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METRIC SPACES AND MULTIPLICATION OF BOREL SETS

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ABSTRACT. Let us say that the Borel sets of spaces X and Y multiply if each Borel set in the product space $X \times Y$ is a member of the product σ-algebra generated by Borel rectangles. We show that the Borel sets of a space X and a metric space Y multiply if and only if the Borel sets of X and D multiply, where D is a discrete space having the same weight as Y.

1. Introduction. The Borel sets $\mathcal{B}(X)$ of a topological space X are the smallest σ-algebra containing the open sets of X. If X and Y are topological spaces, then $\mathcal{B}(X) \times \mathcal{B}(Y)$ denotes the smallest σ-algebra on $X \times Y$ containing sets of the form $E \times F$, where E and F are Borel sets of X and Y, respectively. Always $\mathcal{B}(X) \times \mathcal{B}(Y) \subset \mathcal{B}(X \times Y)$; if $\mathcal{B}(X) \times \mathcal{B}(Y) = \mathcal{B}(X \times Y)$, we say that the *Borel sets of* X *and* Y *multiply.* Notice that the Borel sets of X and Y multiply if and only if each open set in $X \times Y$ is a member of $\mathcal{B}(X) \times \mathcal{B}(Y)$.

Lemma 1.1. *If* Z *is a subset of* Y *and* $\mathcal{B}(X \times Y) = \mathcal{B}(X) \times \mathcal{B}(Y)$, *then* $\mathcal{B}(X \times Z) = \mathcal{B}(X) \times \mathcal{B}(z)$.

Proof. (Compare with [**6**, Theorem 7.1].) It is easily seen that since $\mathcal{B}(X) \times \mathcal{B}(Z)$ is a σ-algebra of subsets of $X \times Z$, the family

$$\mathcal{M} = \{M \subset X \times Y : (X \times Z) \cap M \in \mathcal{B}(X) \times \mathcal{B}(Z)\}$$

forms a σ-algebra of subsets of $X \times Y$. If $U \times V$ is an open rectangle in $X \times Y$, then $(X \times Z) \cap (U \times V) = U \times (Z \cap V)$ is an open rectangle in $X \times Z$, so $U \times V \in \mathcal{M}$ and, consequently, $\mathcal{M}$ contains $\mathcal{B}(X) \times \mathcal{B}(Y) = \mathcal{B}(X \times Y)$.

Now suppose W^* is open in $X \times Z$. There exists an open subset W of $X \times Y$ such that $W^* = (X \times Z) \cap W$. Since $W \in \mathcal{M}$, we have that $W^* \in \mathcal{B}(X) \times \mathcal{B}(Z)$. This implies that $\mathcal{B}(X \times Z) = \mathcal{B}(X) \times \mathcal{B}(Z)$. □

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Lemma 1.2. *If Y is the union of a sequence of Borel subsets Y_n, then $\mathcal{B}(X \times Y) = \mathcal{B}(X) \times \mathcal{B}(Y)$ if and only if for each n, $\mathcal{B}(X \times Y_n) = \mathcal{B}(X) \times \mathcal{B}(Y_n)$. Hence, if Y is the union of a finite family of Borel sets Y_i, then $\mathcal{B}(X \times Y) = \mathcal{B}(X) \times \mathcal{B}(Y)$ if and only if for each i, $\mathcal{B}(X \times Y_i) = \mathcal{B}(X) \times \mathcal{B}(Y_i)$.*

Proof. Necessity follows from Lemma 1.1. To prove sufficiency, suppose $W \in \mathcal{B}(X \times Y)$. Then $W = \bigcup(W \cap (X \times Y_n))$. For each n, the set $W \cap (X \times Y_n) \in \mathcal{B}(X \times Y_n) = \mathcal{B}(X) \times \mathcal{B}(Y_n) \subset \mathcal{B}(X) \times \mathcal{B}(Y)$. Hence, $W \in \mathcal{B}(X) \times \mathcal{B}(Y)$. □

Lemma 1.3. *Let $f : X \to X'$ and $g : Y \to Y'$ be Borel measurable functions, and let $(f \otimes g)$ be defined on $X \times Y$ by $(f \otimes g)(x, y) = (f(x), g(y))$. Then $(f \otimes g)^{-1}(M) \in \mathcal{B}(X) \times \mathcal{B}(Y)$ for each $M \in \mathcal{B}(X') \times \mathcal{B}(Y')$.*

Proof. Because $(f \otimes g)^{-1}$ preserves countable unions and complements, we see that

$$\mathcal{M} = \{M \in \mathcal{B}(X') \times \mathcal{B}(Y') : (f \otimes g)^{-1}(M) \in \mathcal{B}(X) \times \mathcal{B}(Y)\}$$

is a σ-algebra of subsets of $X' \times Y'$. Because $\mathcal{M}$ contains each Borel rectangle in $X' \times Y'$, we have $\mathcal{M} = \mathcal{B}(X') \times \mathcal{B}(Y')$, which completes the proof. □

Lemma 1.4. *If Z is second countable and $\mathcal{B}(X \times Y) = \mathcal{B}(X) \times \mathcal{B}(Y)$, then $\mathcal{B}(X \times (Y \times Z)) = \mathcal{B}(X) \times \mathcal{B}(Y \times Z)$.*

Proof. Suppose G is open in $X \times (Y \times Z)$ and that $\{W_n\}$ is a countable base for Z. For each n and for each open set U in X, let

$$V(n, U) = \bigcup\{V : V \text{ is open in } Y \text{ and } U \times (V \times W_n) \subset G\}.$$

Then each $V(n, U)$ is open in Y, and

$$G = \bigcup_n \bigcup_U U \times V(n, U) \times W_n.$$

For each n, let

$$H_n = \bigcup_U U \times V(n,U).$$

Since H_n is open in $X \times Y$, we have $H_n \in \mathcal{B}(X) \times \mathcal{B}(Y)$ by hypothesis. Hence, $H_n \times W_n \in \mathcal{B}(X) \times \mathcal{B}(Y \times Z)$. Because $G = \bigcup_n H_n \times W_n$, we see that $G \in \mathcal{B}(X) \times \mathcal{B}(Y \times Z)$. □

Lemma 1.5. *$\mathcal{B}(X \times \prod_{i=1}^{\infty} Y_i) = \mathcal{B}(X) \times \mathcal{B}(\prod_{i=1}^{\infty} Y_i)$ if and only if $\mathcal{B}(X \times \prod_{i=1}^{n} Y_i) = \mathcal{B}(X) \times \mathcal{B}(\prod_{i=1}^{n} Y_i)$ for each positive integer n. Hence, $\mathcal{B}((\prod_{i=1}^{\infty} Y_i) \times X) = \mathcal{B}(\prod_{i=1}^{\infty} Y_i) \times \mathcal{B}(X)$ if and only if $\mathcal{B}((\prod_{i=1}^{n} Y_i) \times X) = \mathcal{B}(\prod_{i=1}^{n} Y_i) \times \mathcal{B}(X)$ for each positive integer n.*

Proof. Necessity follows from Lemma 1.1. We prove sufficiency. For each $n \in N$, define

$$f_n((x,(y_1,y_2,\dots)) = ((x,(y_1,\dots,y_n)),(y_{n+1},y_{n+2},\dots))$$

and

$$\mathcal{A}_n = \left\{A \in \mathcal{B}(X) \times \mathcal{B}\left(\prod_{i=1}^{n} Y_i\right) : f_n^{-1}\left(A \times \prod_{i=n+1}^{\infty} Y_i\right) \in \mathcal{B}(X) \times \mathcal{B}\left(\prod_{i=1}^{\infty} Y_i\right)\right\}.$$

Using standard computations, one easily checks that $\mathcal{A}_n$ is a σ-algebra containing all open rectangles $U \times V$ where $U \subset X$ and $V \subset \prod_{i=1}^{n} Y_i$. Since $\mathcal{B}(X \times \prod_{i=1}^{n} Y_i) = \mathcal{B}(X) \times \mathcal{B}(\prod_{i=1}^{n} Y_i)$, we obtain that $\mathcal{A}_n = \mathcal{B}(X \times \prod_{i=1}^{n} Y_i)$ for any n. Let W be open in $X \times \prod_{i=1}^{\infty} Y_i$. For each $n \in N$, put

$$W_n = \bigcup \left\{U \subset X \times \prod_{i=1}^{n} Y_i : U \text{ is open and } U \times \prod_{i=n+1}^{\infty} Y_i \subset f_n(W)\right\}.$$

Without any difficulties one can show that $W = \bigcup_n f_n^{-1}(W_n \times \prod_{i=n+1}^{\infty} Y_i)$. Since $W_n \in \mathcal{A}_n$ for each n, we see that $W \in \mathcal{B}(X) \times \mathcal{B}(\prod_{i=1}^{\infty} Y_i)$, which completes the proof. □

2. Main Theorem. In this section we prove the following main theorem.

Theorem 2.1. *Let X be a space, and let Y be a metrizable space with weight κ. Then $\mathcal{B}(X \times Y) = \mathcal{B}(X) \times \mathcal{B}(Y)$ if and only if $\mathcal{B}(X \times D) = \mathcal{B}(X) \times \mathcal{B}(D)$ for the discrete space D of cardinality κ.*

We shall, in fact, prove in Theorem 2.5 that the conclusion of Theorem 2.1 holds for a slightly larger class of spaces called quasi-developable spaces.

Definition 2.2. (Compare with [**4**, Definition 8.1, **12**, Definition 3].) A collection $\mathcal{G} = \bigcup_n \mathcal{G}_n$ of open sets in Y is a θ-base if and only if for each open set V and each $y \in V$, there exist n and $G \in \mathcal{G}_n$ such that $y \in G \subset V$ and such that y belongs to only finitely many members of $\mathcal{G}_n$.

Definition 2.3. (Compare with [**1**, Definition 2.1, **4**, Definition 8.4].) A space Y is said to be *quasi-developable* if and only if there exists a sequence $\{\mathcal{G}_n\}$ of collections of open sets such that for each open set V and each $y \in Y$, there exists n such that $y \in \bigcup\{G \in \mathcal{G}_n : y \in G\} \subset V$.

Remark 2.4. (Compare with [**2**, Theorem 8].) A space Y has a θ-base if and only if Y is quasi-developable.

Theorem 2.5. *Let X be a space, and let Y be a quasi-developable space with weight κ. Then $\mathcal{B}(X \times Y) = \mathcal{B}(X) \times \mathcal{B}(Y)$ if and only if $\mathcal{B}(X \times D) = \mathcal{B}(X) \times \mathcal{B}(D)$ for the discrete space D of cardinality κ.*

Proof. Let $\mathcal{G} = \bigcup_n \mathcal{G}_n$ be a θ-base for Y. For each pair of natural numbers n and m, let

$$G_{n,m} = \{y \in Y : y \text{ belongs to at least } m \text{ distinct members of } \mathcal{G}_n\}.$$

It is easy to see that each $G_{n,m}$ is open. Let $Y_{n,m} = G_{n,m} \backslash G_{n,m+1}$. Notice that

$$Y_{n,m} = \{y \in Y : y \text{ belongs to exactly } m \text{ distinct members of } \mathcal{G}_n\},$$

and notice that $Y_{n,m}$ is a Borel set since it is the difference of two open sets. Because $\mathcal{G} = \bigcup_n \mathcal{G}_n$ is a θ-base, we see that $Y = \bigcup_{n,m} Y_{n,m}$. For each pair of natural numbers n and m, let $\mathcal{V}_{n,m}$ be the trace in $Y_{n,m}$ of the family of all intersections of m distinct sets in $\mathcal{G}_n$. Equivalently, for each $y \in Y_{n,m}$, let $V_{y,n,m} = \bigcap\{G \in \mathcal{G}_n : y \in G\} \cap Y_{n,m}$, and let $\mathcal{V}_{n,m}$ be the family of all sets of the form $V_{y,n,m}$ for some $y \in Y_{n,m}$. Notice that the members of $\mathcal{V}_{n,m}$ are relatively open sets in $Y_{n,m}$ and that $\mathcal{V}_{n,m}$ is a partition of $Y_{n,m}$. Let $\pi_{n,m}$ be the quotient mapping of $Y_{n,m}$ onto the quotient space $D_{n,m}$ obtained by identifying the points belonging to the same $V \in \mathcal{V}_{n,m}$. Of course, $\pi_{n,m} : Y_{n,m} \to D_{n,m}$ is continuous and, thus, Borel measurable. Because the weight of Y equals κ, each $\mathcal{G}_n$ and, hence, each $\mathcal{V}_{n,m}$ has cardinality less than or equal to κ. Hence, each $D_{n,m}$ has cardinality less than or equal to κ.

Suppose now that W is open in $X \times Y$. For each $V \in \mathcal{V}_{n,m}$, let U_V be the union of all open sets U in X such that $U \times V \subset W$. In other words, U_V is the largest open set in X such that $U_V \times V \subset W$. Let $W_{n,m} = \bigcup\{U_V \times V : V \in \mathcal{V}_{n,m}\}$, and let

$$W^*_{n,m} = \bigcup_{V \in \mathcal{V}_{n,m}} U_V \times \{\pi_{n,m}(V)\}.$$

Clearly, $W^*_{n,m}$ is open in $X \times D_{n,m}$. Hence, $W^*_{n,m} \in \mathcal{B}(X) \times \mathcal{B}(D)$ by hypothesis. Because $W_{n,m} = (\delta \times \pi_{n,m})^{-1}(W^*_{n,m})$, where δ is the identity mapping, we have $W_{n,m} \in \mathcal{B}(X) \times \mathcal{B}(Y_{n,m}) \subset \mathcal{B}(X) \times \mathcal{B}(Y)$ by Lemma 1.3. That $W \in \mathcal{B}(X) \times \mathcal{B}(Y)$ follows from the fact that W is a countable union of the sets $W_{n,m}$.

Since the weight of X equals κ, at least one $\mathcal{V}_{n,m}$ has cardinality κ. The corresponding set $Y_{n,m}$ thus contains a discrete subspace with cardinality κ, and necessity follows from Lemma 1.1. □

As an immediate consequence of Theorem 2.5, we obtain

Corollary 2.6. (Compare with [**11**, Théorème 1].) *Let* $\kappa \le 2^\omega$. *The following conditions are equivalent*:

(i) $\mathcal{B}(D \times D) = \mathcal{B}(D) \times \mathcal{B}(D)$ *for the discrete space* D *of cardinality* κ.

(ii) $\mathcal{B}(X \times Y) = \mathcal{B}(X) \times \mathcal{B}(Y)$ *for each pair of quasi-developable spaces* X *and* Y *of weight* $\le \kappa$.

(iii) $\mathcal{B}(X \times Y) = \mathcal{B}(X) \times \mathcal{B}(Y)$ *for each pair of metrizable spaces* X *and* Y *of weight* $\le \kappa$.

(iv) $\mathcal{B}(X \times X) = \mathcal{B}(X) \times \mathcal{B}(X)$ *for every quasi-developable space* X *of weight* $\leq \kappa$.

(v) $\mathcal{B}(X \times X) = \mathcal{B}(X) \times \mathcal{B}(X)$ *for every metrizable space* X *of weight* $\leq \kappa$.

In Corollary 2.6, condition (i) is clearly the same as the statement that $\mathcal{P}(D \times D) = \mathcal{P}(D) \times \mathcal{P}(D)$ if D has cardinality κ. This equation holds if $\kappa \leq \omega_1$ ([**8**, Theorem 12.5(ii), **9**, Theorem 2]), does not hold if $\kappa > 2^\omega$ ([**8**, Lemma 12.2(ii), **10**]), and depends on one's set theory if $\omega_1 < \kappa \leq 2^\omega$ ([**8**, Theorem 12.8]). Assuming Martin's Axiom, Theorem 2.5 tells us that if X is any quasi-developable space with weight less than c and Y is any quasi-developable space, then $\mathcal{B}(X \times Y) = \mathcal{B}(X) \times \mathcal{B}(Y)$. This follows from Lemma 1.3 and the fact that under Martin's Axiom, $2^\kappa \leq c$ whenever $\kappa < c$.

We caution that quasi-developability (or something similar) is needed in Theorem 2.5 and Corollary 2.6. That is, it is possible that $\mathcal{B}(D \times D) = \mathcal{B}(D) \times \mathcal{B}(D)$ for a discrete space D with cardinality equal to weight of X even though $\mathcal{B}(X \times X) \neq \mathcal{B}(X) \times \mathcal{B}(X)$.

Example 2.7. Let X be the set of ordinals less than or equal to the first uncountable ordinal ω_1, and let X have the order topology. Clearly the weight of X equals ω_1. By the remarks following Corollary 2.6, $\mathcal{B}(D \times D) = \mathcal{B}(D) \times \mathcal{B}(D)$ if D is a discrete space with cardinality ω_1. However, $\mathcal{B}(X \times X) \neq \mathcal{B}(X) \times \mathcal{B}(X)$ (cf. [**6**, Theorem 2]).

It is interesting that Theorem 2.1 has a fairly easy proof in terms of a universal metric space. We close with that proof.

Alternate Proof of Theorem 2.1. The most familiar universal metrizable space is the product of countably many copies of the hedgehog space $\mathcal{J}(\kappa)$ of spininess κ (cf. [**3**, Example 4.1.5 and Theorem 4.4.3]). Since Y is topologically equivalent to some subspace of $\mathcal{J}(\kappa)^\omega$, it suffices by virtue of Lemma 1.1 to prove that $\mathcal{B}(X \times \mathcal{J}(\kappa)^\omega) = \mathcal{B}(X) \times \mathcal{B}(\mathcal{J}(\kappa)^\omega)$.

Since there exists a point $z_0 \in \mathcal{J}(\kappa)$ such that $\mathcal{J}(\kappa)\backslash\{z_0\}$ is homeomorphic to $(0,1] \times D$, it follows that each $\mathcal{J}(\kappa)^n$ can be expressed as the union of finitely many Borel sets Y_i, where for each Y_i, there exists $m \leq n$ such that Y_i is homeomorphic to $\{z_0\}^{n-m} \times ((0,1]^m \times D^m)$. Because $[0,1]^m$ is second countable and D^m is a discrete space of cardinality κ, we have $\mathcal{B}(X \times Y_i) = \mathcal{B}(X) \times \mathcal{B}(Y_i)$ by Lemma 1.4. Then Lemma 1.2 says that $\mathcal{B}(X \times \mathcal{J}(\kappa)^n) = \mathcal{B}(X) \times \mathcal{B}(\mathcal{J}(\kappa)^n)$ for each positive integer n. Lemma 1.5 implies that $\mathcal{B}(X \times \mathcal{J}(\kappa)^\omega) = \mathcal{B}(X) \times \mathcal{B}(\mathcal{J}(\kappa)^\omega)$.

Because a metrizable space Y contains a discrete subspace whose cardinality equals the weight of Y [**5**, Theorem 8.1(d)], necessity follows from Lemma 1.1. □

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