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SUBGROUP SEPARABILITY OF CERTAIN HNN EXTENSIONS

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ABSTRACT. We show that certain HNN extensions are subgroup separable and then apply the result to get a characterization for the Baumslag-Solitar groups to be subgroup separable and some other results.

1. The residual finiteness and hopficity of the one-relator groups $G_{k,l} = \langle t, a; t^{-1}a^k t = a^l\rangle$, now called the Baumslag-Solitar groups, were exhaustively studied and completely characterized by Baumslag and Solitar [**2**], Meskin [**7**] and Collins and Levin [**3**]. Their results can be summarized as follows:

Theorem 1. *Let $G_{k,l} = \langle t, a; t^{-1}a^k t = a^l\rangle$. Then $G_{k,l}$ is residually finite if and only if $|k| = 1$ or $|l| = 1$ or $|k| = |l|$ and $G_{k,l}$ is hopfian if and only if $|k| = 1$ or $|l| = 1$ or $\pi(k) = \pi(l)$, where $\pi(n)$, for a nonzero integer n, denotes the set of prime divisors of n.*

In the note we shall characterize the groups $G_{k,l}$ with regards to subgroup separability. We shall prove the following:

Theorem 2. *Let $G_{k,l} = \langle t, a; t^{-1}a^k t = a^l\rangle$. Then $G_{k,l}$ is subgroup separable if and only if $|k| = |l|$.*

Theorem 2 will follow from Theorems 1, 3 and 4. Theorem 3, which is our main result, partially extends Theorem 1 of Andreadakis, Raptis and Varsos [**1**].

The notations used here are standard. In addition, the following will be used. Let G be a group.

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$N \triangleleft_f G$ means N is a normal subgroup of finite index in G.

$\langle g \rangle$ means the cyclic subgroup generated by the element g in G.

f.g. means finitely generated.

s.s. means subgroup separable.

$G = \langle t, K; t^{-1}At = B, \varphi \rangle$ denotes the HNN extension where K is the base group, A, B are the associated subgroups and φ is the associated isomorphism $\varphi : A \to B$.

Finally recall that a group is subgroup separable if for each f.g. subgroup M and for each $x \in G \backslash M$, there exists $N \triangleleft_f G$ such that $xM \cap N = \phi$. It is well known that polycyclic groups (and hence f.g. abelian groups) are s.s. (Mal'cev [**6**]).

2. We prove Theorem 3 in this section. We begin with a lemma which will be used in the proof of Theorem 3.

Lemma. *Let $G = \langle t, K; t^{-1}At = B, \varphi \rangle$ be an* HNN-*extension where K is a finite group. Then G is subgroup separable.*

Proof. The group G is free-by-finite (Hall [**4**], Karass, Pietrowski and Solitar [**5**]). But free groups are s.s. (Hall [**4**]) and finite extension of s.s. groups are again s.s. (Romanovski [**8**], Scott [**9**]). Hence, G is s.s. □

Now we prove Theorem 3.

Theorem 3. *Let $G = \langle t, K; t^{-1}At = B, \varphi \rangle$ be an* HNN *extension where K is a finitely generated abelian group and A, B have finite index in K. If there exists a subgroup H of finite index in K and H is normal in G, then G is subgroup separable.*

Proof. Let M be an f.g. subgroup of G and $x \in G \backslash M$. If $x \notin MH$, then $xH \notin MH/H$. Now $G/H \simeq \langle t, K/H; t^{-1}(A/H)t = (B/H), \bar{\varphi} \rangle$ where $\bar{\varphi} : (A/H) \to (B/H)$ is the isomorphism induced by φ. Therefore, G/H is s.s. by the Lemma. Thus, there exists $N/H \triangleleft_f G/H$ such that $xH(MH/H) \cap N/H = \phi$, namely there exists

$N \lhd_f G$ such that $xM \cap N = \phi$.

Suppose that $x \in MH$. Then $x = mh$, $m \in M$, $h \in H$ but $h \notin H \cap M$ (since $x \notin M$). Now H and $H \cap M$ are f.g. abelian. Since H is s.s. (Mal'cev [**6**]), there exists a characteristic subgroup R of H of finite index in it such that $h(H \cap M) \cap R = \phi$. If $xR \in MR/R$, then $x = mh = m_1 r$, $m_1 \in M$, $r \in R$. Hence $hr^{-1} = m^{-1}m_1 \in H \cap M$ (since $R < H$) and so $h(H \cap M) \cap R \neq \phi$, a contradiction. So $xR \notin MR/R$. Now, by the Lemma, the group G/R is s.s. So we can argue, as before, with R in place of H and find $N \lhd_f G$ such that $xM \cap N = \phi$. This completes the proof of the theorem. □

3. We complete the proof of Theorem 2 by proving Theorem 4 in this section.

Theorem 4. *Let* $G = \langle t, a; t^{-1}at = a^m \rangle$, $|m| \neq 1$. *Then* G *is not subgroup separable.*

Proof. Clearly $a \notin \langle a^m \rangle$ in G. Let $G\psi$ denote a homomorphic image of G of order n. Then $a\psi = t^{-n}\psi a\psi t^n \psi = a^{m^n}\psi \in \langle a^m \psi \rangle$. So G is not s.s. □

4. We show other applications of Theorems 3 and 4 in this section.

Corollary. *Let* $G = \langle t, K; t^{-1}At = A, \varphi \rangle$ *be an* HNN *extension where* K *is a finitely generated abelian group,* $K \neq A$ *and* A *has finite index in* K. *Then* G *is subgroup separable.*

Proof. This follows directly from Theorem 3. □

Theorem 5. *Let*

$$G = \langle t, a_1, a_2, \dots, a_n; t^{-1}a_i^{d_i}t = a_i^{k_i}, i = 1, 2, \dots, n, [a_i, a_j] = 1 \rangle$$

where $d_i, k_i \neq 0$, $i = 1, 2, \dots, n$. *Then* G *is subgroup separable if and only if* $|d_i| = |k_i|$, $i = 1, 2, \dots, n$.

Proof. This follows directly from Theorems 3, 4 above and Corollary

3 of Andreadakis, Raptis and Varsos [**1**]. □

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