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AUTOMORPHISMS OF THE INTEGRAL GROUP RING OF THE WREATH PRODUCT OF A p-GROUP WITH S_n

ANTONIO GIAMBRUNO AND SUDARSHAN K. SEHGAL

Introduction. Let S_n be the symmetric group on n symbols, and let $G = H \operatorname{wr} S_n$ be the wreath product of a finite p-group H and S_n. Then G is the semidirect product $H^n \rtimes S_n = \{(a_1, \dots, a_n; \sigma) \mid a_i \in H, \sigma \in S_n\}$ with the product rule

$$(a_1, \dots, a_n; \sigma)(b_1, \dots, b_n; \tau) = (a_1 b_{\sigma^{-1}(1)}, \dots, a_n b_{\sigma^{-1}(n)}; \sigma\tau).$$

Let $\mathbf{Z}G$ be the integral group ring and $\mathbf{Q}G$ the rational group algebra of G. If u is a unit of $\mathbf{Q}G$, denote by τ_u the inner automorphism of $\mathbf{Q}G$ induced by u. In this paper, we verify a conjecture of Zassenhaus for these groups G by proving the

Theorem. *Let G be the wreath product $H \operatorname{wr} S_n$ of a finite p-group H and S_n, $n \geq 3$. Then every normalized automorphism θ of $\mathbf{Z}G$ can be written in the form $\theta = \tau_u \circ \lambda$ where λ is an automorphism of G and u is a suitable unit of $\mathbf{Q}G$.*

This result is known if $G = A \operatorname{wr} S_n$ where A is abelian [**1**] and if $G = S_k \operatorname{wr} S_n$ [**5**]. In order to prove the theorem, it suffices [**4**, Proposition III.7.2] to find an automorphism $\mu \in \operatorname{Aut}(G)$ such that for all $g \in G$, $\theta(C_g) = C_{\mu(g)}$. Here, by C_g is understood the sum of elements in the conjugacy class $\mathcal{C}(g)$ of g in G. We shall find in Section 3, as a consequence of a Theorem of Weiss [**6**], $\lambda \in \operatorname{Aut}(H^n)$ such that $\theta(C_h) = C_{\lambda(h)}$ for all $h \in H^n$. However, it is not at all clear if one can extend λ to an automorphism of G. Due to the special structure of G, it is possible to use λ to construct an automorphism of G having the desired effect on the classes in H^n. Then we invoke a result of Valenti [**5**] to complete the proof.

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2. Some basic lemmas. We describe the conjugacy classes $\mathcal{C}_G(g)$, of elements g of G. Let $\sigma \in S_n$ and let $c(\sigma)$ denote the number of disjoint cycles of σ, including 1-cycles, and s_μ their lengths ($\mu = 1, \dots, c(\sigma)$). If i_μ is the smallest number appearing in the μ^{th} cycle, then we can write

$$\sigma = \prod_{\mu=1}^{c(\sigma)} (i_\mu \sigma(i_\mu) \cdots \sigma^{s_\mu - 1}(i_\mu)).$$

For each μ, define the product

$$b^\mu = a_{i_\mu} a_{\sigma(i_\mu)} \cdots a_{\sigma^{s_\mu - 1}(i_\mu)},$$

and if $\mathcal{C}_1, \dots, \mathcal{C}_t$ are the distinct conjugacy classes of H, for every k with $1 \le k \le n$ and $1 \le j \le t$, set

$$x_{jk}(a_1, \dots, a_n; \sigma) = |\{b^\mu \mid s_\mu = k \text{ and } b^\mu \in \mathcal{C}_j\}|.$$

These elements determine the conjugacy classes of G, in fact, by [**2**, Theorem 4.2.8] two elements $(a_1, \dots, a_n; \sigma)$ and $(b_1, \dots, b_n; \tau)$ are conjugate in G if and only if the matrices $x_{jk}(a_1, \dots, a_n; \sigma)$ and $x_{jk}(b_1, \dots, b_n; \tau)$ are equal. In particular for an element $h = (h_1, \dots, h_n; I)$ to be conjugate to $h' = (h'_1, \dots, h'_n; I)$ it is necessary and sufficient that for every $g \in H$, the number of $h_i \sim g$ equals the number of $h'_i \sim g$. We shall often write $(h_1, \dots, h_n)$ for $(h_1, \dots, h_n; I)$. The next three lemmas are obvious. All automorphisms we consider are normalized, i.e., they preserve augmentation.

Lemma 2.1. *Let $a = (h_1, \dots, h_n) \in H^n$ and suppose $h_1 \sim h_2 \cdots \sim h_{t_1}$, $h_{t_1+1} \sim \cdots \sim h_{t_2}, \dots, h_{t_{m-1}+1} \sim \cdots \sim h_{t_m} = h_n$ belong to m distinct conjugacy classes in H. Then*

$$\begin{aligned} |\mathcal{C}_G(a)| &= \binom{n}{t_1}\binom{n - t_1}{t_2} \cdots \binom{n - (t_1 + \cdots + t_{m-1})}{t_m} \\ &\qquad \cdot |\mathcal{C}_H(h_{t_1})|^{t_1} \cdots |\mathcal{C}_H(h_{t_m})|^{t_m} \\ &= \frac{n!}{t_1! \cdots t_m!} |\mathcal{C}_H(h_{t_1})|^{t_1} \cdots |\mathcal{C}_H(h_{t_m})|^{t_m}. \end{aligned}$$

Lemma 2.2. *Let $t_1 \geq t_2 \geq \cdots \geq t_m \geq 0$ be integers such that $t_1 + \cdots + t_m = n > 1$. If $m > 1$, then $n \leq n!/(t_1! \cdots t_m!)$ and equality holds if and only if $t_1 = (n-1)$, $t_2 = 1$.*

Proof. Induction on n. □

Lemma 2.3. *For $x, y \in G$, $|\mathcal{C}_G(xy)| \leq |\mathcal{C}_G(x)||\mathcal{C}_G(y)|$.*

Lemma 2.4. *Let $\theta \in \mathrm{Aut}\,(\mathbf{Z}G)$. Then we have*

(i) *$g \in G \Rightarrow \theta(C_g) = C_x$ for some $x \in G$ with $0(g) = 0(x)$, $|\mathcal{C}(g)| = |\mathcal{C}(x)|$.*

(ii) *$\theta(C_g) = C_x \Rightarrow \theta(C_{g^k}) = \theta(C_{x^k})$ for all integers k.*

(iii) *If $\theta(C_g) = C_x$, $\theta(C_h) = C_y$ then for some t, $z \in G$, we have $\theta(C_{gh}) = C_{xy^t} = C_{x^z y}$.*

Proof. See [**3**]. □

Denote by $\zeta(G)$ the center of G and let θ be a normalized automorphism of $\mathbf{Z}G$.

Lemma 2.5. *If $a \in \zeta(H^n)$, then $\theta(C_a) = C_b$ for some $b \in \zeta(H^n)$.*

Proof. First observe that if $g \in G$ then $|\mathcal{C}_G(g)| = n$ if and only if $g \sim (z, z', \ldots, z')$ with $z, z' \in \zeta(H)$, $z \neq z'$. As a consequence, if we write $a = (z_1, \ldots, z_n)$, $z_i \in \zeta(H)$, then for all i,

$$C_{(z_i,1,\ldots,1)} \xrightarrow{\theta} C_{(z_i', s_i, \ldots, s_i)}, \quad \text{for some} \quad z_i', s_i \in \zeta(H).$$

Thus, since $(z_1, \ldots, z_n) = (z_1, 1, \ldots, 1)(1, z_2, \ldots, 1) \cdots (1, 1, \ldots, z_n)$, we have $C_a = C_{(z_1,\ldots,z_n)} \xrightarrow{\theta} C_b$ when $b = (z_1', s_1, \ldots, s_1)(z_2', s_2, \ldots, s_2)^{x_2} \cdots (z_n', s_n, \ldots, s_n)^{x_n} \in \zeta(H^n)$ by (2.4), proving the result. □

For a normal subgroup N of G, denote by $\Delta(G, N)$ the kernel of the natural map $\mathbf{Z}G \to \mathbf{Z}(G/N)$. Thus, $\Delta(G, N)$ is the ideal generated by all $1 - x$, $x \in N$. We have

Lemma 2.6. *Suppose that H is nilpotent and $\theta \in \operatorname{Aut} \mathbf{Z}G$, $G = H \operatorname{wr} S_n$. Then*

(i) $\theta(\Delta(G, H^n)) = \Delta(G, H^n)$.

(ii) *if* $\theta(C_{(a_1,\dots,a_n;\sigma)}) = C_{(b_1,\dots,b_n;\tau)}$ *then σ is conjugate to τ ($\sigma \sim \tau$).*

Proof. (i) We use induction on $|H|$. If $H = 1$, then $\Delta(G, 1) = \Delta(G)$, the augmentation ideal of $\mathbf{Z}G$, and $G = S_n$. Since θ is normalized we have nothing to prove. Now let $\zeta = \zeta(H)$, $\overline{H} = H/\zeta$, $\overline{G} = G/\zeta^n = (H/\zeta) \operatorname{wr} S_n = \overline{H} \operatorname{wr} S_n$. Let $h \in H^n$. We know by (2.5) that $\theta(\Delta(G, \zeta^n)) = \Delta(G, \zeta^n)$. This gives an induced automorphism $\bar{\theta}$ of $\mathbf{Z}\overline{G}$. Then, by induction, $\bar{\theta}(\bar{h} - 1) \in \Delta(\overline{G}, \overline{H}^n)$. So $\theta(h - 1) \in \Delta(G, H^n) + \Delta(G, \zeta^n) = \Delta(G, H^n)$ as desired.

(ii) Since θ induces an automorphism of $\mathbf{Z}S_n$ by (i), the claim follows by Peterson [**3**]. □

3. A consequence of a theorem of Weiss. We fix notation and recall some known facts. For an element $\alpha = \sum \alpha(g)g$ of a group ring RG of G over a commutative ring R, we set $\tilde{\alpha}(g) = \sum_{h \sim g} \alpha(h)$, the sum of coefficients of α over the conjugacy class of g. We denote by $[RG, RG]$ the additive group generated by the Lie products $[\alpha, \beta] = \alpha\beta - \beta\alpha$, α, $\beta \in RG$. Then

$$(3.1) \qquad [RG, RG] = \left\{ \sum_{x \in G} \alpha(x)x \in RG \mid \tilde{\alpha}(g) = 0 \text{ for all } g \in G \right\}.$$

Moreover, if we pick a set of representatives T of the conjugacy classes of G, then we have as modules

$$(3.2) \qquad RG/[RG, RG] \cong \sum_{x \in T}^{\oplus} Rx.$$

Let $M = (a_{ij}) \in (RG)_m$ be an $m \times m$ matrix over, the not necessarily commutative ring, RG. Then the trace, namely the sum of diagonal elements modulo $[RG, RG]$,

$$\operatorname{tr}(M) = \sum_i \overline{a_{ii}} \in RG/[RG, RG]$$

is well defined and has the expected properties

$$\operatorname{tr}(M_1 + M_2) = \operatorname{tr}(M_1) + \operatorname{tr}(M_2) \tag{3.3}$$

$$\operatorname{tr}(rM) = r\operatorname{tr}(M), \qquad r \in R \tag{3.4}$$

$$\operatorname{tr}(M_1 M_2) = \operatorname{tr}(M_2 M_1). \tag{3.5}$$

Now, let G be a split extension $A \rtimes X$ where $|X| = m$ and A is not necessarily abelian. Then there is an imbedding β of RG into $(RA)_m$. We shall describe this map. We have $RG = \sum_{i=1}^{m} RAx_i$, where $X = \{x_1, \dots, x_m\}$ and $x_1 = 1$. Let $u = \sum u(g)g \in RG$. Suppose

$$x_i u = \sum_j f_{ij}(u) x_j.$$

Then $u \xrightarrow{\beta} U = [f_{ij}(u)] \in (RA)_m$. In particular, for $a \in A$,

$$a \xrightarrow{\beta} \begin{bmatrix} a^{x_1} & & & \\ & a^{x_2} & & \\ & & \ddots & \\ & & & a^{x_m} \end{bmatrix}.$$

We write $u = \sum_g u(g)g \equiv \sum_{g \in T} \tilde{u}(g)g \bmod [RG, RG]$ by (3.1). We need a formula for $\operatorname{tr} U$:

Lemma 3.6. *Let T' be a set of class representatives of A. Then*

$$\operatorname{tr} U = \sum_{h \in T'} s_h \tilde{u}(h) h$$

where s_h is the index of the centralizers $(C_G(h) : C_A(h))$.

Proof. We have $u \equiv \sum_{g \in T} \tilde{u}(g)g \bmod [RG, RG]$. Further, $\operatorname{tr} \beta[RG, RG] \subseteq [RA, RA]$ as $\operatorname{tr} \beta[\alpha_1, \alpha_2] = \operatorname{tr} \beta(\alpha_1\alpha_2 - \alpha_2\alpha_1) = \operatorname{tr}(\beta(\alpha_1)\beta(\alpha_2) - \beta(\alpha_2)\beta(\alpha_1)) = 0$ by (3.5). To prove the lemma, it suffices to do so for $u = g \in G$, $U = \beta(g)$. Since both sides of the formula are zero for $g \notin A$, we may assume that $u = a \in A$. Then

$$u = a \to U = \begin{bmatrix} a^{x_1} & & \\ & a^{x_2} & \\ & & \ddots \end{bmatrix}$$

$x_1 = 1$ and $\operatorname{tr} U = \sum_{x\in X} a^x$. Suppose that $\{a^{x_1}, \dots, a^{x_{r_1}}\}$ are A-conjugates of a among the a^x, $x \in G$ and $\{a^{x_{r_1+1}}, \dots, a^{x_{r_1+r_2}}\}$ are A-conjugates of $a^{x_{r_1+1}}$ among the a^x, $x \in G$, etc. Then we write

$$\begin{aligned}\operatorname{tr} U &= (a + a^{x_2} + \cdots + a^{x_{r_1}}) + (a^{x_{r_1+1}} + \cdots + a^{x_{r_1+r_2}}) + \cdots \\ &= r_1 a + r_2 a^{x_{r_1+1}} + \cdots \\ &= r_1 \tilde{u}(a)a + r_2 \tilde{u}(a^{x_{r_1+1}})a^{x_{r_1+1}} + \cdots, \qquad \text{as } \tilde{u}(a^x) = \tilde{u}(a) = 1.\end{aligned}$$

It remains to observe that $r_1 = (C_G(a) : C_A(a))$ and similarly for the other r_i. This we do now.

(3.7) The number of conjugates of a fixed element $a \in A$ by $x \in X$ which are conjugate in A is $(C_G(a) : C_A(a))$.

Proof. It suffices to set up a 1–1 correspondence between $S_1 = \{$right cosets of $C_A(a)$ in $C_G(a)\}$ and $S_2 = \{x \in X \mid a^x = a^h \text{ some } h \in A\} = \{x \in X \mid x \in C_G(a)h \text{ some } h \in A\}$. Let $C_A(a)b \in S_1$. Write $b = x_1 h_1 \in C_G(a)$ with $x_1 \in X$, $h \in A$. Then set $\phi : C_A(a)b \to x_1$. Notice that $x_1 = bh_1^{-1} \in C_G(a)h_1^{-1}$ and thus $x_1 \in S_2$. One now verifies easily that indeed ϕ is 1–1 and onto. □

We need the strong form of a Theorem by Weiss [**6**]. Let ε be the augmentation map of a group ring. We may extend it to matrices over the group ring by applying it to the entries and call it ε^*. An invertible matrix M such that $\varepsilon^*(M)$ is the identity matrix is called an I-unit. The result then is

Lemma 3.8. *Let $\mathbf{Z}_p P$ be the group ring of a finite p-group P over the p-adic integers $\mathbf{Z}_p$. Suppose that Q is a finite p-group of I-units contained in $(\mathbf{Z}_p P)_m$. Then there exists a matrix $M \in (\mathbf{Z}_p P)_m$ such that every element $M^{-1}qM$, $q \in Q$ is a diagonal matrix $(g_1, \dots, g_m)$, $g_i \in P$.*

Lemma 3.9. *Let $G = P \rtimes X$ where P is a p-group. Suppose that θ is an automorphism of $\mathbf{Z}G$ such that $\theta(\Delta(G,P)) = \Delta(G,P)$. Then there is a group automorphism λ of P such that*

$$\theta(C_g) = C_{\lambda(g)} \qquad \textit{for all } g \in P.$$

Proof. Write $|X| = m$ and let β be as defined. We know $\theta(P) \in 1+\Delta(G,P)$. It follows by the definition of β that $\beta\Delta(G,P) \subseteq (\Delta P)_m$. Thus, $\beta\theta(P) \subseteq I+(\Delta P)_m$ and we can apply Weiss's theorem. It follows that there exists $M \in (\mathbf{Z}_p P)_m$ such that

$$M\beta\theta(P)M^{-1} \subseteq \begin{bmatrix} P & & & \\ & P & & \\ & & \ddots & \\ & & & P \end{bmatrix}. \tag{3.10}$$

For $g \in P$, we have $M\beta\theta(g)M^{-1} = \operatorname{diag}(g_1, \dots, g_m)$. Moreover,

$$\theta(g) = u = \sum_{x\in G} u(x)x \equiv \sum_{x\in T} \tilde{u}(x)x \mod [\mathbf{Z}G, \mathbf{Z}G].$$

We observe that $\operatorname{tr}(\beta\theta(g)) = \sum_{i=1}^m g_i$ and, by (3.6), $\operatorname{tr}(\beta\theta(g)) = \sum_{x\in T'} s_x\tilde{u}(x)x$, s_x natural numbers, T' representatives of classes of P.

Compare the two expressions for $\operatorname{tr}(\beta\theta(g))$ and use (3.2) to conclude that $\tilde{u}(x)$ are natural numbers for $x \in P$. Thus we can write

$$\theta(g) \equiv \sum_{x\in P} \tilde{u}(x)x + \sum_{x\notin P} \tilde{u}(x)x \mod [\mathbf{Z}G, \mathbf{Z}G].$$

Since $\theta(g) \in 1+\Delta(G,P)$, it follows that the augmentation contribution from the second sum, $\sum_{x\notin P} \tilde{u}(x)$, is zero. We conclude that there is a unique $\tilde{u}(x_0)$, $x_0 \in T$, $x_0 \in P$ which equals one and the other $\tilde{u}(x)$ for $x \in T$, $x \in P$ are zero. We have thus

$$\theta(g) = x_0 + \sum_{y\notin P} \tilde{u}(y)y + \lambda_0, \qquad \lambda_0 \in [\mathbf{Z}G, \mathbf{Z}G].$$

Writing $x^G = \sum_{g\in G} x^g$, we have

$$\begin{aligned} \theta(g^G) = \theta(g)^{\theta(G)} &= x_0^{\theta(G)} + \sum_{y\notin P} \tilde{u}(y)y^{\theta(G)} + \lambda_1, \qquad \lambda_1 \in [\mathbf{Z}G, \mathbf{Z}G] \\ &= |G|x_0 + \sum_{y\notin P} |G|\tilde{u}(y)y + \lambda_2, \qquad \lambda_2 \in [\mathbf{Z}G, \mathbf{Z}G]. \end{aligned}$$

The left hand side is a multiple of a class sum $\theta(C_g)$; so should be the right hand side. Also, $x_0 \in P$ and its class does not disappear from the right hand side. Therefore, $\theta(C_g) = C_{x_0}$. Moreover, $\sum_{y \notin P} |G|\tilde{u}(y)y \in [\mathbf{Z}G, \mathbf{Z}G]$ which implies that $\tilde{u}(y) = 0$ for all $y \notin P$. Hence, we may write

$$\theta(g) = x_0 + \lambda_0, \qquad \lambda_0 \in [\mathbf{Z}G, \mathbf{Z}G], \qquad \theta(C_x) = C_{x_0}.$$

We may replace x_0 by any element conjugate to it. We shall fix this choice by

$$M^{-1}\beta\theta(g)M = \begin{bmatrix} x_0 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \ddots \end{bmatrix},$$

namely, we take x_0 to be the first entry in the matrix $M^{-1}\beta\theta(g)M$. Define

$$\lambda : P \to P \quad \text{by} \quad \lambda(g) = x_0.$$

Then λ is a homomorphism. Suppose $\lambda(g) = 1$, then $\lambda(g) - 1 \in [\mathbf{Z}G, \mathbf{Z}G]$ and $\theta(g) = \sum \alpha_g g$, $\alpha_1 \neq 0$. It follows by [**4**, p. 45] that $\theta(g) = 1$, $g = 1$. Thus, λ is an automorphism of P and $C_{\lambda(g)} = C_{x_0} = \theta(C_g)$ for all $g \in P$ as desired. $\square$

4. Proof of the theorem. We need to find $\mu \in \mathrm{Aut}\,(G)$ such that $\theta(C_g) = C_{\mu(g)}$. This we proceed to do in a number of steps. We shall let θ denote a normalized automorphism of $\mathbf{Z}G$.

Lemma 4.1. *For $h \in H$, we have $\theta(C_{(h,1,\dots,1)}) = C_{(h',x,\dots,x)}$ for some h', $x \in H$.*

Proof. We may clearly assume that $h \neq 1$ and $n > 2$. Let $C_{(h,1,\dots,1)} \xrightarrow{\theta} C_{(h_1,\dots,h_n)}$; since $|\mathcal{C}_G(h,1,\dots,1)| = |\mathcal{C}_G(h_1,\dots,h_n)|$, by (2.1), we get

$$n|\mathcal{C}_H(h)| = \frac{n!}{t_1!\cdots t_m!}|\mathcal{C}_H(h_{t_1})|^{t_1}\cdots|\mathcal{C}_H(h_{t_m})|^{t_m}.$$

If $m = 1$, then $h_1 \sim h_2 \sim \cdots \sim h_n$ and we are done. If $m > 1$, then $n!/(t_1! \cdots t_m!) \geq n$ and this implies $|\mathcal{C}_H(h)| \geq |\mathcal{C}_H(h_{t_1})|^{t_1} \cdots |\mathcal{C}_H(h_{t_m})|^{t_m}$. If we prove that $|\mathcal{C}_H(h)| = |\mathcal{C}_H(h_{t_1})|^{t_1} \cdots |\mathcal{C}_H(h_{t_m})|^{t_m}$, then $n!/(t_1! \cdots t_m!) = n$, and so, by (2.2), it will follow that $t_1 = n-1$, $t_2 = 1$, $t_3 = \cdots = t_n = 0$, the desired conclusion. The proof will be by induction on $|\mathcal{C}_H(h)|$. If $|\mathcal{C}_H(h)| = 1$, then $h \in \zeta(H)$ and, by Lemma 2.5, $|\mathcal{C}_H(h_1)| = \cdots = |\mathcal{C}_H(h_n)| = 1$. Since $|\mathcal{C}_G(h, 1, \ldots, 1)| = n$, we are done in this case. Suppose now that $|\mathcal{C}_H(h)| > 1$ and assume, by contradiction, that for some $\theta \in \operatorname{Aut}(\mathbf{Z}G)$, $m > 1$ and

$$|\mathcal{C}_H(h)| \gneqq |\mathcal{C}_H(h_{t_1})|^{t_1} \cdots |\mathcal{C}_H(h_{t_m})|^{t_m}.$$

This implies that $|\mathcal{C}_H(h_{t_i})| \lneqq |\mathcal{C}_H(h)|$, for all i; hence, by induction, $C_{(h_{t_i},1,\ldots,1)} \stackrel{\eta}{\to} C_{(h'_{t_i},x_{t_i},\ldots,x_{t_i})}$ for all i, for all $\eta \in \operatorname{Aut}(\mathbf{Z}G)$, where h'_{t_i}, x_{t_i} depend upon η. Thus, in particular, by taking $\eta = \theta^{-1}$, one gets:

$$\begin{aligned} C_{(h,1,\ldots,1)} &\xrightarrow{\theta} C_{(h_1,\ldots,h_n)} = C_{(h_1,1,\ldots,1)(1,h_2,1,\ldots,1)\cdots(1,\ldots,1,h_n)} \\ &\xrightarrow{\theta^{-1}} C_{(h'_1,x_1,\ldots,x_1)(h'_2,x_2,\ldots,x_2)^{k_2}\cdots(h'_n,x_n,\ldots,x_n)^{k_n}} \end{aligned}$$

for suitable elements h'_i, $x_i \in H$, $k_i \in H^n$. Thus,

$$(h'_1, x_1, \ldots, x_1)(h'_2, x_2, \ldots, x_2)^{k_2} \cdots (h'_n, x_n, \ldots, x_n)^{k_n} \sim (h, 1, \ldots, 1)$$

and this implies that $u_1 u_2 \cdots u_n \sim h$ where, for all i, either $u_i \sim h'_i$ or $u_i \sim x_i$. Also, notice that, since $C_{(h_i,1,\ldots,1)} \stackrel{\theta^{-1}}{\to} C_{(h'_i,x_i,\ldots,x_i)}$, then $|\mathcal{C}_G(h_i, 1, \ldots, 1)| = |\mathcal{C}_G(h'_i, x_i, \ldots, x_i)|$. Now, if $h'_i \sim x_i$, then we get $n|\mathcal{C}_H(h_i)| = |\mathcal{C}_H(x_i)|^n$, whereas if $h'_i \not\sim x_i$, then $|\mathcal{C}_H(h_i)| = |\mathcal{C}_H(h'_i)||\mathcal{C}_H(x_i)|^{n-1}$. In any case, since $n > 2$, $|\mathcal{C}_H(u_i)| \leq |\mathcal{C}_H(h_i)|$, for all i. From the above relations, we obtain:

$$\begin{aligned} |\mathcal{C}_H(h)| &= |\mathcal{C}_H(u_1 u_2 \cdots u_n)| \leq |\mathcal{C}_H(u_1)| \cdots |\mathcal{C}_H(u_n)| \\ &\leq |\mathcal{C}_H(h_1)| \cdots |\mathcal{C}_H(h_n)| = |\mathcal{C}_H(h_{t_1})|^{t_1} \cdots |\mathcal{C}_H(h_{t_m})|^{t_m}, \end{aligned}$$

a contradiction. □

Lemma 4.2. *Let $n > 2$, and suppose that for every $\theta \in \operatorname{Aut}(\mathbf{Z}G)$ and $h \in H$, $\theta(C_{(h,\ldots,h)}) = C_{(xz_1, xz_2, \ldots, xz_m)}$, for some $x \in H$, $z_i \in \zeta(H)$. Then $\theta(C_{(h,1,\ldots,1)}) = C_{(h',z,\ldots,z)}$ where $h' \in H$, $z \in \zeta(H)$.*

Proof. By the previous lemma, $C_{(h,1,\dots,1)} \overset{\theta}{\to} C_{(h',x,\dots,x)}$; notice that if $h' \sim x$, then $C_{(x,\dots,x)} \overset{\theta^{-1}}{\to} C_{(h,1,\dots,1)} = C_{(yz_1,\dots,yz_n)} \Rightarrow yz_i = 1$ for some $i \Rightarrow y \in \zeta(H) \Rightarrow$ by Lemma 4.1, $x \in \zeta(H)$ and we are done. Thus, we may assume that $h' \not\sim x$, and so, since $|\mathcal{C}_G(h,1,\dots,1)| = |\mathcal{C}_G(h',x,\dots,x)|$, we get $|\mathcal{C}_H(h)| = |\mathcal{C}_H(h')||\mathcal{C}_H(x)|^{n-1}$. The proof will be by induction on $|\mathcal{C}_H(h)|$. If $|\mathcal{C}_H(h)| = 1$, then $h \in \zeta(H)$ and, by Lemma 4.1, h', $x \in \zeta(H)$, proving the lemma. Suppose $|\mathcal{C}_H(h)| > 1$ and write $(h',x,\dots,x) = (h'x^{-1},1,\dots,1)(x,x,\dots)$. Since $n > 2$, by applying (2.3), we get

$$\begin{aligned}|\mathcal{C}_H(h'x^{-1})| &\le |\mathcal{C}_H(h')||\mathcal{C}_H(x^{-1})| = |\mathcal{C}_H(h')||\mathcal{C}_H(x)| \\ &\lneqq |\mathcal{C}_H(h')||\mathcal{C}_H(x)|^{n-1} = |\mathcal{C}_H(h)|.\end{aligned}$$

Hence, by induction we can write

$$C_{(h'x^{-1},1,\dots,1)} \overset{\theta^{-1}}{\to} C_{(h'',z,\dots,z)} \quad \text{for some } z \in \zeta(H), h'' \in H.$$

Also, by hypothesis, $C_{(x,\dots,x)} \overset{\theta^{-1}}{\to} C_{(yz_1,\dots,yz_n)}$ where $y \in H$, $z_i \in \zeta(H)$. Thus,

$$C_{(h',x,\dots,x)} = C_{(h'x^{-1},1,\dots,1)(x,\dots,x)} \overset{\theta^{-1}}{\to} C_{(h'',z,\dots,z)(yz_1,\dots,yz_n)^a}.$$

It follows that $(h,1,\dots,1) \sim (h'',z,\dots,z)(yz_1,\dots,yz_n)^a = (h'',z,\dots,z)(y^{a_1}z_{i_1},\dots,y^{a_n}z_{i_n}) = (h''y^{a_1}z_{i_1}, y^{a_2}zz_{i_2},\dots,y^{a_n}zz_{i_n})$ where $a_1,\dots,a_n \in H$ and $\{i_1,\dots,i_n\}$ is a permutation of $\{1,\dots,n\}$. Hence, since $n > 2$, $y^{a_j}zz_{i_j} = 1$, for some j; this forces $y \in \zeta(H)$, and so, since $C_{(x,\dots,x)} \overset{\theta^{-1}}{\to} C_{(yz_1,\dots,yz_n)}$, by Lemma 2.5, $x \in \zeta(H)$. This completes the proof of the lemma. □

Lemma 4.3. *Let $n > 2$ and suppose that H is a nilpotent group. Then, for all $\theta \in \mathrm{Aut}\,(\mathbf{Z}G)$ and $h \in H$,*

$$\theta(C_{(h,\dots,h)} = C_{(x,\dots,x)}, \quad \textit{for some } x \in H.$$

Proof. The proof will be by induction on $|H|$. If $|H| = 1$, the lemma is trivially true. Suppose then that $|H| > 1$. By looking at class orders, one immediately checks the lemma when $h \in \zeta(H)$. Suppose then that $h \notin \zeta(H) = \zeta$.

If $\theta \in \mathrm{Aut}\,(\mathbf{Z}G) = \mathrm{Aut}\,(H \mathrm{wr}\, S_n)$, then θ induces an automorphism $\bar{\theta}$ of $\mathbf{Z}[(H\mathrm{wr}\, S_n)/(\zeta^n \mathrm{wr}\,\{1\})] \cong \mathbf{Z}[H/\zeta \mathrm{wr}\, S_n]$: if $C_{(h,\dots,h)} \xrightarrow{\theta} C_{(x_1,\dots,x_n)}$ and $- : H \to H/\zeta$ is the projection map, then $C_{(\bar{h},\dots,\bar{h})} \xrightarrow{\bar{\theta}} C_{(\bar{x}_1,\dots,\bar{x}_n)}$; also, since $|H/\zeta| \lneqq |H|$, by the inductive hypothesis $C_{(\bar{h},\dots,\bar{h})} \xrightarrow{\bar{\theta}} C_{(\bar{y},\dots,\bar{y})}$ for some $\bar{y} \in H/\zeta$. It follows that $(\bar{x}_1,\dots,\bar{x}_n) \sim (\bar{y},\dots,\bar{y})$; hence, $\bar{x}_1 \sim \bar{x}_2 \sim \cdots \sim \bar{x}_n \sim \bar{y}$ and this says that there exist $z_1,\dots,z_n \in \zeta$ such that $x_i \sim yz_i$ for $i = 1,\dots,n$. Thus, $C_{(h,\dots,h)} \xrightarrow{\theta} C_{(yz_1,\dots,yz_n)}$, $z_i \in \zeta$. By the previous lemma, then $C_{(h,1,\dots,1)} \xrightarrow{\theta} C_{(h',z,\dots,z)}$ where $h' \in H$, $z \in \zeta$. Thus

$$C_{(h,\dots,h)} = C_{(h,1,\dots,1)(1,h,1,\dots,1)(1,\dots,1,h)} \xrightarrow{\theta} C_{(h',z,\dots,z)(h',z,\dots,z)^{a_2}\cdots(h',z,\dots,z)^{a_n}}.$$

It follows that

$$(h',z,\dots,z)(h',z,\dots,z)^{a_2}\cdots(h',z,\dots,z)^{a_n} \sim (yz_1,\dots,yz_n).$$

Now, if for some i, $yz_i \sim z^n$, then $y \in \zeta$ and by Lemma 2.5, $h \in \zeta$, a contradiction. Thus, $yz_i \not\sim z^n$, for all i. It follows that $yz_i \sim h'^{b_j} z^{n-1}$ for all i, for some $b_j \in H$. Thus, for all i,k, $yz_i \sim h'^{b_j} z^{n-1} \sim h'^{b_l} z^{n-1} \sim yz_k$; this says that $(yz_1,\dots,yz_n) \sim (yz_1,\dots,yz_1)$ and the lemma is proved. □

A consequence of the last two lemmas is the

Corollary. *Let $n > 2$ and suppose that H is a nilpotent group. Then, for all $\theta \in \mathrm{Aut}\,(\mathbf{Z}G)$ and $h \in H$, $\theta(C_{(h,1,\dots,1)}) = C_{(h',z,\dots,z)}$, for some $h' \in H$, $z \in \zeta(H)$.*

We recall

Lemma 3.9. *Let H be a p-group. If $\theta \in \mathrm{Aut}\,(\mathbf{Z}G)$, then there exists $\lambda \in \mathrm{Aut}\,(H^n)$ such that if $\theta(C_{(h_1,\dots,h_n)}) = C_{(x_1,\dots,x_n)}$, then $\lambda(h_1,\dots,h_n) = (y_1,\dots,y_n)$ where $(y_1,\dots,y_n) \sim (x_1,\dots,x_n)$.*

Now let H be a p-group, $\theta \in \mathrm{Aut}\,(\mathbf{Z}G)$ and $n > 2$. If $h \in H$, by the Corollary above, we can write $\theta(C_{(h,1,\dots,1)}) = C_{z_h(k,1,\dots,1)}$ where $k \in H$ and $z_h \in \zeta(H)$. Then, by (3.9), there exists $\lambda \in \mathrm{Aut}\,(H^n)$ such that $\lambda(h,1,\dots,1) = z_h(1,\dots,1,\underset{i}{h'},1,\dots,1)$ where $h' \sim k$.

We claim that i is independent of the element h; in fact, let $x, y \in H$, $x \neq 1$, $y \neq 1$ and suppose by contradiction that $\lambda(x,1,\dots,1) = z_x(1,\dots,1,\underset{i}{x'},1,\dots,1)$ and $\lambda(y,1,\dots,1) = z_y(1,\dots,1,\underset{j}{y'},1,\dots,1)$ with $i \lneqq j$. Since λ is a homomorphism we get

$$z_{xy}(1,\dots,1,(xy)',1,\dots,1) = z_x z_y(1,\dots,1,\underset{i}{x'},1,\dots,1,\underset{j}{y'},1,\dots,1).$$

Since $n > 2$, $z_{xy} = z_x z_y$ and, therefore, either $x' = 1$ or $y' = 1$. Suppose $x' = 1$; since $\theta(C_{(x,1,\dots,1)}) = C_{z_x}$, by Lemma 4.1, $x = 1$, a contradiction; same for y.

Let η be the automorphism of H^n which switches the first and the i^{th} component of H^n. By working with $\eta \circ \lambda$ instead of λ, we may assume that

$$\lambda(h,1,\dots,1) = z_h(h',1,\dots,1)$$

for all $h \in H$, where $z_h \in \zeta(H)$ and $h' \in H$.

Let now $\mu : H^n \to H^n$ be the map defined by $\mu(h_1,h_2,\dots,h_n) = z_{h_1}z_{h_2}\cdots z_{h_n}(h'_1,h'_2,\dots,h'_n)$ where $\lambda(h_i,1,\dots,1) = z_{h_i}(h'_i,1,\dots,1)$. Since λ is a homomorphism, μ is also a homomorphism. Also, μ is injective; in fact, suppose $z_{h_1}\cdots z_{h_n}(h'_1,\dots,h'_n) = (1,\dots,1)$; then $z_{h_1}\cdots z_{h_n}h'_i = 1$ for all i and this forces $h'_1 = \cdots = h'_n = h' \in \zeta(H)$. Hence, because $\lambda \in \mathrm{Aut}\,(H^n)$, $z_{h_1} = \cdots = z_{h_n} = z_h$ and $z_h^n h' = 1$. Now, since $\lambda(h,1,\dots,1) = z_h(h',1,\dots,1)$, then $C_{(h,1,\dots,1)} \overset{\theta}{\to} C_{z_h(h',1,\dots,1)}$, it follows that $C_{(h,h,\dots,h)} = C_{(h,1,\dots,1)(1,h,1,\dots,1)\cdots(1,\dots,1,h)} \overset{\theta}{\to} C_{z_h}(h',1,\dots,1)z_h(h',1,\dots,1)^{a_2}\cdots z_h(h',1,\dots,1)^{a_n}$. Since, by Lemma 4.4, $C_{(h,\dots,h)} \overset{\theta}{\to} C_{(x,\dots,x)}$, for some $x \in H$, and $h' \in \zeta(H)$ we get $z_h^n(h',\dots,h') \sim (x,\dots,x)$. Thus, since $z_h^n h' = 1$, $x = 1$. This implies that $h = 1$ and the claim is established. We have proved that $\mu \in \mathrm{Aut}\,(H^n)$.

We now extend μ to an automorphism of $H \operatorname{wr} S_n$ by defining $\bar{\mu}(h_1,\dots,h_n;\sigma) = \mu(h_1,\dots,h_n)\cdot(1,\dots,1;\sigma)$. It is easy to check that $\bar{\mu} \in \mathrm{Aut}\,(H \operatorname{wr} S_n) = \mathrm{Aut}\,(G)$. Also, for each $h \in H$,

$$\bar{\mu}^{-1} \circ \theta(C_{(h,1,\dots,1)}) = \bar{\mu}^{-1}(C_{z_h(h',1,\dots,1)}) = C_{(h,1,\dots,1)}.$$

Therefore, by working with $\bar{\mu}^{-1}\cdot\theta$ instead of θ, we will assume from now on that $\theta(C_{(h,1,\dots,1)}) = C_{(h,1,\dots,1)}$ for all $h \in H$.

Lemma 4.4. *If $n > 2$ and H is a p-group, then* $\theta(C_{(h_1,\dots,h_n)}) = C_{(h_1,\dots,h_n)}$.

Proof. Let s be such that for all $h_1,\dots,h_s \in H$, $\theta(C_{(h_1,\dots,h_s,1,\dots,1)}) = C_{(h_1,\dots,h_s,1,\dots,1)}$ and $\theta^{-1}(C_{(h_1,\dots,h_s,1,\dots,1)}) = C_{(h_1,\dots,h_s,1,\dots,1)}$. The proof will be by induction on s.

If $s = 1$, this is the previous result. So, suppose $s > 1$ and let $h_1,\dots,h_s \in H$. Then, by induction,

$$\begin{aligned} C_{(h_1,\dots,h_s,1,\dots,1)} &= C_{(h_1,\dots,h_{s-1},1,\dots,1)(1,\dots,1,h_s,1,\dots,1)} \\ &\overset{\theta}{\longrightarrow} C_{(h_1,\dots,h_{s-1},1,\dots,1)(1,\dots,1,h_s^x,1,\dots,1)} \end{aligned}$$

for some $x \in H$. Now if $(h_1,\dots,h_{s-1},1,\dots,1)(1,\dots,h_s^x,\dots,1) \sim (h_1,\dots,h_s,1,\dots,1)$, we are done; so suppose $(h_1,\dots,h_{s-1},1,\dots,1)\cdot(1,\dots,h_s^x,\dots,1) \sim (h_1,\dots,h_ih_s^x,\dots,h_{s-1},1,\dots,1)$ for some i, $1 \le i \le s-1$. But then, by induction, $C_{(h_1,\dots,h_ih_s^x,\dots,h_{s-1},1,\dots,1)} \overset{\theta^{-1}}{\to} C_{(h_1,\dots,h_ih_s^x,\dots,h_{s-1},1,\dots,1)}$ which implies that $(h_1,\dots,h_ih_s^x,\dots,h_{s-1},1,\dots,1) \sim (h_1,\dots,h_s,1,\dots,1)$, a contradiction. $\square$

We have proved that $\theta(C_a) = C_a$ for all $a \in H^n$, after θ has been modified by a suitable automorphism of G. It follows by the Proposition in Valenti [**5**] that $\theta(C_g) = C_g$ for all $g \in G$. The theorem is proved by [**4**, Proposition III.7.2].

Added in proof. The case $n = 2$ has now been proved by M. Parmenter and S.K. Sehgal and will appear in the same journal.

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Dipartimento di Matematica, Universitá di Palermo, 90123 Palermo, Italy

Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1