(s) = [D(b)a - bD(a)]/a^2$ in $k(A)$, where a^2 is a regular nonunit in A, since a satisfies (2). So, according to Proposition 2.1, it is enough to show that $[D(b)a - bD(a)] \in (a^2A)_{\Delta}$, i.e., there is an ideal $K' \in \Delta$ such that:

$$[D(b)a - bD(a)]K' \subset a^2K'. \tag{4}$$

Since K satisfies (3), for each $k \in K$ there exists $k' \in K$ such that $bk = ak'$, so in $k(A)$ one has $k' = (bk/a)$, and by applying D one obtains $bD(k) + kD(b) = D(a)k' + aD(k') = D(a)(bk/a) + aD(k')$ in $k(A)$. So, in A one has $abD(k) + akD(b) = D(a)bk + a^2D(k')$, i.e., $k[aD(b) - bD(a)] = -abD(k) + a^2D(k')$. Now $D(k') \in K$ and $D(k) \in K$, since K is D-differential; so $a^2D(k') \in a^2K$, and $abD(k) \in abK$, so that $abD(k) \in a^2K$ since b and K satisfy (3). Then $k[aD(b) - bD(a)] \in a^2K$ for each $k \in K$, so that

$$[aD(b) - bD(a)]K \subset a^2K. \tag{5}$$

If we put $K' = K$, we obtain (4). □

Remark 2.4. The converse of Proposition 2.3 is not true, generally. Indeed, we exhibit A, A^{Δ}, $s \in A^{\Delta}$ such that $D(s) \in A^{\Delta}$, but there is no $K \in \Delta$ such that K is D-differential and satisfies (3).

Let $A = \mathbf{Z}_p[x, y]$ where p is prime, $p \neq 2$ and $y^p = x^p(x+1)$. According to [**8**, n. 5], we have: A is an integral domain, y/x is integral over A, and $\overline{A} = A[y/x]$. Let $A^{\Delta} = \overline{A}$, $D \in \mathrm{Der}\,(A)$ be defined by $D(x) = 0$, $D(y) = 1$ (see [**8**, n. 5]). Take $s = (y^2/2x) \in \overline{A}$. Then

$$y^2K \subset (2x)K \quad \text{for an ideal } K \in \Delta, \tag{6}$$

according to (3).

Now we show that each K satisfying (6) cannot be D-differential. In fact, if there is $K \in \Delta$, K D-differential and satisfying (6), then, according to the proof of Proposition 2.3, one obtains (5) for $a = 2x$,

$b = y^2$, i.e., $[2xD(y^2) - y^2 D(2x)]K \subset (4x^2)K$, so $yK \subset xK$ (since A is a domain). This means that $y/x \in A^\Delta$ (Proposition 2.1 and (3)), and K satisfies the assumptions of Proposition 2.3 for $s = y/x$. Then, according to the result of Proposition 2.3, one has $D(y/x) \in A^\Delta$, a contradiction, since $D(y/x) = (1/x) \notin \overline{A} = A^\Delta$ (see [**8**, n.5]).

If A^Δ is finitely generated over A, it is possible to find a particular $K \in \Delta$ such that K is an ideal of A^Δ. In fact, if $A^\Delta = A[x_1, \dots, x_n]$, the above K can be obtained as follows (see [**7**, proof of Theorem (6.4)]): for $i = 1, \dots, n$, let $x_i = b_i/a_i$ and $K_i \in \Delta$ be such that $b_i K_i \subset a_i K_i$ (see (3)); then, $K = \Pi K_i$ has the requested property. This ideal K satisfies the assumptions of the following

Proposition 2.5. *Let A, Δ, A^Δ, D be as usual; moreover, let $K \in \Delta$ be an ideal of A^Δ. Then:*

if K is D–differential, then $D(A^\Delta) \subset A^\Delta$.

Proof. Let $s = (b/a) \in A^\Delta$. Since K is an ideal of A^Δ, one has $sK \subset K$, i.e., $bK \subset aK$. Then K satisfies the assumptions of Proposition 2.3; since K is D-differential, Proposition 2.3 shows $D(s) \in A^\Delta$. □

Remark 2.6. Generally the converse of Proposition 2.5 is not true, as the following examples show.

1) Let $A = k[t^5, t^6]$ where k is a field of characteristic $p = 5$, $A^\Delta = A[t^8] \subset \overline{A}$. If $K = (t^{16}, t^{24})A$, then $K = (t^{16})A^\Delta$; so $K \in \Delta$ according to Proposition 2.2. Further, K satisfies the assumption of Proposition 2.5. Let $D = t^{10}(\partial/\partial t) \in \mathrm{Der}\,(A)$. One has $D(A^\Delta) \subset A^\Delta$, since $D(t^8) = 8 \cdot t^{17} \in A$. Nevertheless, $D(K) \not\subset K$; in fact, $D(t^{16}) = 16 \cdot t^{25} \notin K$.

2) Let $A = k[t^3, t^7, t^8]$, where k is a field of characteristic zero, and $A^\Delta = A[t^4] \subset \overline{A}$. According to Proposition 2.2, it can be seen that $K = (t^8, t^{12})A$ belongs to Δ and is an ideal of A^Δ. If $D = t^6(\partial/\partial t) \in \mathrm{Der}\,(A)$, one has

$$-D(A^\Delta) \subset A^\Delta, \qquad \text{since } D(t^4) = 4 \cdot t^9 \in A,$$
$$-D(K) \not\subset K, \qquad \text{since} D(t^8) = 8 \cdot t^{13} \notin K.$$

We note that Examples 1), 2) of Remark 2.6 can be seen as particular cases of a larger class of examples we shall consider in Section 3.

Corollary 2.7. *Let β be the conductor of A in A^{Δ}, $D \in \mathrm{Der}\,(A)$. If $\beta \in \Delta$, the following conditions are equivalent*:

1) $D(A^{\Delta}) \subset A^{\Delta}$;

2) *β is D-differential.*

Proof. 1) $\Rightarrow$ 2). For each $s \in A^{\Delta}$, $x \in \beta$, one has $sx \in A$ so that $D(sx) \in A$, i.e., $xD(s) + sD(x) \in A$. Now $xD(s) \in A$ since $D(s) \in A^{\Delta}$ and $x \in \beta$; then, $sD(x) \in A$ (for each $s \in A^{\Delta}$), so $D(x) \in \beta$.

2) $\Rightarrow$ 1). Let $s = (b/a) \in A^{\Delta}$. Since β is an ideal of A^{Δ}, the result follows from Proposition 2.5. □

When A satisfies the (S_1)-property and $A^{\Delta} = \overline{A}$, under suitable assumptions one has $\beta \in \Delta$ (Corollary 2.8). We recall a ring A satisfies the (S_1)-property if and only if A has no embedded prime ideals associated with (0).

Corollary 2.8. *Let β be the conductor of A in $\overline{A}$, $D \in \mathrm{Der}\,(A)$. If $\overline{A}$ is a finitely generated A-module and if A satisfies the (S_1)-property, the following conditions are equivalent*:

1) $D(\overline{A}) \subset \overline{A}$;

2) *β is D-differential.*

Proof. Since A satisfies (S_1), each $P \in \mathrm{Ass}\,(A/\beta)$ has height ≥ 1 since it is associated to an ideal I generated by a regular element (see [**2**, Proposition 5.21] where only the assumption (S_1) is needed). Then $ht(\beta) \geq 1$, so $\beta \in \Delta$, according to Proposition 2.2. Now the result follows from Corollary 2.7.

Remark 2.9. 1) In Corollary 2.7 the assumption $\beta \in \Delta$ is needed only as regards 2) $\Rightarrow$ 1).

2) Under the assumptions of Corollary 2.8, the condition $\beta \in \Delta$ of Corollary 2.7 is satisfied. Generally, for a Δ-closure A^{Δ}, it is

not true that $\beta \in \Delta$. Consider, for example, $A = k[X^2, XY, Y^3]$, $A^\Delta = A[X+Y] \subset \overline{A}$, $D = X(\partial/\partial X) + XY^2(\partial/\partial Y)$. In [**4**, ex. 2.6], it is proved that $D(A^\Delta) \not\subset A^\Delta$ and β is D-differential; according to Corollary 2.7, we have necessarily that $\beta \notin \Delta$.

Section 3. In this section we show some classes of Δ-closures A^Δ and of ideals $K \in \Delta$ satisfying the assumptions of Proposition 2.5, such that K is D-differential with respect to some $D \in \mathrm{Der}\,(A)$. In this case, according to Proposition 2.5, one has $D(A^\Delta) \subset A^\Delta$. First, we show the following

Lemma 3.1. *Let A be a noetherian ring, $A^\Delta = A[x] \subset \overline{A}$, where x^2 and x^3 belong to A, $K = (x^2, x^3)A$. Then $k \in \Delta$ and K is an ideal of A^Δ contained in A.*

Proof. If $a = x^2$, then $aA^\Delta = K$. In fact, $(x^2, x^3) \subset (x^2)A^\Delta$ obviously; on the other hand, for each $y \in (x^2)A^\Delta$ one has $y = x^2(a_0 + a_1 x)$, where $a_0, a_1 \in A$, so that $y = a_0 x^2 + a_1 x^3 \in K$. So $K = aA^\Delta \cap A$, then we have also $k \in \Delta$, according to Proposition 2.2. $\square$

So, in Lemma 3.1, we construct an ideal $K \in \Delta$ that satisfies the assumptions of Proposition 2.5 and also generalizes the examples of Remark 2.6. Now we refer to rings of the particular type $A = k[t^{\alpha_1}, \dots, t^{\alpha_m}]$, in order to show ideals K as above and certain $D \in \mathrm{Der}\,(A)$ such that K is D-differential. So let $A = k[t^{\alpha_1}, \dots, t^{\alpha_m}]$, where k is a field of characteristic zero.

In this section we denote with S the semigroup $\langle \alpha_1, \dots, \alpha_m \rangle$ and suppose that $(\alpha_1, \dots, \alpha_m) = 1$, i.e., that there exists $s \in S$ such that $s + n \in S$ for each $n \in \mathbf{N}$; the least of these integers, s, is called the *conductor* of S. Now, according to Lemma 3.1, take $x = t^a \in \overline{A}$ such that $t^a \notin A$, but t^{2a}, $t^{3a} \in A$, and let $A^\Delta = A[x] = A[t^a]$, $K = (t^{2a}, t^{3a})$. Further, take $D = t^\alpha(\partial/\partial t)$ with $\alpha \geq 0$ ($D \in \mathrm{Der}\,(\overline{A})$). Then the conditions we are interested in are the following:

i) $D \in \mathrm{Der}\,(A)$

ii) $D(K) \subset K$,

that are respectively equivalent to

I) $\alpha_i - 1 + \alpha \in S$ for $i = 1, \dots, m$

II) $2a - 1 + \alpha \in S$ (since ii holds if and only if $D(x^2) \in K$).

Now, in order to have II, the following condition is sufficient

III) $a - 1 + \alpha = s + ka$, for some $s \in S$ and $k \geq 1$.

In fact, if III holds, one has: $2a - 1 + \alpha = s + (k+1)a$, with $s \in S$ and $k + 1 \geq 2$, so that $2a - 1 + \alpha \in S$. Nevertheless, II doesn't imply III. Take, for example, $\alpha_1 = 3$, $\alpha_2 = 7$, $\alpha_3 = 8$, $a = 8$, $\alpha = 6$; conditions I and II are satisfied, but $a - 1 + \alpha = 13$ and $13 \neq 8k + s$ for each $s \in S$, $k \geq 1$.

Now we show that conditions I and III are satisfied for a particular choice of the integers $\alpha_1, \dots, \alpha_m$.

Proposition 3.2. *Let $A = k[t^2, t^{2n+1}]$ with k field of characteristic zero, $n \geq 1$, $t^a \in \overline{A}\backslash A$ be such that t^{2a}, $t^{3a} \in A$. Moreover, let $A^\Delta = A[t^a]$, $K = (t^{2a}, t^{3a})$. If $D = t^\alpha(\partial/\partial t) \in \mathrm{Der}\,(A)$, then $D(K) \subset K$.*

Proof. Let $D = t^\alpha(\partial/\partial t) \in \mathrm{Der}\,(A)$. According to the above notations and remarks, it is enough to show that III holds. In this case, the assumption over D means:

$$\text{(I)} \qquad 1 + \alpha \in S, \qquad 2n + \alpha \in S.$$

Moreover, $S = \{0, 2, \dots, 2n+1, \dots\}$ and, since $t^a \notin A$, one has that a is odd and $a < 2n + 1$.

If α is even, then I is satisfied if and only if $\alpha \geq 2n$. Then, $a - 1 + \alpha \geq a - 1 + 2n > a - 1 + a - 1$ (since $a < 2n + 1) = 2(a-1)$. So, $a - 1 + \alpha$ is even and greater than $2(a-1)$, so that $a - 1 + \alpha = 2(a-1) + 2h$ for some $h \geq 1$, i.e., $a - 1 + \alpha = 2a + 2(h-1)$ where $h - 1 \geq 0$. Then III holds since $2(h-1) \in S$.

If α is odd, then I is trivially satisfied for each $\alpha \geq 1$. Further, $a - 1 + \alpha = a + (-1 + \alpha)$, where $-1 + \alpha$ belongs to S since it is even; then III holds.

Remark 3.3. Let $A = k[t^{\alpha_1}, \dots, t^{\alpha_m}]$, A^{Δ}, K, be as before. If c is the conductor of $S = \langle \alpha_1, \dots, \alpha_m \rangle$, then each $D = t^{\alpha}(\partial/\partial t)$ with $\alpha \geq c+1$ is such that $D \in \mathrm{Der}(A)$ and $D(K) \subset K$. In fact, if $\alpha \geq c+1$, one has for $i = 1, \dots, m$, $\alpha_i - 1 + \alpha \geq \alpha_i + c \in S$; $a - 1 + \alpha \geq a + c$, so that $a - 1 + \alpha = a + s$, for some $s \in S$. Then I and III hold so that the result follows. □

REFERENCES

1. J.W. Brewer, *Seminormality in power series rings*, J. Algebra **82** (1983), 282–284.

2. S. Greco, *Normal varieties*, Inst. Math. **IV**, Academic Press (1978).

3. H. Matsumura, *Integrable derivations*, Nagoya Math. J. **87** (1982), 227–245.

4. S. Molinelli, *On differentiations and birational integral extensions*, Comm. Algebra **16** (8) (1988), 1663–1671.

5. S. Molinelli and G. Tamone, *On some invariant ideals and on extension of differentiations to seminormalization*, J. Pure Appl. Algebra, to appear.

6. N. Onoda, T. Sugatani and K. Yoshida, *Local quasinormality and closedness type criteria*, Houston J. Math. **11** (1985), 247–256.

7. L.J. Ratliff, Jr., Δ*-closures on ideals and rings*, Trans. Amer. Math. Soc. **313** (1989), 221–247.

8. A. Seidenberg, *Derivations and integral closure*, Pacific J. Math. **16** (1966), 167–173.

9. T. Sugatani and K. Yoshida, *On t-closures*, C.R. Math. Rep. Acad. Sci. Canada **VI** (1984), 55–57.

10. J.J. Watkins, *Root and integral closure for* $R[[X]]$, J. Algebra **75** (1982), 43–58.

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