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STURMIAN THEORY FOR NONSELFADJOINT SYSTEMS

E.C. TOMASTIK

ABSTRACT. The theory of μ_0-positive operators is used to systematically develop the Sturmian properties of the second order system (1) $(r(t)x')' + q(t)x = 0$, where $r(t)$ and $q(t)$ are $n \times n$ matrices of continuous functions on $[a, b]$. Since no symmetry assumptions are made on either of the matrices $r(t)$ or $q(t)$, (1) will in general be nonselfadjoint. However, all results are new even if (1) is selfadjoint. It is assumed that $r^{-1}(t)$ and $q(t)$ are *positive* with respect to some cone, K, in Euclidean space with nonempty interior K^0. With some additional assumptions on $r(t)$, the following basic result is given. If b is the first conjugate point to a, then there exists a unique (up to multiplication by a constant) nontrivial solution, $x(t)$, to (1) with $x(a) = 0 = x(b)$ and $x(t) \in K^0$ on (a, b).

1. Introduction. In this paper the theory of μ_0-positive operators defined on a Banach space equipped with a cone is used to develop certain Sturmian properties of the system of second order differential equations

$$(r(t)x')' + q(t)x = 0, \tag{1}$$

where $r(t)$ and $q(t)$ are $n \times n$ matrices of continuous functions on $[a, b]$, $a \geq 0$, and $r(t)$ is nonsingular for all $t \in [a, b]$ and $\int_a^t r^{-1}(s)\,ds$ is nonsingular for all $t \in (a, b]$. Since no symmetry assumptions are made on either of the matrices $r(t)$ or $q(t)$, (1) will in general be nonselfadjoint. However, all results presented here are new even if (1) is selfadjoint.

Equation (1) with $r(t) \equiv E$, the identity matrix, has been studied recently by a number of people (see [**1**–**12**, **14**, **16**–**21**]). It needs to be emphasized, however, that nobody has obtained results for conjugate points for the more general equation (1). Keener and Travis [**10**] used μ_0-positive operators to study conjugate and focal points of (1) when

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$r(t) \equiv E$ and the author [**19, 21**] used μ_0-positive operators to study focal points of (1) in the general case.

Throughout this paper it is assumed that some cone, K, in the Euclidean space R^n, with nonempty interior, has been given. One thinks of this cone as the *positive* cone.

Also, throughout this paper it is assumed that $r^{-1}(t)$ and $q(t)$ satisfy the following *positivity* condition. For all $t \in [a,b]$,

$$r^{-1}(t) : K^0 \to K^0, \qquad q(t) : K \to K;$$

and given any $\alpha < \beta$ with $(\alpha, \beta) \subset [a,b]$, there exists $\tau \in (\alpha, \beta)$ such that

$$q(\tau) : K - \{0\} \to K^0,$$

where K^0 denotes the interior of K. A further condition on $r(t)$ is given later.

Notice that all these conditions permit $r(t)$ to be the identity matrix. Also notice that the hypothesis on $q(t)$ is not quite as restrictive as that found in [**10**], and thus the results are new when $r(t) \equiv E$.

A number of authors, when considering (1) with $r(t) \equiv E$, have assumed that all the elements of $q(t)$ are nonnegative. This is the case when the cone K is the first quadrant. A more general cone than this (but along these same lines) is to assume that some partition of $\{I, J\}$ of the integers $\{1, \dots, n\}$ has been given, i.e., $I \cup J = \{1, \dots, n\}$ with $I \cap J = \phi$, and that the set K is given by

$$K = \{(z_1, \dots, z_n) : i \in I \Longrightarrow z_i \geq 0,\ i \in J \Longrightarrow z_i \leq 0\}. \tag{2}$$

Then if $q = (q_{ij})$, one can assume that for any point at which $q_{ij} \neq 0$ that sign $\{q_{ij}\} = \delta_i \delta_j$ where $\delta_i = 1$ if $i \in I$ and $\delta_i = -1$ if $i \in J$, (see [**19–21**]). (In this context, one obtains the first quadrant by having $J = \phi$.) Notice that for this type of cone, none of the elements of $q(t)$ can ever be identically zero on any subinterval or change sign. These latter two facts follow from the simple observation that if e_j is the j-th unit basis vector, then $qe_j = (q_{1j}, \dots, q_{nj})^*$ where "$*$" indicates transpose.

But notice that, in general, the cone K may overlap quadrants and some of the components of r^{-1} or q may oscillate, unlike the examples given in the previous paragraph.

A point $c(a) \in (a, b]$ is called the (first) conjugate point of a relative to (1) if there exists a nontrivial solution $x(t)$ of (1) such that $x(a) = x(c(a)) = 0$, and there is no nontrivial solution $z(t)$ of (1) with $z(a) = z(\beta) = 0$ with $a < \beta < c(a)$. If (1) does not possess such a conjugate point on $(a, b]$, then (1) is said to be disconjugate on $[a, b]$.

Consider now the differential operator

$$D(t) = -(r(t)x'(t))'$$

subject to the conjugate point boundary condition

$$x(a) = x(b) = 0. \tag{3}$$

It is easy to see that the Green's matrix for this differential operator subject to the boundary condition (3) is given by

$$g(b,t,s) = \begin{cases} \int_t^b r^{-1}(\xi)\,d\xi(\int_a^b r^{-1}(\xi)\,d\xi)^{-1}\int_a^s r^{-1}(\xi)\,d\xi, \\ \qquad a \le s \le t \le b \\ \int_a^t r^{-1}(\xi)\,d\xi(\int_a^b r^{-1}(\xi)\,d\xi)^{-1}\int_s^b r^{-1}(\xi)\,d\xi, \\ \qquad a \le t \le s \le b \end{cases} \tag{4}$$

It has been assumed that $\int_a^t r^{-1}(\xi)\,d\xi$ is nonsingular for $t \in (a, b]$. This assumption is necessary since there exist examples where $r(t)$ is nonsingular on $[a, b]$ and $r^{-1}(t)$ maps K^0 into K^0 and even $r(t)$ is symmetric but that $\int_a^b r^{-1}(\xi)\,d\xi$ is singular. There is one important case where this is not so. If $r(t)$ is symmetric and positive definite on $[a, b]$, then it is very easy to see that $\int_a^t r^{-1}(\xi)\,d\xi$ is positive definite and thus nonsingular for $t > a$.

Let $\mathcal{B}$ be a real Banach space and $\mathcal{K}$ a (positive) cone in $\mathcal{B}$. We say that $u \le v$ if $v - u \in \mathcal{K}$. A bounded linear operator $L : \mathcal{B} \to \mathcal{B}$ is said to be μ_0-positive with respect to the cone $\mathcal{K}$ provided there exists a nonzero element $\mu_0 \in \mathcal{K}$ such that for every nonzero element $u \in \mathcal{K}$, there exist positive constants k_1 and k_2 and a positive integer ν such that

$$k_1\mu_0 \le L^\nu u \le k_2\mu_0$$

with respect to the cone $\mathcal{K}$. The following is a fundamental result on μ_0-positive operators (cf. [**13**]).

Theorem 1. *If L is a compact μ_0-positive linear operator with respect to the cone $\mathcal{K}$, then L has exactly one (normalized) eigenvector in $\mathcal{K}$ and the corresponding eigenvalue is simple, positive, and larger than the absolute value of any other eigenvalue.*

2. Additional hypotheses on $r(t)$. One critical fact about the Green's matrix, $g(b,t,s)$, that will be needed for the proofs given here to work is that $g(b,t,s)$ must map K^0 into K^0. It is natural to wonder if $r^{-1}(t) : K^0 \to K^0$ for all $t \in [a,b]$, then does $g(b,t,s)$ also? Examples indicate that the answer is no. Additional hypotheses will be needed on r^{-1}.

Define the sets K_t and D_t by

$$K_t = \left(\int_a^t r^{-1}(\xi)\, d\xi \right)(K),$$
$$D_t = \left(\int_a^t r^{-1}(\xi)\, d\xi \right)^{-1}(K).$$

The following condition on $r(t)$ will be assumed throughout the rest of this paper:

$$K_t \subset K_b, \qquad D_b \subset D_t \qquad \text{for all } t \in (a,b].$$

Examples indicate that these two conditions are independent of each other.

Also notice that if $r(t)$ is a constant matrix, then $K_t \equiv K_b$ and $D_t \equiv D_b$ for $t \in (a,b]$; thus, these two conditions are trivially true in this case.

As Lemma 5 indicates, the above conditions are sufficient to assure that $g(b,t,s)$ maps K^0 into K^0 for all $s,t \in (a,b)$. The point here is that these conditions are easily checked for many standard cones. For example, set $n = 2$ and K as the first quadrant. Then ∂K_t, the boundary of K_t, is determined by $(\int_a^t r^{-1}(\xi)\, d\xi)e_1$ and $(\int_a^t r^{-1}(\xi)\, d\xi)e_2$, and these vectors are just the columns of $\int_a^t r^{-1}(\xi)\, d\xi$. Also, in this case, ∂D_t is just determined by the columns of $(\int_a^t r^{-1}(\xi)\, d\xi)^{-1}$.

The following example further illustrates these points. Let

$$r^{-1}(t) = \begin{pmatrix} 1+2t & 1 \\ 1 & 2+2t \end{pmatrix}$$

and let $a = 0$, $b = 1$, and let K be the first quadrant. Then

$$\int_0^t r^{-1}(\xi)\, d\xi = t \begin{pmatrix} 1+t & 1 \\ 1 & 2+t \end{pmatrix}$$

and K_t is the cone in the first quadrant bounded by the two rays determined by the vectors $(1+t,1)^*$ and $(1,2+t)^*$. It follows that $K_t \subset K_1$ for all $t \in (0,1]$. Furthermore,

$$t^2(1+3t+t^2)\bigg(\int_0^t r^{-1}(\xi)\, d\xi\bigg)^{-1} = \begin{pmatrix} 2+t & -1 \\ -1 & 1+t \end{pmatrix}$$

and D_t is the region that includes K and is bounded by the two rays determined by the vectors $(2+t,-1)^*$ and $(-1,1+t)^*$. It follows that $D_1 \subset D_t$ for all $t \in (0,1]$.

3. Sturmian theory.

Consider now the integral operator

$$(Lx)(t) = \int_a^b g(b,t,s)q(s)x(s)\, ds$$

defined on the Banach space

$$\mathcal{B} = \{x \in C([a,b]) : x(a) = 0\}$$

equipped with the usual sup norm. The cone $\mathcal{K}(b) \subset \mathcal{B}$ is defined by

$$\mathcal{K}(b) = \{x \in \mathcal{B} : x(t) \in K \text{ for } t \in [a,b]\}.$$

In this section, L will be shown to be compact and μ_0-positive. It is first convenient to give some lemmas. The first lemma can be found in [**10**].

Lemma 2. *If $f : [a,b] \to K$ is continuous and $f(t) \in K^0$ for some $t \in [a,b]$, then $\int_a^b f(s)\, ds \in K^0$.*

Lemma 3. *If $K_s \subset K_b$ for all $s \in (a,b]$, then for $s \in (a,b]$,*

$$\bigg(\int_a^b r^{-1}(\xi)\, d\xi\bigg)^{-1} \int_a^s r^{-1}(\xi)\, d\xi : K^0 \to K^0.$$

The proof of the lemma follows from the observation that if $K_s \subset K_b$, then

$$\left(\int_a^b r^{-1}(\xi)\, d\xi \right)^{-1} (K_s^0) \subset K^0.$$

Lemma 4. *If $D_b \subset D_t$ for all $t \in (a,b]$, then for $t \in (a,b]$,*

$$\int_a^t r^{-1}(\xi)\, d\xi \left(\int_a^b r^{-1}(\xi)\, d\xi \right)^{-1} : K^0 \to K^0.$$

The proof follows from the observation that if $D_b \subset D_t$, then

$$\int_a^b r^{-1}(\xi)\, d\xi (D_b^0) \subset K^0.$$

Lemma 5. *For $s,t \in (a,b)$,*

$$g(b,t,s) : K^0 \to K^0.$$

The proof follows readily from applying Lemmas 1, 2, and 3 to the expression for $g(b,t,s)$ given by (4).

The following theorem can now be proved.

Theorem 6. *The operator L is compact and μ_0-positive with respect to the cone $\mathcal{K}(b)$.*

The compactness of L is clear.

To show that L is μ_0-positive with respect to $\mathcal{K}(b)$, let $x \in \mathcal{K}(b) - \{0\}$, i.e., $x(t) \not\equiv 0$ on $[a,b]$ and $x(t) \in K$ on $[a,b]$. Notice that

$$-(Lx)'(b) = \int_a^b r^{-1}(b) \left(\int_a^b r^{-1}(\xi)\, d\xi \right)^{-1} \int_a^s r^{-1}(\xi)\, d\xi q(s) x(s)\, ds.$$

Since $K_s \subset K_b$, it follows from Lemma 2 and the hypothesis on r^{-1} that for $s > a$,

$$r^{-1}(b)\left(\int_a^b r^{-1}(\xi)\,d\xi\right)^{-1}\int_a^s r^{-1}(\xi)\,d\xi : K^0 \to K^0.$$

Now $x(s) \in K$ and thus $q(s)x(s) \in K$ for all $s \in [a,b]$. Furthermore, $x(s) \not\equiv 0$, and thus, by the hypothesis on $q(t)$, there exists at least one $\tau \in [a,b]$ such that $q(\tau)x(\tau) \in K^0$. Therefore, from Lemma 1, $-(Lx)'(b) \in K^0$.

Also notice that

$$(Lx)'(a) = \int_a^b r^{-1}(a)\left(\int_a^b r^{-1}(\xi)\,d\xi\right)^{-1}\int_s^b r^{-1}(\xi)\,d\xi q(s)x(s)\,ds.$$

It will now be shown that if $D_b \subset D_t$ for all $t \in (a,b]$, then

$$r^{-1}(a)\left(\int_a^b r^{-1}(\xi)\,d\xi\right)^{-1} : K^0 \to K^0.$$

To see this, suppose that $w \in K^0$ and

$$r^{-1}(a)\left(\int_a^b r^{-1}(\xi)\,d\xi\right)^{-1} w = y \in \partial K$$

Since $(\int_a^b r^{-1}(\xi)\,d\xi)^{-1}w = u \in D_b^0$, $r^{-1}(a)u = y \in \partial K$. Now

$$\frac{1}{t-a}\int_a^t r^{-1}(\xi)\,d\xi : \partial D_t \xrightarrow{\text{onto}} \partial K$$

for $t \in (a,b]$. Thus, there exists $u(t) \in \partial D_t$ such that

$$\frac{1}{t-a}\left(\int_a^t r^{-1}(\xi)\,d\xi\right)u(t) = y.$$

In fact,

$$u(t) = \left(\frac{1}{t-a}\int_a^t r^{-1}(\xi)\,d\xi\right)^{-1} y$$

and

$$\lim_{t\to a} u(t) = (r^{-1}(a))^{-1}y = r(a)y = u \in D_b^0.$$

But $u(t) \in \partial D_t \subset (D_b^0)^c$, where "$c$" indicates complement. Since this last set is closed, one concludes that $u \in (D_b^0)^c$, which is contrary to what has already been established. Now, proceeding as in the proof of $-(Lx)'(b) \in K^0$, one can conclude that $(Lx)'(a) \in K^0$.

Now take any $\mu \in K^0$ and define

$$\mu_0(t) = \bigg(\int_a^b g(b,t,s)\,ds\bigg)\mu.$$

Then L is μ_0-positive. To see this, notice that since $\mu_0(a) = 0$, Taylor's Theorem indicates that for any constant k_a,

$$(Lx)(t) - k_a\mu_0(t) = (t-a)\{[(Lx)'(a) - k_a\mu_0'(a)] + \dots\}.$$

Since $(Lx)'(a) \in K^0$, we can pick k_a sufficiently small so that

$$(Lx)'(a) - k_a\mu_0'(a) \in K^0$$

and, thus, there exists δ_a such that

$$(Lx)(t) - k_a\mu_0(t) \in K^0$$

for all $t \in (a, \delta_a]$. In the same way, there exists k_b and $\delta_b \in (\delta_a, b)$ such that

$$(Lx)(t) - k_b\mu_0(t) \in K^0$$

for all $t \in (\delta_b, b)$. Using familiar arguments and Lemma 5, we readily see that $(Lx)(t) \in K^0$ on (a,b). Then, by continuity, the graph of $(Lx)(t)$ is bounded away from the boundary of K on $[\delta_a, \delta_b]$. Thus there exists sufficiently small $k_c > 0$ such that $(Lx)(t) - k_c\mu_0(t) \in K^0$ for all $t \in [\delta_a, \delta_b]$.

Then if $k_1 = \min\{k_a, k_b, k_c\}$, $(Lx)(t) - k_1\mu_0(t) \in K$ for all $t \in [a,b]$, i.e.,

$$k_1\mu_0 \le Lx.$$

In the same way, one can show that there exists $k_2 > 0$ such that

$$Lx \le k_2\mu_0.$$

This then shows that L is μ_0-positive.

The following theorem is an immediate consequence of Theorems 1 and 6.

Theorem 7. *The conjugate point eigenvalue problem*

$$(r(t)y')' + \lambda q(t)y = 0, \qquad y(a) = 0 = y(b), \tag{5}$$

has a real eigenvalue $\lambda_0(b)$ which is simple, positive and smaller than the absolute value of any other eigenvalue. The normalized eigenvector associated with this eigenvalue is contained in the cone $\mathcal{K}(b)$ and is the only eigenvalue with this property.

The following theorem gives an extremal characterization of the smallest positive eigenvalue $\lambda_0(b)$. The proof follows as in [**9, 10**] since L has already been shown to be compact and μ_0-positive with respect to $\mathcal{K}(b)$.

Theorem 8. *The smallest eigenvalue $\lambda_0(b)$ of* (5) *is given by*

$$\lambda_0^{-1}(b) = \max_{x\in\mathcal{K}(b),x\neq 0} \frac{\int_a^b \int_a^b x^*(t)g(b,t,s)q(s)x(s)\,ds\,dt}{\int_a^b x^*(t)x(t)\,dt}. \tag{6}$$

The unique vector function, except for a constant multiple, which maximizes (6) *is a positive* (*with respect to $\mathcal{K}(b)$*) *eigenvector corresponding to the eigenvalue λ_0.*

If one is to proceed and prove the theorems that are to follow, then $\lambda_0(b)$ needs to be a strictly decreasing function of b. A critical factor in the proof given here of this property is the need for the Green's matrix to be nondecreasing with respect to K.

In order to proceed, the following lemma is now needed.

Lemma 9. *For all $s, t \in (a, b)$,*

$$\frac{\partial g}{\partial b}(b,t,s) : K^0 \to K^0.$$

To prove the lemma, first notice that a calculation shows that

$$\frac{\partial g}{\partial b}(b,t,s) = \bigg(\int_a^t r^{-1}\bigg)\bigg(\int_a^b r^{-1}\bigg)^{-1} r^{-1}(b)\bigg(\int_a^b r^{-1}\bigg)^{-1}\bigg(\int_a^s r^{-1}\bigg).$$

Then the result follows from an application of Lemmas 2 and 3.

With the previous lemma now established, the following corollary of Theorem 8 can now be given.

Corollary 10. *The smallest positive eigenvalue* $\lambda_0(b)$ *is a continuous, strictly decreasing function of* b *with the property that* $\lim_{b\to a+}\lambda_0(b) = +\infty$.

To prove the corollary, suppose that b_1 and b_2 are given such that $a < b_1 < b_2 \le b$. For $i = 1, 2$, Theorem 7 implies that there exists $\lambda_0(b_i) > 0$, and nontrivial solutions $x_i(t) \in K$ for $t \in [a, b]$ of

$$(r(t)x')' + \lambda_0(b_i)q(t)x = 0, \qquad x(a) = 0 = x(b_i).$$

Then

$$\lambda_0^{-1}(b_i)x_i(t) = \int_a^{b_i} g(b_i,t,s)q(s)x_i(s)\,ds.$$

Let

$$x(t) = \begin{cases} x_1(t) & a \le t \le b_1 \\ 0 & b_1 \le t \le b_2. \end{cases}$$

Then

$$\begin{aligned} \lambda_0^{-1}(b_1)x(t) &= \int_a^{b_1} g(b_1,t,s)q(s)x(s)\,ds \\ &< \int_a^{b_2} g(b_2,t,s)q(s)x(s)\,ds \end{aligned}$$

by virtue of Lemma 9, which assures that $g(b,t,s)$ is strictly increasing on K. Thus

$$\begin{aligned} \lambda_0^{-1}(b_1) &< \frac{\int_a^{b_2}\int_a^{b_2} x^*(t)g(b_2,t,s)q(s)x(s)\,ds\,dt}{\int_a^{b_2} x^*(t)x(t)\,dt} \\ &\le \lambda_0^{-1}(b_2) \end{aligned}$$

by Theorem 8.

The main result can now be given. Having established the previous basic results, the proof now follows along the same lines as found in [**10**], and will not be given here.

Theorem 11. *If* (1) *has a conjugate point* $c(a)$, *then the extremal solution* $x(t)$ *of* (1) *corresponding to the conjugate point is the cone* $\mathcal{K}(b)$ *and, furthermore,* $x(t) \in K^0$ *for* $t \in (a, c(a))$.

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University of Connecticut, Storrs, CT 06269