ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 4, Fall 1992

REMARKS ON SOME THEOREMS OF BANACH, McSHANE, AND PETTIS

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ABSTRACT. In [**12**] B.J. Pettis gives several results which sharpen results of McShane [**7**]. These results use properties related to the ideal of meager or first category subsets. An *ideal* is a collection of subsets closed under the operations of subset and finite union. In this paper we identify the crucial properties of the ideal of meager sets which make these results possible and extend the results to other ideals which possess these properties. A well-known result of Banach [**1**], concerning subgroups of a topological group is extended and two theorems concerning continuity of homomorphisms and linear transformations are given as applications.

1. Introduction. In [**12**] B.J. Pettis gives several results which sharpen results of E.J. McShane [**7**] which in turn extend a well-known theorem of Banach [**1**]. These results use properties related to the ideal of meager or first category subsets. In this paper we identify the crucial properties of this ideal which make these results possible and extend the results to other ideals which possess these properties.

An *ideal* ([**6, 15, 16**]) $\mathcal{I}$ on a topological space (X, τ) is a collection of subsets of X which satisfies the following two properties: (1) $A \in \mathcal{I}$ and $B \subseteq A$ implies $B \in \mathcal{I}$ (heredity), and (2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ (finite additivity). If, in addition, $\mathcal{I}$ satisfies (3) $\{A_n : n = 1, 2, 3, \dots\} \subseteq \mathcal{I}$ implies $\cup\{A_n : n = 1, 2, 3, \dots\} \in \mathcal{I}$ (countable additivity), then $\mathcal{I}$ is called a *σ-ideal*.

Let $\mathcal{P}(X)$ denote the power set of X. Given a topological space (X, τ) and an ideal $\mathcal{I}$ on X, a set operator $(\quad)^*(\mathcal{I}, \tau)$: $\mathcal{P}(X) \to \mathcal{P}(X)$, called the *local function of $\mathcal{I}$ with respect to τ* in [**16**] is defined as follows: for $A \subseteq X$, $(A)^*(\mathcal{I}, \tau) = \{x \in X : U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$. A Kuratowski closure

Received by the editors on October 22, 1989.
This research was partially supported by a grant from East Central University.
1980 AMS *Mathematics Subject Classification.* 54H99, 54C50.
Key words and phrases. Ideal, Baire set, measure, meager, sets of measure zero, topological group, topological vector space, homomorphism, linear transformation, continuous function, compatible ideal, τ-boundary ideal.

operator Cl^* for a topology $\tau^*(\mathcal{I})$ finer than τ is defined as follows: $\mathrm{Cl}^*(A) = A \cup A^*(\mathcal{I}, \tau)$ [**16**]. A basis $\beta(\mathcal{I}, \tau)$ for $\tau^*(\mathcal{I})$ can be described as follows: $\beta(\mathcal{I}, \tau) = \{U - I : U \in \tau, I \in \mathcal{I}\}$ [**15**]. When no ambiguity is present, we simply write A^*, τ^*, and β for $A^*(\mathcal{I}, \tau)$, $\tau^*(\mathcal{I})$, and $\beta(\mathcal{I}, \tau)$, respectively.

In [**8**] Natkaniec defines a set operator $\psi : \mathcal{P}(X) \to \tau$ as follows: for $A \subseteq X$, $\psi(A) = \{x :$ there exists a $U \in \tau(x)$ such that $U - A \in \mathcal{I}\}$, and observes that $\psi(A) = X - (X - A)^*$. The operator ψ is studied extensively in [**3**] where it is observed that $\psi(A) = \cup\{U \in \tau : U - A \in \mathcal{I}\}$.

In [**12**] Pettis defines three set operators, which he denotes as $I(S)$, $II(S)$, and $III(S)$ for $S \subseteq (X, \tau)$. Observe that $I(S) = X - S^* = \psi(X - S)$, $II(S) = S^*$, and $III(S) = \mathrm{Int}(S^*) \cap \psi(S)$ where Int denotes the interior operator with respect to τ.

In what follows, we will denote by $(X, \tau, \mathcal{I})$ a topological space (X, τ) and an ideal $\mathcal{I}$ on X. Given a space $(X, \tau, \mathcal{I})$ and $A \subseteq X$, we denote by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ the closure and interior of A with respect to τ, respectively, and by $\mathrm{Cl}^*(A)$ and $\mathrm{Int}^*(A)$ the closure and interior of A with respect to τ^*, respectively. We abbreviate "implies" or "this implies" or "which implies" by "$\to$", "if and only if" by "iff," and "neighborhood" by "nbhd."

2. Compatible and τ-boundary ideals. Given a space $(X, \tau, \mathcal{I})$, $\mathcal{I}$ is said to be *compatible with* τ [**10**], denoted $\mathcal{I} \sim \tau$, if the following holds for every $A \subseteq X$: if for every $a \in A$ there exists a $U \in \tau(a)$ such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$. Ideals having this property with respect to a topology are also called "supercompact" [**16**], "adherence ideals" [**15**], and "strong locally Banach" [**14**], in the literature. One significant consequence of $\mathcal{I} \sim \tau$ is that this implies $\beta = \tau^*$ [**10**]. It is known that in any topological space (X, τ) the ideal of meager sets, denoted $\mathcal{I}_m$, is compatible with τ ([**11, 16**]), the Banach Category Theorem), and the ideal of nowhere dense sets, denoted $\mathcal{I}_n$, is compatible with τ [**16**]. It is also known [**5**] that in a hereditarily Lindelöf space, any σ-ideal is compatible with the topology. The last result applies for instance to the σ-ideal of subsets of measure zero, the *null* sets, with respect to a complete measure in a hereditarily lindelöf space. See [**5**] for more information.

Given a space $(X,\tau,\mathcal{I})$, $\mathcal{I}$ is said to be *τ-boundary* [**9**], if $\tau\cap\mathcal{I}=\{\phi\}$. Recall that an equivalent formulation of (X,τ) being a *Baire space* is $\tau\cap\mathcal{I}_m=\{\phi\}$.

Given a topological group $(X,\tau,\cdot)$ and an ideal $\mathcal{I}$ on X, denoted $(X,\tau,\mathcal{I},\cdot)$, and $x\in X$, we denote by $x\mathcal{I}=\{xI : I\in\mathcal{I}\}$. We will say $\mathcal{I}$ is *left translation invariant* if for every $x\in X$ we have $x\mathcal{I}\subseteq\mathcal{I}$. Observe that if $\mathcal{I}$ is left translation invariant then $x\mathcal{I}=\mathcal{I}$ for every $x\in X$. We define $\mathcal{I}$ to be right translation invariant in a similar way and observe that $\mathcal{I}$ is right translation invariant if and only if $\mathcal{I}x=\mathcal{I}$ for every $x\in X$.

Note that if $\mathcal{I}$ is left or right translation invariant, $X\notin\mathcal{I}$, and $I\sim\tau$, then $\mathcal{I}$ is τ-boundary. The well-known result that a second category topological group is a Baire space follows immediately from this observation.

3. Extending some results of Pettis [**12**]. We first prove some preliminary lemmas which lead to a theorem extending the theorem in [**12**] and apply the theorem to topological groups.

Lemma 1. *Let $(X,\tau,\mathcal{I})$ be a space with $\mathcal{I}\sim\tau$, $S\subseteq X$. If N is a nonempty open subset of $S^*\cap\psi(S)$, then $N-S\in\mathcal{I}$ and $N\cap S\notin\mathcal{I}$.*

Proof. If $N\subseteq S^*\cap\psi(S)$, then $N-S\subseteq\psi(S)-S\in\mathcal{I}$ [**3**] $\rightarrow N-S\in\mathcal{I}$ by heredity. Since $N\in\tau-\{\phi\}$ and $N\subseteq S^*$, we have $N\cap S\notin\mathcal{I}$ by definition of S^*. □

Given a space $(X,\tau,\mathcal{I})$, we will follow the notation of Pettis [**12**] and define $\mathcal{B}_r(X,\tau,\mathcal{I})=\{S\subseteq X$: there exists $G\in\tau$ such that $S\Delta G\in\mathcal{I}\}$, where $S\Delta G=(S-G)\cup(G-S)$ is the "symmetric difference" of S with G. We will call $\mathcal{B}_r(X,\tau,\mathcal{I})$ the *Baire sets of (X,τ) with respect to $\mathcal{I}$*, denoted simply $\mathcal{B}_r(X)$ when no ambiguity is present. We denote by $\mathcal{B}(X)=\mathcal{B}_r(X)-\mathcal{I}$.

Lemma 2. *Let $(X,\tau,\mathcal{I})$ be a topological space with ideal $\mathcal{I}$.*

(1) $B\in\mathcal{B}(X)\rightarrow$ *there exists a $G\in\tau-\{\phi\}$ such that $B\Delta G\in\mathcal{I}$.*

(2) *If* $\mathcal{I} \cap \tau = \{\phi\}$, *then* $B \in \mathcal{B}(X)$ *if and only if there exists a* $G \in \tau - \{\phi\}$ *such that* $B \Delta G \in \mathcal{I}$.

Proof. (1) Assume $B \in \mathcal{B}(X)$, then $B \in \mathcal{B}_r(X)$. Now if there does not exist a $G \in \tau - \{\phi\}$ such that $B\Delta G \in \mathcal{I}$, we have $B = B\Delta\phi \in \mathcal{I}$ which is a contradiction.

(2) Assume there exists a $G \in \tau - \{\phi\}$ such that $B\Delta G \in \mathcal{I}$. Then $G = (B - J) \cup I$ where $J, I \in \mathcal{I}$. If $B \in \mathcal{I}$, then $G \in \mathcal{I}$ by heredity and additivity, which contradicts the τ-boundary assumption. □

Lemma 3. *Let* $(X, \tau, \mathcal{I})$ *be a space with* $\mathcal{I}$ τ*-boundary. If* $B \in \mathcal{B}(X)$, *then* $\psi(B) \cap \mathrm{Int}(B^*) \neq \phi$.

Proof. Assume $B \in \mathcal{B}(X)$. Then, by Lemma 2, (1), there exists a $G \in \tau - \{\phi\}$ such that $B\Delta G \in \mathcal{I} \to \phi \neq G \subseteq G^* = B^*$. Also, $\phi \neq G \subseteq \psi(G) = \psi(B)$ so that $G \subseteq \psi(B) \cap \mathrm{Int}(B^*)$. □

Given a space $(X, \tau, \mathcal{I})$, let $\mathcal{U}(X, \tau, \mathcal{I})$ denote $\{A \subseteq X$: there exists a $B \in \mathcal{B}(X)$ such that $B \subseteq A\}$. Following our convention we denote $\mathcal{U}(X, \tau, \mathcal{I})$ as $\mathcal{U}(X)$ when no ambiguity is present.

Lemma 4. *let* $(X, \tau, \mathcal{I})$ *be a space with* $\tau \cap \mathcal{I} = \{\phi\}$. *The following are equivalent*: (1) $S \in \mathcal{U}(X)$; (2) $\mathrm{Int}(S^*) \cap \psi(S) \neq \phi$; (3) $S^* \cap \psi(S) \neq \phi$; (4) $\psi(S) \neq \phi$; (5) $\mathrm{Int}^*(S) \neq \phi$; *and* (6) *there exists* $N \in \tau - \{\phi\}$ *such that* $N - S \in \mathcal{I}$ *and* $N \cap S \notin \mathcal{I}$.

Proof. (1) $\to$ (2). Let $B \in \mathcal{B}(X)$ such that $B \subseteq S$. Then $\mathrm{Int}(B^*) \subseteq \mathrm{Int}(S^*)$ and $\psi(B) \subseteq \psi(S) \to \phi \neq \mathrm{Int}(B^*) \cap \psi(B) \subseteq \mathrm{Int}(S^*) \cap \psi(S)$ by Lemma 3.

(2) $\to$ (3). Obvious.

(3) $\to$ (4). Obvious.

(4) $\to$ (5). If $\psi(S) \neq \phi$, then let $U \in \tau - \{\phi\}$ such that $U - S \in \mathcal{I}$. Since $U \notin \mathcal{I}$ and $U = (U - S) \cup (U \cap S)$, we have $U \cap S \notin \mathcal{I}$. In particular, $\phi \neq U \cap S \subseteq \psi(U) \cap S = \mathrm{Int}^*(S)$.

(5) $\rightarrow$ (6). If $\text{Int}^*(S) \neq \phi$ then there exists $N \in \tau - \{\phi\}$, $I \in \mathcal{I}$, such that $\phi \neq N - I \subseteq S$. We have $N - S \in \mathcal{I}$, $N = (N - S) \cup (N \cap S)$, and $N \notin \mathcal{I} \rightarrow N \cap S \notin \mathcal{I}$.

(6) $\rightarrow$ (1). Let $B = N \cap S \notin \mathcal{I}$ with $N \in \tau - \{\phi\}$ and $N - S \in \mathcal{I}$. Then $B \in \mathcal{B}(X)$ since $B \notin \mathcal{I}$ and $B \Delta N = N - S \in \mathcal{I}$. □

A function (or mapping) $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be *open* if $U \in \tau \rightarrow f(U) \in \sigma$. *In what follows $\mathcal{F}$ always denotes a nonempty collection of open mappings from a space X to a space Y.* We do not restrict the domains of members of $\mathcal{F}$ as does Pettis [**12**].

Lemma 5. *Given spaces $(X, \tau), (Y, \sigma)$ and a collection $\mathcal{F}$ of open mappings from X to Y, let $N \in \tau - \{\phi\}$ and let A be a nonempty subset of N. If $f(N) \subseteq \mathcal{F}(A) = \cup\{f(A) : f \in \mathcal{F}\}$ for every $f \in \mathcal{F}$, then $\mathcal{F}(A) \in \sigma - \{\phi\}$.*

Proof. Apply Lemma 3 of [**12**]. □

We are now ready to prove a theorem similar to the theorem of Pettis in [**12**]. Let $\mathcal{F}^{-1}(y) = \cup\{f^{-1}(y) : f \in \mathcal{F}\}$

Theorem 1. *Let $(X, \tau, \mathcal{I})$ and (Y, σ) be spaces with $\mathcal{I} \sim \tau$, $\tau \cap \mathcal{I} = \{\phi\}$, $N \in \tau - \{\phi\}$, $S \subseteq X$, $N \subseteq S^* \cap \psi(S)$, and $\mathcal{F}$ a nonempty collection of open mappings from X to Y. If $y \in \mathcal{F}(N) \rightarrow N \cap \mathcal{F}^{-1}(y) \notin \mathcal{I}$, then $\mathcal{F}(N \cap S) \in \sigma - \{\phi\}$.*

Proof. By Lemma 1, $N - S \in \mathcal{I}$ and $N \cap S \notin \mathcal{I}$. Now we have $N \cap S \cap \mathcal{F}^{-1}(y) \notin \mathcal{I}$ for every $y \in \mathcal{F}(N)$, since $N \cap \mathcal{F}^{-1}(y) \subseteq [N \cap \mathcal{F}^{-1}(y) \cap S] \cup [N - S]$. In particular, $y \in \mathcal{F}(N) \rightarrow y \in \mathcal{F}(N \cap S) \rightarrow \mathcal{F}(N \cap S) \in \sigma - \{\phi\}$ by Lemma 5. □

In light of Lemma 4, it is clear that Theorem 1 will only apply when $S \in \mathcal{U}(X)$. Also note that $\mathcal{U}(X) \neq \phi$ only if $X \notin \mathcal{I}$. Theorem 1 implies the Theorem of Pettis [**12**] if X is a Baire space; however, Pettis does not assume X is a Baire space. Our next corollary which is analogous to

Corollary 2 of [**12**] requires that $X \notin \mathcal{I}$. Here Pettis requires $X \notin \mathcal{I}_m$, which in a topological group implies X is a Baire space.

First we need a preliminary lemma.

Lemma 6. *Let $g : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be a continuous mapping with $g^{-1}(\mathcal{J}) \subseteq \mathcal{I}$. Then $g[A^*(\mathcal{I}, \tau)] \subseteq [g(A)]^*(\mathcal{J}, \sigma)$.*

Proof. Assume $y \notin [g(A)]^*$. Then there exists $V \in \sigma(y)$ such that $V \cap g(A) \in \mathcal{J} \to g^{-1}[V \cap g(A)] \in g^{-1}(\mathcal{J}) \subseteq \mathcal{I} \to g^{-1}(V) \cap A \in \mathcal{I}$ by heredity. Since $g^{-1}(V) \in \tau$ and $g^{-1}(y) \subseteq g^{-1}(V)$, we have $g^{-1}(y) \cap A^* = \phi \to y \notin g(A^*)$, and the proof is complete. □

Corollary 1. *Let $(X, \tau, \mathcal{I}, \cdot)$ be a topological group with $\mathcal{I} \sim \tau$. Let $S \in \mathcal{U}(X)$, $R \in \mathcal{P}(X) - \mathcal{I}$. Let $G, H \in \tau$ such that $G \cap R^* \neq \phi \neq H \cap \operatorname{Int}(S^*) \cap \psi(S)$. If $A = G \cap R \cap R^*$ and $B = H \cap S \cap \operatorname{Int}(S^*) \cap \psi(S)$, then*

(1) *If $\mathcal{I}$ is right translation invariant, then $A^{-1}B$ is a nonempty open subset of $R^{-1}S$; and*

(2) *If $\mathcal{I}$ is left translation invariant, then BA^{-1} is a nonempty open subset of SR^{-1}.*

Proof. We will prove (1) only, since the proof of (2) is similar. Observe that $R \notin \mathcal{I} \to X \notin \mathcal{I} \to \mathcal{I} \cap \tau = \{\phi\}$ since X is a topological group and $\mathcal{I} \sim \tau$.

For every $a \in A$ define $f_a : X \to X$ by $f_a(x) = a^{-1}x$, and let $\mathcal{F} = \{f_a : a \in A\}$. Note that $G \cap R^* \subseteq (G \cap R)^*$ [**6**, p. 64] $= (G \cap R \cap R^*)^*$ (since $\mathcal{I} \sim \tau$), and the hypothesis $G \cap R^* \neq \phi$ then implies $A \neq \phi$ and $\mathcal{F} \neq \phi$. Note that $A \subseteq A^*$ and that each f_a is a homomorphism. Let $N = H \cap \operatorname{Int}(S^*) \cap \psi(S)$. Now if it is shown that $N \cap \mathcal{F}^{-1}(y) \notin \mathcal{I}$ for every $y \in \mathcal{F}(N)$, Theorem 1 applies and $\mathcal{F}(N \cap S) = \mathcal{F}(B) = A^{-1}B$ is a nonempty open subset of $R^{-1}S$.

Let $y \in \mathcal{F}(N)$, then $y = a^{-1}x$ for some $a \in A$, $x \in N \to \mathcal{F}^{-1}(y) = Aa^{-1}x$. Thus, $x \in A\ a^{-1}x \subseteq A^* a^{-1}x \subseteq (Aa^{-1}x)^*$ (Lemma 6) $= [\mathcal{F}^{-1}(y)]^* \to U \cap \mathcal{F}^{-1}(y) \notin \mathcal{I}$ for every $U \in \tau(x)$; in particular, $N \in \tau(x) \to N \cap \mathcal{F}^{-1}(y) \notin \mathcal{I}$.

Thus, by Theorem 1, $A^{-1}B$ is a nonempty open subset of $R^{-1}S$, and the proof is complete. $\square$

Observe that Corollary 2 of [**12**] is a special case of Corollary 1 where $\mathcal{I} = \mathcal{I}_m$.

Letting $G = H = X$ in the previous corollary, we have the following result.

Corollary 2. *Let $(X, \tau, \mathcal{I}, \cdot)$ be a topological group with $\mathcal{I} \sim \tau$. Let $S \in \mathcal{U}(X)$ and $R \in \mathcal{P}(X) - \mathcal{I}$, then*

(1) *If $\mathcal{I}$ is right translation invariant then $[R \cap R^*]^{-1}$ $[S \cap \mathrm{Int}(S^*) \cap \psi(S)]$ is a nonempty open subset of $R^{-1}S$; and*

(2) *If $\mathcal{I}$ is left translation invariant, then $[S \cap \mathrm{Int}(S^*) \cap \psi(S)][R \cap R^*]^{-1}$ is a nonempty open subset of SR^{-1}.*

Proof. Let $G = H = X$ and apply Corollary 1. $\square$

Observe that Corollary 3 of [**12**] is a special case of Corollary 2 where $\mathcal{I} = \mathcal{I}_m$.

Letting $R = S$ in Corollary 2, we obtain the following result.

Corollary 3. *Let $(X, \tau, \mathcal{I}, \cdot)$ be a topological group with identity e, $\mathcal{I} \sim \tau$, and $S \in \mathcal{U}(X)$.*

(1) *If $\mathcal{I}$ is right translation invariant, then $e \in \mathrm{Int}(S^{-1}S)$.*

(2) *If $\mathcal{I}$ is left translation invariant, then $e \in \mathrm{Int}(SS^{-1})$.*

(3) *If $\mathcal{I}$ is translation invariant, then $e \in \mathrm{Int}(S^{-1}S) \cap \mathrm{Int}(SS^{-1})$.*

Proof. We will prove (1) only. The proof of (2) is similar and (3) obviously follows from (1) and (2).

Observe that $S \cap \mathrm{Int}(S^*) \cap \psi(S) \subseteq S \cap S^*$ so that in Corollary 2, (1), we have $e \in [S \cap \mathrm{Int}(S^*) \cap \psi(S)][S \cap S^*]^{-1}$. $\square$

Corollary 5 of [**12**] is a special case of Corollary 3 where $\mathcal{I} = \mathcal{I}_m$. We could continue the development here to extend a well-known result of Banach, utilizing the hypothesis $\mathcal{I} \sim \tau$. However, the previous corollary can be strengthened as the next theorem shows. First, we state an easily established lemma without proof.

4. Extending a theorem of Banach and applications.

Lemma 7. *If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is a homomorphism with $f(\mathcal{I}) = \mathcal{J}$, then $f(\psi(A)) = \psi(f(A))$ for every $A \subseteq X$.*

The following theorem strengthens Corollary 3 by weakening the hypothesis $\mathcal{I} \sim \tau$ to $\mathcal{I} \cap \tau = \{\phi\}$.

Theorem 2. *Let $(X, \tau, \mathcal{I}, \cdot)$ be a topological group with identity e, $\mathcal{I} \cap \tau = \{\phi\}$ and $S \in \mathcal{U}(X)$.*

(1) *If $\mathcal{I}$ is right translation invariant, then $e \in \operatorname{Int}(S^{-1}S)$.*

(2) *If $\mathcal{I}$ is left translation invariant, then $e \in \operatorname{Int}(SS^{-1})$.*

(3) *If $\mathcal{I}$ is translation invariant, then $e \in \operatorname{Int}(S^{-1}S) \cap \operatorname{Int}(SS^{-1})$.*

Proof. We prove only (2) since the proof of (1) is similar.

(3) follows immediately from (1) and (2).

Let $S \in \mathcal{U}(X)$; then there exists $A \subseteq S$ such that $A \in \mathcal{B}(X)$. Since left translation by any element is a homomorphism, we have by Lemma 7 that, for any $x \in X$, $x\psi(A) = \psi(xA)$ and, hence, $x\psi(A) \cap \psi(A) = \psi(xA \cap A)$ (it is shown in [**3**] that the operator ψ distributes over intersection). Thus if $x\psi(A) \cap \psi(A) \neq \phi$, then $xA \cap A \neq \phi$ since $\psi(\phi) = \cup(\tau \cap \mathcal{I}) = \phi$. Now we have

$$\begin{aligned}[\psi(A)][\psi(A)]^{-1} &= \{x : x\psi(A) \cap \psi(A) \neq \phi\} \\ &\subseteq \{x : xA \cap A \neq \phi\} \\ &= AA^{-1} \\ &\subseteq SS^{-1}.\end{aligned}$$

By Lemma 4, $\psi(A) \neq \phi$. Since $\psi(A)$ is open for any $A \subseteq X$, we have $e \in [\psi(A)][\psi(A)]^{-1} \subseteq \operatorname{Int}(SS^{-1})$. □

Observe that Corollary 3 could also be obtained from Theorem 2.

The following corollaries extend a well-known result of Banach [**1**].

Corollary 4. *Let* $(X, \tau, \mathcal{I}, \cdot)$ *be a topological group with* $\mathcal{I} \cap \tau = \{\phi\}$ *and* $\mathcal{I}$ *right or left translation invariant. If* S *is a subgroup and* $S \in \mathcal{U}(X)$*, then* $S = \mathrm{Int}(S) = \mathrm{Cl}(S)$*.*

Proof. $S = \mathrm{Int}(S)$ since, by Theorem 2, $e \in \mathrm{Int}(S^{-1}S)$ (or $e \in \mathrm{Int}(SS^{-1})) \subseteq \mathrm{Int}S$. □

Corollary 5. *Let* $(X, \tau, \mathcal{I}, \cdot)$ *be a topological group with* $\mathcal{I} \sim \tau$ *and* $\mathcal{I}$ *right or left translation invariant. If* S *is a subgroup and* $S \in \mathcal{U}(X)$*, then* $S = \mathrm{Int}(S) = \mathrm{Cl}(S)$*.*

Proof. $S \in \mathcal{U}(X) \to X \notin \mathcal{I}$. $X \notin \mathcal{I}$ and $\mathcal{I} \sim \tau \to \tau \cap \mathcal{I} = \{\phi\}$. Now apply Corollary 4. □

The well-known result of Banach follows from Corollary 4 or Corollary 5 by letting $\mathcal{I} = \mathcal{I}_m$.

An interesting special case of Corollary 4 is to take the subsets of Haar measure zero on a locally compact group, the *null* sets, denoted $\mathcal{H}_0$. If $\mathcal{H}_0 \cap \tau = \{\phi\}$, then Corollary 4 applies establishing a *symmetry* between Banach's result for meager sets and the null sets.

The following theorem extends a result of McShane [**7**, Corollary 7].

Theorem 3. *Let* $f : (X, \tau, \mathcal{I}, \cdot) \to (Y, \sigma, \cdot)$ *be an onto homomorphism with* $\mathcal{I}$ *a right or left translation invariant* σ*-ideal,* $\mathcal{I}$ τ*-boundary,* $X \notin \mathcal{I}$*, and* Y *separable. If* $f^{-1}(V) \in \mathcal{B}_r(X)$ *for every* $V \in \sigma$ *then* f *is continuous.*

Proof. We assume that $\mathcal{I}$ is right translation invariant. The proof for left translation invariant is similar.

Let $\{y_n\}$ be a countable dense subset of Y, and let $V \in \sigma - \{\phi\}$. Then $\{Vy_n\}$ is an open cover of Y and hence $\{f^{-1}(Vy_n)\}$ is a cover of X.

Since $X \notin \mathcal{I}$, there exists a natural number K such that $f^{-1}(Vy_K) \notin \mathcal{I}$. Now let $x \in f^{-1}(y_K^{-1})$, then $f^{-1}(Vy_K)x \subseteq f^{-1}(V) \rightarrow f^{-1}(V) \notin \mathcal{I}$.

Now let $U \in \sigma(e_Y)$, where e_Y is the identity element in Y. There exists $U_i \in \sigma(e_Y)$ such that $U_1U_1^{-1} \subseteq U$. By the above comments, $f^{-1}(U_1) \in \mathcal{B}(X)$, thus (by Theorem 2) $e_X \in \text{Int}[f^{-1}(U_1)][f^{-1}(U_1)]^{-1} = G$. Let $g \in G$, then $g = ab^{-1}$ where $f(a), f(b) \in U_1$. Thus, $f(g) = f(ab^{-1}) = f(a)f(b)^{-1} \in U_1U_1^{-1} \subseteq U$, showing $f(G) \subseteq U$ and the proof is complete. □

Observe that the hypothesis "$\mathcal{I}$ τ-boundary" can be replaced by the stronger hypothesis "$\mathcal{I} \sim \tau$" in the previous theorem. Also observe that if we have a complete measure on a topological group G and if the null sets are right or left translation invariant (such as a Haar measure on a locally compact group) then the null sets are a σ-ideal.

Our next result is an application of Theorem 2 to functional analysis.

Given a real (or complex) topological vector space $(X, \tau, +)$ and an ideal $\mathcal{I}$ on X, denoted $(X, \tau, \mathcal{I}, +)$, $\mathcal{I}$ is said to be *multiplication invariant* if for any scalar r and any $I \in \mathcal{I}$, $rI \in \mathcal{I}$.

Theorem 4. *Let $(X, \tau, \mathcal{I}, +)$ and $(Y, \sigma, +)$ be real (or complex) topological vector spaces, $\mathcal{I}$ a σ-ideal, $\mathcal{I}$ τ-boundary, $\mathcal{I}$ translation invariant and multiplication invariant, and $X \notin \mathcal{I}$. If $T : X \rightarrow Y$ is a linear transformation such that $T^{-1}(V) \in \mathcal{B}_r(X)$ for every $V \in \sigma(0)$, then T is continuous.*

Proof. Let V be a nbd of 0 in Y and let V_1 be a nbd of 0 in Y such that $V_1 - V_1 \subseteq V$. Now $\{nV_1 : n = 1, 2, 3, \dots\}$ covers Y [**13**, Theorem 1.15], hence $\{T^{-1}(nV_1) : n = 1, 2, 3, \dots\}$ covers X. Since $X \notin \mathcal{I}$, we must have $T^{-1}(KV_1) \notin \mathcal{I}$ for some K. Since $(1/K)T^{-1}(KV_1) \subseteq T^{-1}(V_1)$, we have that $T^{-1}(V_1) \notin \mathcal{I}$. By Theorem 2, $0 \in \text{Int}[T^{-1}(V_1) - T^{-1}(V_1)] = G$. We claim that $T(G) \subseteq V$; indeed, if $z \in G$, $z = w - y$ for $w, y \in T^{-1}(V_1) \rightarrow T(z) = T(w - y) = T(w) - T(y) \in V_1 - V_1 \subseteq V$. Thus, T is continuous at 0 and the proof is complete. □

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