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A DIRECT WEAKENING OF NORMALITY FOR FILTERS

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ABSTRACT. Weakly normal filters have been defined and studied for filters on cardinals κ by Kanamori and for filters on $P_\kappa\lambda$ by Abe. Both versions of weak normality require functions regressive on some set of measure one to be bounded on a set of measure one; which is not a direct weakening of Solovay's notion of normality, where functions regressive on some set of positive measure are constant on a set of positive measure. Consequently, Abe's definition of weak normality is not a property possessed by the closed unbounded filter over $P_\kappa\lambda$.

Here, a weak version of normality, called quasi-normal, is presented which is a direct weakening of normality for filters. Functions regressive on some set of positive measure must be bounded on a set of positive measure. The final section filter and the strongly closed unbounded filter on $P_\kappa\lambda$ are studied for quasi-normality. Whether or not these filters are quasi-normal depends on the cofinality of λ with respect to κ.

Introduction. Fodor's theorem states that if $f : \kappa \to \kappa$ is regressive on a stationary set B, with $B \subseteq \kappa$, then there exists a stationary set B' with $B' \subseteq B$, such that f is constant on B', see [**3**]. The stationary sets are the sets of positive measure with respect to the closed unbounded filter over κ. In [**9**], Solovay introduced the notion of a normal filter by generalizing the property described in Fodor's theorem. A filter F on κ is normal if whenever $f : \kappa \to \kappa$ is regressive on a set B of positive measure with respect to F, then there is a set of positive measure B', with $B' \subseteq B$ such that f is constant on B'.

A generalization of this property was provided by Jech in [**4**], where the notions of filters, closed unbounded sets, stationary sets, and regressive functions were extended to $P_\kappa\lambda$.

The property of normality was weakened by Kanamori, with a twist, see [**6**]. Instead of requiring regressive functions to be constant somewhere, now they are required only to be bounded somewhere. The

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twist is that the somewhere is a measure one set. The property of weak normality was generalized by Abe in [**1**], to filters over $P_\kappa\lambda$.

This paper studies the direct weakening of normality to filters on $P_\kappa\lambda$. It is organized into three sections. Section one introduces the property of quasi-normality for filters over $P_\kappa\lambda$ and compares results with some in [**1**]. At this point weakly normal filters and quasi-normal filters on $P_\kappa\lambda$ begin to diverge. Whether or not the final section filter and the strongly closed unbounded filter can be quasi-normal will depend on the cofinality of λ with respect to κ. Section two investigates the existence of filters and ultrafilters which are not quasi-normal. (Note: quasi-normality and weak normality are equivalent for ultrafilters.) Finally, Section three shows that the *minimal cover* $q*(\mu)$ of [**8**] is quasi-normal and discusses the *large cardinal* strength of quasi-normality.

For basic notation and background, please see [**5**]. For any filter F on S, denote

$$F^+ = \{A : A \subseteq S \text{ and } A \cap B \neq \varnothing \text{ for all } B \in F\},$$

called the sets of positive measure with respect to F, or just the sets of positive measure when no confusion will result. For every $x \in P_\kappa\lambda$, let $\hat{x} = \{y \in P_\kappa\lambda : x \subseteq y\}$. The final segment filter, denoted $FSF_{\kappa\lambda}$, is defined

$$FSF_{\kappa\lambda} = \{A : A \subseteq P_\kappa\lambda \text{ and } \exists\, x \in P_\kappa\lambda \text{ such that } \hat{x} \subseteq A\}.$$

For the rest of this paper, any filter F on $P_\kappa\lambda$ will be an extension of $FSF_{\kappa\lambda}$.

A subset C of $P_\kappa\lambda$, is unbounded, if for every $x \in P_\kappa\lambda$, there exists a y in C, such that $x \subseteq y$. And C is closed, if whenever $\eta < \kappa$ and $\{x_\gamma : \gamma < \eta\} \subset C$ such that $x_\gamma \subset x_{\gamma'}$ for $\gamma < \gamma'$, then $\cup_{\gamma<\eta} x_\gamma \in C$. The closed unbounded filter on $P_\kappa\lambda$, denoted $CF_{\kappa\lambda}$, is defined:

$$CF_{\kappa\lambda} = \{B : B \subseteq P_\kappa\lambda \text{ and } \exists\, C \text{ which is closed unbounded and } C \subseteq B\}.$$

A subset S of $P_\kappa\lambda$ is stationary, if $S \in (CF_{\kappa\lambda})^+$.

Finally, let D be an unbounded subset of $P_\kappa\lambda$. D is strongly closed, if whenever $\eta < \kappa$ and $\{x_\gamma : \gamma < \eta\}$ is a subset of D, then $\cup_{\gamma<\eta} x_\gamma \in D$.

The strongly closed unbounded filter on $P_\kappa\lambda$, denoted $SCF_{\kappa\lambda}$, is defined

$$SCF_{\kappa\lambda} = \{A : A \subseteq P_\kappa\lambda \text{ and } \exists\, D \text{ strongly closed unbounded and } D \subseteq A\}.$$

Section 1. It is possible to define a notion of quasi-normality for filters over an uncountable cardinal κ and get comparable results to Proposition 1.2(i), (iii) and (iv) in [**6**]. Both Kanamori's version of weak normality and quasi-normality coincide with normality for κ-complete filters. For this reason, the choice was made to introduce quasi-normality for κ-complete filters over $P_\kappa\lambda$. The interested reader is directed to [**6,7**], where the regularity of measures on κ is studied in the context of weak normality.

Definition 1.1. A filter F on $P_\kappa\lambda$ is *quasi-normal*, if whenever $f : P_\kappa\lambda \to \lambda$ is regressive on a set $B \in F^+$, then there exists a subset B' of B such that $B' \in F^+$ and f is bounded on B'.

As in the case of normality, the following characterization in terms of closure under diagonal intersection exists.

Theorem 1.2. *Let F be a filter on $P_\kappa\lambda$. F is quasi-normal if and only if whenever $\{A_\gamma : \gamma < \lambda\}$ is a subset of F such that $A_\beta \subseteq A_\alpha$ when $\alpha < \beta$, then*

$$\Delta\{A_\gamma : \gamma < \lambda\} = \{x \in P_\kappa\lambda : \gamma \in x \Rightarrow x \in A_\gamma\} \in F.$$

Proof. Similar proofs are standard throughout the literature, see [**4**].

Assuming F is quasi-normal, suppose $\{A_\gamma : \gamma < \lambda\}$ is a subset of F where $A_\beta \subseteq A_\alpha$ when $\alpha < \beta$. Suppose for all $C \in F$ there exists an $x \in C$ such that there exists $\delta \in x$ and $x \notin A_\delta$. Define $f : P_\kappa\lambda \to \lambda$ such that $f(x) = \delta$ for the first such δ in x. Then $\{x \in P_\kappa\lambda : f(x) \in x\} \in F^+$. By the quasi-normality of F, there exists a $B \subset \{x \in P_\kappa\lambda : f(x) \in x\}$ and a $\gamma < \lambda$ such that $B \in F^+$ and $f(x) \leq \gamma$ for all $x \in B$. Let $x \in B \cap A_{\gamma+1}$. Then $x \in A_{\gamma+1}$ and

$x \notin A_{f(x)}$ since $x \in B$. But $A_{\gamma+1} \subseteq A_{f(x)}$, since $f(x) < \gamma + 1$. A contradiction.

Next, assuming F is closed under the diagonal intersection of nested λ-sequences of measure one sets, suppose $\{x \in P_\kappa\lambda : f(x) \in x\} \in F^+$. But for all $\beta < \lambda$, $\{x \in P_\kappa\lambda : f(x) \leq \beta\} \notin F^+$. Then for all $\beta < \lambda$, $\{x \in P_\kappa\lambda : f(x) > \beta\} \in F$. This gives a nested λ-sequence of measure one sets and eventually a contradiction. The details are straightforward. □

In [**2**], Carr proves that for all $\lambda > \kappa$, $FSF_{\kappa\lambda} \subset SCF_{\kappa\lambda} \subset CF_{\kappa\lambda}$ and $CF_{\kappa\lambda} \neq SCF_{\kappa\lambda} \neq FSF_{\kappa\lambda}$; and if F is a normal filter on $P_\kappa\lambda$, then $CF_{\kappa\lambda} \subseteq F$. Hence, $SCF_{\kappa\lambda}$ and $FSF_{\kappa\lambda}$ are not normal. It is also true that $FSF_{\kappa\lambda}$, $SCF_{\kappa\lambda}$ and $CF_{\kappa\lambda}$ are not weakly normal. However, clearly $CF_{\kappa\lambda}$ is quasi-normal, and as this section will demonstrate, the quasi-normality of $FSF_{\kappa\lambda}$ and $SCF_{\kappa\lambda}$ depends on the cofinality of λ with respect to κ.

Theorem 1.3. *If $cf(\lambda) < \kappa$, then every κ-complete filter over $P_\kappa\lambda$ is quasi-normal.* (*In particular, $FSF_{\kappa\lambda}$ and $SCF_{\kappa\lambda}$ are quasi-normal.*)

Proof. Note: The author is grateful to the referee for pointing out a simple argument which greatly strengthened the original theorem.

Let F be a κ-complete filter over $P_\kappa\lambda$ and suppose f is a function mapping $P_\kappa\lambda$ into λ such that $\{x \in P_\kappa\lambda : f(x) \in x\} \in F^+$. Let $\delta = cf(\lambda) < \kappa$ and $\{\gamma_\alpha : \alpha < \delta\}$ be cofinal in λ. Suppose for each $\alpha < \delta$, $\{x \in P_\kappa\lambda : f(x) \leq \gamma_\alpha\} \notin F^+$. Then for each $\alpha < \delta$ there exists $A_{\gamma_\alpha} \subset \{x \in P_\kappa\lambda : f(x) > \gamma_\alpha\}$ such that $A_{\gamma_\alpha} \in F$. By the κ-completeness of F, $\cap\{A_{\gamma_\alpha} : \alpha < \delta\} \in F$. But clearly,

$$\{x \in P_\kappa\lambda : f(x) \in x\} \cap \{\cap\{A_{\gamma_\alpha} : \alpha < \delta\}\} = \varnothing. \qquad \square$$

Definition 1.4. Let $w : \lambda \to P_\kappa\lambda$. Then denote

$$C(\{w\}) = \{x : x \subseteq P_\kappa\lambda \text{ and } \alpha \in x \Rightarrow w(\alpha) \subseteq x\},$$

the collection of all sets in $P_\kappa\lambda$ closed under w.

The following proposition is distilled from results in [**8**].

Proposition 1.5 (Menas). *Let A be a subset of $P_\kappa\lambda$. Then,*

$$A \in SCF_{\kappa\lambda} \iff \text{ there exists } w : \lambda \to P_\kappa\lambda \text{ and } C(\{w\}) \subseteq A.$$

Proof. If $A \in SCF_{\kappa\lambda}$ and C is strongly closed unbounded and a subset of A, define $w : \lambda \to P_\kappa\lambda$ by letting $w(\alpha)$ be any member x in C such that $\alpha \in x$. □

Next, given $w : \lambda \to P_\kappa\lambda$ define $w' : \lambda \to P_\kappa\lambda$ as follows: given

$$\begin{gathered} \eta < \lambda, \text{ let} \\ y_0(\eta) = w(\eta) \cup \{\eta\}; \\ y_{n+1}(\eta) = \cup\{w(\delta) : \delta \in y_n(\eta)\}; \text{ then} \\ w'(\eta) = \cup\{y_n(\eta) : n \in \omega\}. \end{gathered}$$

Set $\beta_w = \{w'(\eta) : \eta < \lambda\}$. These definitions yield the following.

Proposition 1.6. *Let $w : \lambda \to P_\kappa\lambda$ and $v : \lambda \to P_\kappa\lambda$. Then,*

i) $C(\{w\}) = C(\{w'\})$;

ii) $w'(\eta) \in C(\{w'\})$ *and* $\eta \in w'(\eta)$ *for all* $\eta < \lambda$;

iii) $w'(\eta) = \cup\{w'(\alpha) : \alpha \in w'(\eta)\}$;

iv) $\alpha \in w'(\eta) \cap w'(\gamma)$ *implies* $w'(\alpha) \subset w'(\eta) \cap w'(\gamma)$;

v) $C(\{v\}) \subseteq C(\{w\})$ *if and only if* $w'(\eta) \subseteq v'(\eta)$, *for all* $\eta < \lambda$;

vi) $C(\{w\}) = \{x \in P_\kappa\lambda : x = \cup D$ *where* $D \subset \beta_w\}$;

vii) $\beta_w = \beta_v$ *implies* $C(\{w\}) = C(\{v\})$.

Theorem 1.7. *If $\kappa \leq cf(\lambda)$, then $FSF_{\kappa\lambda}$ is not quasi-normal.*

Proof. Since $\kappa \leq cf(\lambda)$, $f : P_\kappa\lambda \to \lambda$ can be defined by $f(x) = \sup x$. For every $A \in FSF_{\kappa\lambda}$, there exists a $\hat{x} \subset A$ for some $x \in P_\kappa\lambda$. This gives $x \cup \{\sup x\} \in \hat{x}$, $x \cup \{\sup x\} \in A$ and $f(x \cup \{\sup x\}) \in x \cup \{\sup x\}$, which means that

$$\{x \in P_\kappa\lambda : f(x) \in x\} \cap A \neq \varnothing \text{ for all } A \in FSF_{\kappa\lambda}.$$

Hence, $\{x \in P_\kappa\lambda : f(x) \in x\} \in FSF^+_{\kappa\lambda}$. Suppose there exists an $\alpha < \lambda$ such that $\{x \in P_\kappa\lambda : f(x) \leq \alpha\} \in FSF^+_{\kappa\lambda}$. Consider $\{\widehat{\alpha + 1}\} \in FSF_{\kappa\lambda}$. Let $y \in \{\widehat{\alpha+1}\} \cap \{x \in P_\kappa\lambda : f(x) \leq \alpha\}$. Then $f(y) = \sup y \leq \alpha$. But $\alpha + 1 \in y$. Hence, $\sup y > \alpha$. This contradiction proves the theorem. □

Theorem 1.8. *If $cf(\lambda) > \kappa$, then $SCF_{\kappa\lambda}$ is not quasi-normal.*

Proof. Let $\{\lambda_\alpha : \alpha < \lambda\}$ be such that $\lambda_\alpha \subset \lambda$, $|\lambda_\alpha| = \lambda$ and $\lambda_\alpha \cap \lambda_\beta = \varnothing$ when $\alpha \neq \beta$. Consider λ_α as $\lambda_\alpha : \lambda \to P_\kappa\lambda$ where $\lambda_\alpha(\eta)$ is the one element set whose only member is the η^{th} member of λ_α. Define $A_\alpha = \cup\{C(\{\lambda_\eta\}) : \alpha \leq \eta\}$. This gives $\{A_\alpha : \alpha < \lambda\} \subset SCF_{\kappa\lambda}$ such that $A_\beta \subset A_\alpha$ for $\alpha < \beta$. Assuming $SCF_{\kappa\lambda}$ is quasi-normal, by Propositions 1.5 and 1.6 (i), there exists $w : \lambda \to P_\kappa\lambda$ such that $C(\{w'\}) \subset \Delta A_\alpha$ (the diagonal intersection of the A_α). These definitions result in the following claim.

Claim. *For all $\alpha < \lambda$ there exists a $\gamma < \lambda$ such that for all $\beta < \lambda$, if $\beta \in \lambda - \lambda_\alpha$ and $\beta > \gamma$ then there exists an η where $\beta \subseteq \eta$ and $\lambda_\eta(\alpha) \in w'(\beta)$.*

Proof. Otherwise: There exists an $\alpha < \lambda$ such that for all $\gamma < \lambda$ there exists a $\beta < \lambda$, with $\beta \in \lambda - \lambda_\alpha$ and $\beta > \gamma$ and for all η if $\beta \leq \eta$ then $\lambda_\eta(\alpha) \notin w'(\beta)$. For such an α, let $\beta_1 > \alpha$ and $\beta_1 \in \lambda - \lambda_\alpha$ such that for all η with $\beta_1 \leq \eta$, $(\lambda_n(\alpha) \notin w'(\beta_1))$. Now $\beta_1 \in w'(\alpha) \cup w'(\beta_1) \in C(\{w'\})$. Hence, $w'(\alpha) \cup w'(\beta_1) \in A_{\beta_1}$. For some η_1 with $\beta_1 \leq \eta_1$, $w'(\alpha) \cup w'(\beta_1) \in C(\{\lambda_{\eta_1}\})$. By assumption, $\lambda_{\eta_1}(\alpha) \notin w'(\beta_1)$. But, $\lambda_{\eta_1}(\alpha) \in w'(\alpha) \cup w'(\beta_1)$ since $\alpha \in w'(\alpha) \cup w'(\beta_1)$. Hence, $\lambda_{\eta_1}(\alpha) \in w'(\alpha)$. Next, choose $\beta_2 > \eta_1$ such that $\beta_2 \in \lambda - \lambda_\alpha$ and for all η with $\beta_2 \leq \eta$ $(\lambda_\eta(\alpha) \notin w'(\beta))$. Similarly, there exists η_2 with $\beta_2 \leq \eta_2$ such that $\lambda_{\eta_2}(\alpha) \in w'(\alpha)$. Next, assume $\delta < \lambda$ and β_ξ, η_ξ have been chosen for $\xi < \delta$. By assumption, there exists $\beta_\xi > \sup\{\eta_\xi : \xi < \delta\}$ with $\beta_\delta \in \lambda - \lambda_\alpha$ such that for al η with $\beta_\delta \leq \eta$ $(\lambda_\eta(\alpha) \notin w'(\beta_\delta))$. Again, $\beta_\delta \in w'(\beta_\delta) \cup w'(\alpha)$ and $w'(\beta_\delta) \cup w'(\alpha) \in A_{\beta_\delta}$. Hence, there exists an η_δ with $\beta_\delta \leq \eta_\delta$ such that $w'(\beta_\delta) \cup w'(\alpha) \in C(\{\lambda_{\eta_\delta}\})$. Since $\alpha \in w'(\beta_\delta) \cup w'(\alpha)$, $\lambda_{\eta_\alpha}(\alpha) \in w'(\beta_\delta) \cup w'(\alpha)$. But $\lambda_{\eta_\delta}(\alpha) \notin w'(\beta_\delta)$. So $\lambda_{\eta_\delta}(\alpha) \in w'(\alpha)$. Now letting $\delta = \kappa$, since $\lambda_{\eta_\xi}(\alpha) \neq \lambda_{\eta_\zeta}(\alpha)$ for $\xi \neq \zeta$ and $\{\lambda_{\eta_\xi}(\alpha) : \xi < \kappa\} \subset w'(\alpha)$, hence $\kappa \leq |w'(\alpha)|$. But this is impossible, since $w'(\alpha) \in P_\kappa\lambda$. This proves the claim. □

Back to the proof of Theorem 1.8. Given $\alpha_1 < \lambda$, choose $\alpha_2 > \gamma_{\alpha_1}$ the γ known to exist by the claim for α_1 such that $\alpha_2 \in \lambda - \lambda_{\alpha_1}$ and $\alpha_2 > \alpha_1$. By the claim, there exists $\eta_{2,1}$ with $\alpha_2 \leq \eta_{2,1}$ such that $\lambda_{\eta_{2,1}} \in w'(\alpha_2)$. Next, let $\alpha_3 > \sup\{\alpha_1, \alpha_2, \gamma_{\alpha_1}, \gamma_{\alpha_2}\}$, where $\alpha_3 \in \lambda - (\lambda_{\alpha_1} \cup \lambda_{\alpha_2})$. The claim gives $\eta_{3,1}$ and $\eta_{3,2}$ with $\alpha_3 \leq \eta_{3,1}$ and $\alpha_3 \leq \eta_{3,2}$ such that $\lambda_{\eta_{3,1}}(\alpha_1) \in w'(\alpha_3)$ and $\lambda_{\eta_{3,2}}(\alpha_2) \in w'(\alpha_3)$. For $\delta < cf(\lambda)$, assume $\alpha_\xi, \gamma_{\alpha_\xi}$ and $\eta_{\xi,\iota}$ have been defined for $\xi < \delta$ and $\iota < \xi$. Choose $\alpha_\delta > \sup\{\alpha_\xi, \gamma_{\alpha_\xi} : \xi < \delta\}$ such that $\alpha_\delta \in \lambda - (\cup_{\xi<\delta}\lambda_{\alpha_\xi})$. By the claim, there exists, for each $\xi < \delta$, an $\eta_{\delta,\xi}$ with $\alpha_\delta \leq \eta_{\delta,\xi}$ such that $\lambda_{\eta_{\delta,\xi}}(\alpha_\xi) \in w'(\alpha_\delta)$, where $\alpha_\xi \neq \alpha_\zeta$ if $\xi \neq \zeta$. Since $cf(\lambda) > \kappa$, letting $\delta = \kappa$ gives $\{\lambda_{\eta_{\kappa,\xi}}(\alpha_\xi) : \xi < \kappa\} \subset w'(\alpha_\kappa)$. But this implies $\kappa \leq |w'(\alpha_k)|$. This contradiction proves the theorem. □

The next theorem is a modified analog to Proposition 3.2 in [**1**] and provides a p-point like characterization for quasi-normal extensions of $SCF_{\kappa\lambda}$.

Theorem 1.9. *An extension F of $SCF_{\kappa\lambda}$ is quasi-normal if and only if whenever $\{x \in P_\kappa\lambda : f(x) > \alpha\} \in F$ for every $\alpha < \lambda$, then there exists an $A \in F$ such that $A \cap f^{-1}(\{\gamma\}) \subset P_\kappa\gamma$ for every $\gamma < \lambda$.*

Proof. $\Rightarrow$ Let $x_\alpha = \{x \in P_\kappa\lambda : f(x) > \alpha\} \in F$. Use the fact that $\Delta x_\alpha \in F$ to establish that $\Delta x_\alpha \cap f^{-1}(\{\gamma\}) \subset P_\kappa\gamma$.

$\Leftarrow$ The method of proof used for Proposition 3.2(ii) in [**1**] works here as well. □

Next, a large class of filters lying between $FSF_{\kappa\lambda}$ and $SCF_{\kappa\lambda}$ will be defined. Like $FSF_{\kappa\lambda}$ and $SCF_{\kappa\lambda}$, whether or not these filters are quasi-normal depends upon the cofinality of λ.

Definition 1.10. For δ a cardinal such that $\kappa \le \delta < \lambda$, let

$$D_\delta = \{w \in {}^\lambda P_\kappa\lambda : |\cup\{w(\beta) : \beta < \lambda\}| \le \delta\};\ \text{and}$$
$$F_\delta = \{A : A \subseteq P_\kappa\lambda \text{ and } C(\{w\}) \subseteq A \text{ for some } w \in D_\delta\}.$$

F_δ is a κ-complete filter on $P_\kappa\lambda$ extending $FSF_{\kappa\lambda}$.

Theorem 1.11. *For $\kappa < \delta_1 < \delta_2 < \lambda$, the following strict inclusion holds*:

$$FSF_{\kappa\lambda} \subset F_\kappa \subset F_{\delta_1} \subset F_{\delta_2} \subset SCF_{\kappa\lambda}$$

and F_κ or F_{δ_i} can replace $SCF_{\kappa\lambda}$ in Theorem 1.8.

Proof. First consider F_κ. Define $w : \lambda \to P_\kappa\lambda$ by

$$w(\beta) = \begin{cases} \beta^{\text{th}} \text{ interval of length } \beta, & \text{if } \beta < \kappa; \\ \varnothing, & \text{if } \kappa \le \beta. \end{cases}$$

For any $x \in P_\kappa\lambda$, if $\kappa > \alpha > \sup(x \cap \kappa)^+$, then $x \cup \{\alpha\} \in x$, but $x \cup \{\alpha\} \notin C(\{w\})$. Hence, $C(\{w\}) \notin FSF_{\kappa\lambda}$, giving $FSF_{\kappa\lambda} \ne F_\kappa$.

Next, given cardinals δ_1 and δ_2 such that $\kappa < \delta_1 < \delta_2 < \lambda$, partition δ_2 into δ_2-many disjoint consecutive intervals of length δ_1, $\{H_\beta : \beta < \delta_2\}$, and partition each H_β into δ_1-many consecutive intervals of length $\kappa, \{H_{\beta,\eta} : \eta < \delta_1\}$. Now, for $\alpha < \delta_2$, let $H(\alpha)$ be the unique interval $H_{\beta,\eta}$ such that $\alpha \in H_{\beta,\eta}$.

Define $u : \lambda \to P_\kappa\lambda$ by

$$u(\alpha) = \begin{cases} \alpha \cap H(\alpha) & \text{if } \alpha < \delta_2; \\ \varnothing & \text{otherwise.} \end{cases}$$

For any $w \in D_{\delta_1}$, there are at most δ_1 many intervals H_β such that for some $\xi < \lambda$, $w(\xi) \cap H_\beta \ne \varnothing$. Let H_β be such that $H_\beta \cap \cup\{w(\alpha) : \alpha < \lambda\} = \varnothing$. Choose $\gamma \in H_{\beta,\eta}$ such that cardinality of $\gamma \cap H_{\beta,\eta}$ is greater than one. Recall the construction developed prior to Proposition 1.6:

$$y_0(\gamma) = w(\gamma) \cup \{\gamma\};$$
$$y_{n+1}(\gamma) = \cup\{w(\delta) : \delta \in y_n(\gamma)\};\ \text{and}$$
$$w'(\gamma) = \cup\{y_n(\gamma) : n < \omega\}.$$

Now $\gamma \in w'(\gamma)$ and $w'(\gamma) \in C(\{w\})$ by Proposition 1.6. But $u(\gamma) = \gamma \cap H_{\beta,\eta}$, where the cardinality of $\gamma \cap H_{\beta,\eta}$ is greater than one. However, $w'(\gamma) \cap H_{\beta,\eta} = \{\gamma\}$ so $u(\gamma) \not\subset w'(\gamma)$, hence $w'(\gamma) \notin C(\{u\})$. This shows that for any $w \in D_{\delta_1}$, $C(\{w\}) \not\subset C(\{u\})$, hence $C(\{u\}) \notin F_{\delta_1}$. Therefore, $F_{\delta_1} \neq F_{\delta_2}$. This argument can be modified to handle $\delta_1 = \kappa$.

Finally, the proof of Theorem 1.8 works for F_{δ_i} and F_κ. □

Section 2. In [**1**] an ultrafilter extending $CF_{\kappa\lambda}$ is constructed which is not weakly normal, assuming κ is strongly compact and λ is regular and greater than κ. Since, for ultrafilters, weak normality and quasi-normality are the same, this provides an example of a nonquasi-normal ultrafilter. In fact, as the next proposition will demonstrate, a variation of the construction used in [**1**] and a different proof will provide an example of a nonquasi-normal filter which is not an ultrafilter on $P_\kappa\lambda$, when λ is a strongly compact cardinal greater than κ a regular cardinal; and a specific instance of the following construction can be used to provide an example of a nonquasi-normal extension of $CF_{\kappa\lambda}$. The proof of the next proposition can be modified for weak normality, giving an example of a nonweakly normal filter which is not an ultrafilter. First, the basic construction from [**1**].

Let

$$a = \{\delta < \lambda : \delta > \kappa \text{ and } cf(\delta) < \kappa\}, \text{ and}$$
$$v = \{x : \lambda \supset x \text{ and } a - x \text{ is not stationary}\}.$$

This makes v a λ-complete filter. In fact, v can be shown to be normal; since if $\{a_\alpha : \alpha < \lambda\}$ is a subset of v, then there exists a closed unbounded subset c_α of $\lambda - (a - a_\alpha)$, so

$$\Delta\{c_\alpha : \alpha < \lambda\} \cap (a - \Delta\{a_\alpha : \alpha < \lambda\}) = \varnothing.$$

Suppose λ is strongly compact; then there exists a λ-complete ultrafilter u extending v such that $CF_\lambda \cup \{a\}$ is extended by u. This means that u is nonnormal, since $a \in u$, hence $\{\alpha < \lambda : \alpha \text{ is inaccessible}\} \notin u$. Therefore, u is nonnormal, hence nonquasi-normal, since u is λ-complete.

Let F_α be a κ-complete filter over $P_\kappa\alpha$ extending $FSF_{\kappa\alpha}$ for each $\alpha < \lambda$. Define F on $P_\kappa\lambda$ by

$$A \in F \quad \text{iff} \quad A \subseteq P_\kappa\lambda \quad \text{and} \quad \{\alpha < \lambda : A \cap P_\kappa\alpha \in F_\alpha\} \in u.$$

Proposition 2.1. *Let λ be strongly compact and let F be a filter as defined above. Then F is not quasi-normal.*

Proof. Let $f : \lambda \to \lambda$ witness that u is not quasi-normal. It is a straightforward exercise to verify that for $\alpha \in a$, $\{x \in P_\kappa\lambda : \sup x = \alpha\} \in F_\alpha$. Set $b = a \cap \{\alpha \in \kappa : f(\alpha) < \alpha\}$. Hence, $b \in u$. For $\alpha \in b$, set

$$B_\alpha = \{x \in P_\kappa\lambda : \sup x = \alpha\} \cap \{x \in P_\kappa\alpha : f(\alpha) \in x\}.$$

Hence, $B_\alpha \in F_\alpha$ for each $\alpha \in b$; and $B_\alpha \cap B_\beta = \varnothing$ when $\alpha \neq \beta$. Next, set $B = \cup\{B_\alpha : \alpha \in b\}$. For $\alpha \in b$, $B_\alpha \subseteq B \cap P_\kappa\alpha$. Hence, $B \in F$. Finally, define $h : P_\kappa\lambda \to \lambda$ by $h(x) = f(\alpha)$, if $x \in B_\alpha$ for some $\alpha \in b$; and $\varnothing$ otherwise. Hence, $h(x) \in x$ for each $x \in B$. Since $B \in F^+$, suppose F is quasi-normal. Then there would exist $C \in F^+$ and $\gamma < \lambda$ such that $B \supset C$ and $h(x) \leq \gamma$ for $x \in C$. Now, $\{\alpha \in b : C \cap P_\kappa\alpha \in F_\alpha^+\} \in u$. But, for $\alpha \in \{\alpha \in b : C \cap P_\kappa\alpha \in F_\alpha^+\}$, let $x \in C \cap P_\kappa\alpha \cap B_\alpha$. Then, $f(\alpha) = h(x) \leq \gamma$. Hence, $\{\alpha < \lambda : f(\alpha) \leq \gamma\} \in u$. But this contradicts the choice of f. $\square$

Remark . Since $CF_\lambda \subset u$, if $F_\alpha = CF_{\kappa\alpha}$ for $\alpha < \lambda$, then F is a nonquasi-normal filter extending $CF_{\kappa\lambda}$.

Section 3. This concluding section investigates the existence of some quasi-normal ultrafilters (hence weakly normal ultrafilters) on $P_\kappa\lambda$ when κ is λ-strongly compact.

Let U be an ultrafilter on $P_\kappa\lambda$. As usual, assume that U extends $FSF_{\kappa\lambda}$. If $q : P_\kappa\lambda \to P_\kappa\lambda$ and $q_*(U) = \{A \subseteq P_\kappa\lambda : q^{-1}(A) \in U\}$, then $q_*(U)$ is an ultrafilter on $P_\kappa\lambda$. And $q_*(U)$ extends $FSF_{\kappa\lambda}$, whenever $\{x \in P_\kappa\lambda : \alpha \in q(x)\} \in U$ for each $\alpha < \lambda$. This notion is derived from the Rudin-Keisler ordering on measures. Furthermore, if for all such q, there is a measure one set A_q in U such that q is one-to-one on A_q, then U is said to be minimal.

Given U an ultrafilter on $P_\kappa\lambda$ and $j : V \to M \cong V^{P_\kappa\lambda}/U$; if $f : P_\kappa\lambda \to V$, let $[f]_U$ denote the member of M corresponding to the equivalence class of f modulo U. Next, let $s : P_\kappa\lambda \to \lambda$ be such that $[s]_U = \sup\{j(\alpha) : \alpha < \lambda\}$. Now, $\{x \in P_\kappa\lambda : \gamma < s(x)\} \in U$ for each $\gamma < \lambda$. And if $\{x \in P_\kappa\lambda : g(x) < s(x)\} \in U$, then there exists a $\gamma < \lambda$ such that $\{x \in P_\kappa\lambda : g(x) < \gamma\} \in U$.

A consequence of this property is that U is weakly normal, hence quasi-normal, if and only if $\{x \in P_\kappa\lambda : s(x) = \sup x\} \in U$ (this fact is mentioned in [**1**]).

In [**8**], a *minimal cover* for an ultrafilter U over $P_\kappa\lambda$, where λ is regular, is defined as follows: Using a result of Solovay [**9**], $\{\alpha < \lambda : cf(\alpha) = \omega\}$ can be partitioned into λ many stationary subsets of λ, $\{A_\alpha : \alpha < \lambda\}$. The *minimal cover* for U is $q_*(U)$, where the function $q : P_\kappa\lambda \to P_\kappa\lambda$ is defined by

$$q(x) = \{\alpha < s(x) : A_\alpha \cap s(x) \text{ is a stationary subset of } s(x)\},$$

for all $x \in P_\kappa\lambda$.

This makes $q_*(U)$ an ultrafilter on $P_\kappa\lambda$ extending $FSF_{\kappa\lambda}$.

Let $j_0 : V \to M_0 \cong V^{P_\kappa\lambda}/U$; $j_1 : V \to M_1 \cong V^{P_\kappa\lambda}/q_*(U)$; and $[s_1]_{q_*(U)} = \sup\{j_1(\alpha) : \alpha < \lambda\}$.

In the proof of Theorem 2.14 in [**8**], where Menas proves that the *minimal cover* $q_*(U)$ is a minimal fine measure on $P_\kappa\lambda$, it is shown that $\{x \in P_\kappa\lambda : s_1(x) = \sup x\} \in q_*(U)$. This, combined with the comment made in the fourth paragraph of this section, gives:

Theorem 3.1. *Let $\lambda > \kappa$ be regular and κ be λ-strongly compact. If U is any ultrafilter on $P_\kappa\lambda$, then the* minimal cover *for U is a minimal, quasi-normal* (*weakly normal*) *ultrafilter on $P_\kappa\lambda$.*

Remark . In [**1**], under the same hypothesis, Abe produces a weakly normal (quasi-normal) ultrafilter which does not extend $SCF_{\kappa\lambda}$.

It would be interesting to determine what conditions yield $CF_{\kappa\lambda} \subset q_*(U)$. (Note: By a result of Solovay (see [**8**]), if $\sup\{\beta : M \supset M^\beta\} > \lambda$, then $q_*(U)$, the *minimal cover* of U, is normal; hence an extension of $CF_{\kappa\lambda}$.)

By another result in [**1**], for any $\lambda > \kappa$, a weakly normal (quasi-normal) ultrafilter can be constructed from any ultrafilter on $P_\kappa\lambda$.

Results such as these demonstrate that a cardinal κ gets no more large cardinal strength from the existence of a quasi-normal ultrafilter on $P_\kappa\lambda$ than that of κ being λ-strongly compact. However, it may be that the consequences of κ being λ-strongly compact can be facilitated knowing that quasi-normal ultrafilters also must exist on $P_\kappa\lambda$.

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