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RELATING DIFFERENT CONDITIONS FOR THE POSITIVITY OF THE SCHRÖDINGER OPERATOR

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ABSTRACT. The following article directs proofs that sufficient conditions for the positivity of the Schrödinger operator due to C. Fefferman and Chang, Wilson, and Wolff imply a necessary and sufficient condition of Kerman-Sawyer. The method is by reduction to dyadic case, Calderon-Zygmund decomposition, and, in one case, the use of Orlicz norms.

This article presents some direct proofs between several different conditions which imply the positivity of the Schrödinger operator, $-\Delta - (1/c)v$, where $v \geq 0$. If $L = -\Delta - (1/c)v$ is essentially self-adjoint, then L being a positive operator is equivalent to the following inequality:

$$(*) \qquad \int_{\mathbf{R}^d} u^2(x)v(x)\,dx \leq c\int_{\mathbf{R}^d} |\nabla u(x)|^2\,dx \qquad \forall\, u \in C_0^\infty,$$

as can be seen by an integration by parts:

$$\begin{aligned}\langle(-\Delta - \frac{1}{c}v)u, u\rangle &= \int(-\Delta u)u - \frac{1}{c}\int u^2 v\\ &= \int|\nabla u|^2 - \frac{1}{c}\int u^2 v\\ &\geq 0 \Leftrightarrow (*) \text{ holds.}\end{aligned}$$

In his paper, "The Uncertainty Principle," [**5**], C. Fefferman raises the question: What conditions on v imply $(*)$? In [**5**] the following condition (a) is shown to be sufficient for $(*)$:

(a) There exists $p > 1$ for all cubes Q

$$\left(\frac{1}{|Q|}\int_Q v^p\right)^{1/p} \leq \frac{c}{l^2(Q)}$$

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(where $|Q|$ = Lebesgue measure of Q and $l(Q)$ = side length of Q).

The proof in [**5**] that (a) $\Rightarrow$ $(*)$ uses A_∞ weight theory since if

$$M_p(v)(x) = \sup_{x\in Q}\left(\frac{1}{|Q|}\int v^p\right)^{1/p}$$

then $M_p(v)(x) \in A_\infty$ for any v; then it is shown that

$$\int |u|^2 v \le \int |u|^2 M_p(v) \le \int S^2(u) M_p(v)$$

for $M_p(v) \in A_\infty$, this gives $(*)$.

Two other conditions for $(*)$ were found by Kerman and Sawyer [**7**] and Chang, Wilson and Wolff [**3**][1]. The proofs that these conditions imply $(*)$, although differing from each other in method, are both more direct in that A_∞ weights are not used. The Kerman-Sawyer condition (b) (see below) is shown to be necessary and sufficient for $(*)$ by a good λ inequality, while Chang, Wilson, Wolff condition (C) is a sufficient condition for $(*)$ which is weaker than Fefferman's condition (a) (in fact, as is easily seen, (c) $\Rightarrow$ (a)). The proof is by dyadic decomposition and L^2 projection of the function u.

(b) If

$$M_1(f)(x) = \sup_{\substack{Q:\, x\in Q\\ Q \text{ any cube}}} \frac{1}{|Q|^{1-1/d}}\int_Q f(y)\,dy,$$

then for all cubes Q,

$$\int_Q M_1^2(\chi_Q v) \le c\int_Q v.$$

(This is shown to be equivalent to having $\int_{\mathbf{R}^d} M_1^2(\chi_Q v)$ on the left hand side by an earlier argument of Sawyer—this seemingly stronger form of inequality (b) is proved below.)

(c) If $\varphi(x)$ is increasingly on $[0,\infty)$ and $\int_1^\infty dx/(x\varphi(x)) < \infty$, then for all cubes

$$\frac{1}{|Q|}\int_Q v(x) l^2(Q)\varphi(v(x) l^2(Q))\,dx \le c.$$

Obviously a) and c) imply b) using (∗). Also see [**7**] for a direct proof that a) ⇒ b) for A^∞ weights[1]. Following are direct proofs that (a) and a particular form of (c) imply (b) and thus give a way to obtain (∗) without using either A_∞ weights or the Haar-type decomposition of u used in both [**3**] and [**5**]. The method of proof in both cases is reduction to the dyadic case, using a Calderon-Zygmund type decomposition (see argument in [**2**] for this part of the proof) then using Hölder's inequality for $(a) \Rightarrow (b)$ while $(c) \Rightarrow (b)$ requires Orlicz norms. Then the extension to the continuous case is by standard arguments.

Theorem 1. *$(a) \Rightarrow (b)$ with* (a) *and* (b) *as above.*

Theorem 1 will be proved by first replacing M_1 in condition (b) by

$$M_1^d(\chi_Q v)(x) = \sup_{\substack{Q:\, x\in Q\\ Q \text{ dyadic}}} \frac{1}{|Q|^{1-1/d}} \int_Q v(y)\, dy$$

to get condition (b′), and then proving (a) ⇒ (b′), (b′) is (b) with M_1^d in place of M_1.

Lemma 1. *There is a Calderon-Zygmund decomposition of*

$$D_k = \{x \mid M_1^d(\chi_Q v)(x) > R_k\}$$

where $R_k = 2^{k(d+1)}$ such that $D_k = \cup_j Q_j^k$ where Q_j^k are the maximal dyadic cubes for which

$$\text{(A)} \qquad R_k < \frac{1}{|Q_j^k|^{1-1/d}} \int_{Q_j^k} (\chi_Q v)(y)\, dy \le 2^{d-1} R_k$$

and

$$\text{(B)} \qquad \left| D_{k+l} \bigcap Q_j^k \right| \le \frac{1}{2^l} |Q_j^k| \qquad \text{for each } l \ge 0.$$

Proof. (A). Divide $\mathbf{R}^d$ into dyadic cubes $\{Q_m\}$ so large that

$$\frac{1}{|Q_m|^{1-1/d}} \int_{Q_m} (\chi_Q v) \le R_k$$

and so that for any dyadic cube $\tilde{Q}_m \supseteq Q_m$,

$$\frac{1}{|\tilde{Q}_m|^{1-1/d}} \int_{\tilde{Q}_m} \chi_Q v \leq R_k.$$

This is always possible since $\chi_Q v \in L^1$ and has compact support.

Now bisect each Q_m into $\{Q_m^i\}$ and if

$$\frac{1}{|Q_m^i|^{1-1/d}} \int_{Q_m^i} (\chi_Q v) > R_k,$$

put Q_m^i into a set, call it Ω. If

$$\frac{1}{|Q_m^i|^{1-1/d}} \int_{Q_m^i} (\chi_Q v) \leq R_k$$

continue to bisect Q_m^i and repeat the selection process. For $Q_j^k \in \Omega$, there exist Q_m^i, $Q_j^k \subseteq Q_m^i$ and $|Q_j^k| = (1/2^d)|Q_m^i|$ and

$$\frac{1}{|Q_m^i|^{1-1/d}} \int_{Q_m^i} (\chi_Q v) \leq R_k.$$

Then

$$\begin{aligned} \frac{1}{|Q_j^k|^{1-1/d}} \int_{Q_j^k} (\chi_Q v) &\leq \left(\frac{|Q_m^i|}{|Q_j^k|} \right)^{1-1/d} \frac{1}{|Q_m^i|^{1-1/d}} \int_{Q_m^i} (\chi_Q v) \\ &\leq (2^d)^{1-1/d} R_k = 2^{d-1} R_k \end{aligned}$$

since $\chi_Q v \geq 0$.

Then $\Omega = \{M_1^d \chi_Q v > R_k\}$, since clearly $\Omega \subseteq \{M_1^d \chi_Q v > R_k\}$, and conversely if $x \in \{M_1^d \chi_Q v > R_k\}$, there exists a dyadic $\tilde{Q}$, $x \in \tilde{Q}$, and

$$\frac{1}{|\tilde{Q}|^{1-1/d}} \int_{\tilde{Q}} \chi_Q v > R_k.$$

So there is such a maximal dyadic cube $\tilde{Q}$ which must belong to Ω by the way $\mathbf{R}^d$ was decomposed. So $\{M_1^d \chi_Q v > R_k\} \subseteq \Omega$.

Note. This decomposition differs from the Calderon-Zygmund decomposition for the Hardy-Littlewood maximal operator in that it is not necessarily true that $\chi_Q v \le R_k$ on Ω^c.

So $D_k = \cup_j Q_j^k$ where Q_j^k are dyadic, disjoint and

$$\text{(A)} \qquad R_k < \frac{1}{|Q_j^k|^{1-1/d}} \int_{Q_j^k} \chi_Q v \le 2^{d-1} R_k.$$

(B). $D_{k+l} \cap Q_j^k = \cup_m Q_{i_m}^{k+l}$ for some i_m since $D_{k+l} = \cup_i Q_i^{k+l}$ and dyadic cubes are either nested or disjoint.

Then

$$\begin{aligned}
|D_{k+l} \cap Q_j^k| &= \left| \bigcup_m Q_{i_m}^{k+l} \right| = \sum_m |Q_{i_m}^{k+l}| \\
&= \sum |Q_{i_m}^{k+l}|^{1-1/d} |Q_{i_m}^{k+l}|^{1/d} \\
&\le \sum_m |Q_{i_m}^{k+l}|^{1-1/d} |Q_j^k|^{1/d}, \qquad \text{since } Q_{i_m}^{k+l} \subseteq Q_j^k, \\
&< |Q_j^k|^{1/d} \sum_m \frac{1}{R_{k+l}} \int_{Q_{i_m}^{k+l}} \chi_Q v \qquad \text{by (A)} \\
&= |Q_j^k|^{1/d} \frac{1}{R_{k+l}} \sum_m \int_{Q_{i_m}^{k+l}} \chi_Q v \\
&\le |Q_j^k|^{1/d} \frac{1}{R_{k+l}} \int_{Q_j^k} \chi_Q v
\end{aligned}$$

since $\chi_Q v \ge 0$ and $\bigcup_m Q_{i_m}^{k+l} \subseteq Q_j^k$,

$$\begin{aligned}
&\le |Q_j^k|^{1/d} \frac{1}{R_{k+l}} 2^{d-1} R_k |Q_j^k|^{1-1/d} \qquad \text{by (A)} \\
&= \frac{2^{d-1} 2^{(d+1)k}}{2^{(d+1)(k+l)}} |Q_j^k| = \frac{2^{d-1}}{2^{(d+1)l}} |Q_j^k| \\
&< \frac{1}{2^l} |Q_j^k| \qquad \text{for } l \ge 1. \qquad \square
\end{aligned}$$

To summarize the notation and facts obtained so far:

$$D_k = \{M_1\chi_Q v > R_k\} = \bigcup_j Q_j^k$$

$$E_k = D_k \backslash D_{k+1} \Rightarrow D_k = \bigcup_{l\geq 0} E_{k+l}$$

and E_k are disjoint, $\{M_1\chi_Q v > 0\} = \cup_{k=-\infty}^{\infty} D_k$ and

(A) $$R_k < \frac{1}{|Q_j^k|^{1-1/d}} \int_{Q_j^k} \chi_Q v \leq 2^{d-1} R_k < R_{k+1}$$

(B) $$\left| D_{k+l} \bigcap Q_j^k \right| < \frac{1}{2^l} |Q_j^k|.$$

Proof of Theorem 1 a) $\Rightarrow$ b$'$). (See [**2**] for the following proof.) Now

$$\begin{aligned}
\int_{\mathbf{R}^d} [M_1\chi_Q v]^2 &\geq \sum_{k=-\infty}^{\infty} R_k^2 |D_k \backslash D_{k+l}| \\
&= \sum_k R_k^2 (|D_k| - |D_{k+1}|) = \sum_k (R_k^2 - R_{k-1}^2)|D_k| \\
&= \sum_k (2^{2k(d+1)} - 2^{2(k-1)(d+1)})|D_k| \\
&= \sum_k 2^{2k(d+1)} (1 - 2^{-2(d+1)})|D_k| = c_1 \sum_k R_k^2 |D_k|.
\end{aligned}$$

Also,

$$\begin{aligned}
\int_{\mathbf{R}^d} [M_1\chi_Q v]^2 &\leq \sum_{k=-\infty}^{\infty} R_{k+1}^2 |D_k \backslash D_{k+1}| \\
&= \sum_k R_{k+1}^2 (|D_k| - |D_{k+1}|) = \sum_k (R_{k+1}^2 - R_k^2)|D_k| \\
&= \sum_k (2^{2(k+1)(d+1)} - 2^{2k(d+1)})|D_k| \\
&= \sum_k 2^{2k(d+1)} (2^{2(d+1)} - 1)|D_k| \\
&= c_2 \sum_k R_k^2 |D_k|.
\end{aligned}$$

So one obtains

$$(+) \qquad c_1 \sum_{k=-\infty}^{\infty} R_k^2 |D_k| \le \int_{\mathbf{R}^d} [M_1(\chi_Q v)]^2 \le c_2 \sum_{k=-\infty}^{\infty} R_k^2 |D_k|.$$

Now

$$\begin{aligned}\sum_{k=-\infty}^{\infty} R_k^2 |D_k| &= \sum_k R_k^2 \sum_j |Q_j^k| \\ &\le \sum_k R_k \bigg(\sum_j \bigg(\frac{1}{|Q_j^k|^{1-1/d}} \int_{Q_j^k} \chi_Q v \bigg) |Q_j^k| \bigg) \qquad \text{by (A)} \\ &= \sum_k R_k \sum_j |Q_j^k|^{1/d} \sum_{l \ge 0} \int_{E_{k+l} \cap Q_j^k} \chi_Q v = (*)\end{aligned}$$

since $Q_j^k = D_k \cap Q_j^k = \cup_{l \ge 0} (E_{k+l} \cap Q_j^k)$ and $\{E_{k+l}\}_{l=0}^{\infty}$ are disjoint.

Using Hölder's inequality with three exponents $1/2 + 1/(2p) + 1/r = 1$ ($p > 1 \Rightarrow r$ exists),

$$\begin{aligned}\int_{E_{k+l} \cap Q_j^k} \chi_Q v \le \bigg(\int_{E_{k+l} \cap Q_j^k} \chi_Q v \bigg)^{1/2} \bigg(\int_{E_{k+l} \cap Q_j^k} [\chi_Q v]^p \bigg)^{1/2p} \\ \cdot \bigg(\int_{E_{k+l} \cap Q_j^k} \chi_{Q_j^k}^r \bigg)^{1/r}\end{aligned}$$

and for p as in condition (a), then

$$\bigg(\int_{Q_j^k} (\chi_Q v)^p \bigg)^{1/p} \le c |Q_j^k|^{1/p - 2/d} \qquad \forall\, Q_j^k$$

and, using

$$\int_{E_{k+l} \cap Q_j^k} (\chi_Q v)^p \le \int_{Q_j^k} (\chi_Q v)^p$$

and

$$\bigg(\int_{E_{k+l} \cap Q_j^k} \chi_{Q_j^k}^r \bigg)^{1/r} = |E_{k+l} \cap Q_j^k|^{1/r} \le \bigg(\frac{1}{2^l} |Q_j^k| \bigg)^{1/r} \qquad \text{from (B)}$$

then

$$\int_{E_{k+l}\cap Q_j^k} \chi_Q v \le \left(\int_{E_{k+l}\cap Q_j^k} \chi_Q v\right)^{1/2} c|Q_j^k|^{1/(2p)-1/d}\frac{1}{2^{l/r}}|Q_j^k|^{1/r}.$$

So, putting this into $(*)$, one obtains (since $1/(2p)+1/r=1/2$)

$$\begin{aligned}&\left(\int_{\mathbf{R}^d}[M_1\chi_Q v]^2\right)\\ &\quad\le c\sum_{k=-\infty}^{\infty} R_k \sum_j |Q_j^k|^{1/d}\sum_{l\ge 0}\left(\int_{E_{k+l}\cap Q_j^k}\chi_Q v\right)^{1/2}\frac{1}{2^{l/r}}|Q_j^k|^{1/2-1/d}\\ &\quad= c\sum_{l\ge 0}\frac{1}{2^{l/r}}\sum_k R_k\sum_j |Q_j^k|^{1/2}\left(\int_{E_{k+l}\cap Q_j^k}\chi_Q v\right)^{1/2}.\end{aligned}$$

Now, using Cauchy-Schwarz on the sum over j, the above is

$$\begin{aligned}&\le c\sum_{l\ge 0}\frac{1}{2^{l/r}}\sum_k R_k\left(\sum_j |Q_j^k|\right)^{1/2}\left(\sum_j\int_{E_{k+l}\cap Q_j^k}\chi_Q v\right)^{1/2}\\ &= c\sum_{l\ge 0}\frac{1}{2^{l/r}}\sum_k R_k|D_k|^{1/2}\left(\int_{E_{k+l}}\chi_Q v\right)^{1/2}\end{aligned}$$

since $|D_k| = \sum_j |Q_j^k|$ and $\cup_j E_{k+l}\cap Q_j^k = E_{k+l}$, $\{Q_j^k\}$ disjoint for k fixed.

Then, using Cauchy-Schwarz again on the sum over k, the above is

$$\begin{aligned}&\le c\sum_{l\ge 0}\frac{1}{2^{l/r}}\left(\sum_k R_k^2|D_k|\right)^{1/2}\left(\sum_k\int_{E_{k+l}}\chi_Q v\right)^{1/2}\\ &\le c\sum_{l\ge 0}\frac{1}{2^{l/r}}\left(\sum_k R_k^2|D_k|\right)^{1/2}\left(\int_{\mathbf{R}^d}\chi_Q v\right)^{1/2}\end{aligned}$$

since E_{k+l} are disjoint for l fixed, $k=-\infty,\dots,+\infty$;

$$\le c\left(\sum_{l\ge 0}\left(\frac{1}{2^{l/r}}\right)\left(\int_{\mathbf{R}}[M_1^d\chi_Q v]^2\right)^{1/2}\left(\int_Q v\right)^{1/2}\right)$$

by the first inequality in (+).

Thus,

$$\int_{\mathbf{R}^d}[M_1^d\chi_Q v]^2 \le c\left(\int_{\mathbf{R}^d}[M_1^d\chi_Q v]^2\right)^{1/2}\left(\int_Q v\right)^{1/2}$$
$$\Rightarrow \left(\int_{\mathbf{R}^d}[M_1^d\chi_Q v]^2\right)^{1/2} \le c\left(\int_Q v\right)^{1/2}$$

when $(\int_{\mathbf{R}^d}[M_1^d\chi_Q v]^2)^{1/2}$ is finite, which it is for $\chi_Q v$ being of compact support and satisfying either (a) or (c), since

$$(M_1\chi_Q v)^2(x) = \sup_{x\in\tilde{Q}}\left(\frac{1}{|\tilde{Q}|^{1-2/d}}\int_{\tilde{Q}}\chi_Q v\right)\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}\chi_Q v\right)$$
$$\le c\cdot M(\chi_Q v)(x)$$

where M is the Hardy-Littlewood maximal function, and the condition for Mf being $L^1(3Q)$ is $|f|\log(2+|f|)\in L^1(3Q)$ (Stein [**10**]). Both a) and c) imply

$$\left(\frac{1}{|\tilde{Q}|^{1-2/d}}\int_{\tilde{Q}}\chi_Q v\right) \le c.$$

Obviously, $\int_{\mathbf{R}^n\setminus 3Q}(M_1\chi_Q v)^2$ is bounded, since $M_1(\chi_Q v) \le I_1(\chi_Q v)$ pointwise almost everywhere where $I_1(\chi_Q v)$ is the Riesz potential of $\chi_Q v$ (Adams [**1**, p. 15]), and

$$||I_1(\chi_Q v)||_2 \le c\cdot ||\chi_Q v||_q, \qquad 1/q = 1/2 + 1/n$$

(Stein [**10**]).

Thus, condition (b) holds for the dyadic maximal function.

Using a standard argument for extending from the dyadic to the continuous case, Theorem 1 holds for M_1 [**6**]. □

Theorem 2. *If, for all cubes Q,*

(d) $$\frac{1}{|Q|}\int_Q l^2(Q)v(y)\log^{2+\varepsilon}(1+l^2(Q)v(y))\,dy \le c$$

then

$$\int_{\mathbf{R}^d} [M_1(\chi_Q v)(x)]^2\, dx \le c \int_Q v(y)\, dy.$$

Note. In Theorem 2, condition (d) is the Chang, Wilson, Wolff condition (c) with $\varphi(x) = \log^{2+\varepsilon}(1+x)$.

To prove Theorem 2, a lemma is needed:

Lemma 2. *Condition* (d) *in Theorem* 2 *implies*

$$\left(\int_{Q_j^k \cap E_{k+l}} l^2(Q_j^k)(\chi_Q v)(y)\, dy\right)^{1/2} \le c\left(\frac{|Q_j^k|}{\log^{2+\varepsilon}(1+2^l)}\right)^{1/2}$$

where Q_j^k *and* E_{k+l} *are the same sets as in the proof of Theorem* 1.

Proof. Let $\Phi(x) = x\log^{2+\varepsilon}(1+x)$ and $\Psi(y)$ be Youngs' functions, i.e.,

$$\Phi(x) = \int_0^x \varphi(t)\, dt \quad \text{and} \quad \Psi(y) = \int_0^y \psi(t)\, dt$$

where $\varphi \circ \psi(t) = t$. The function Ψ exists if $\Phi(x)$ is ≥ 0, convex for $x \ge 0$, $\Phi(0) = 0$ and $\lim_{x\to\infty} \Phi(x)/x = \infty$. These conditions hold for $\Phi(x) = x\log^{2+\varepsilon}(1+x)$ (Zygmund [**12**, p. 24]).

If X is a given space, then the two norms for Orlicz space $L^*(X, dm)$ are

$$||x||_\Phi = \sup_{\int_X \Psi(y) \le 1} \int_X x(t)y(t)\, dm(t)$$

and

$$||x||_{N_\Phi} = \inf\left\{u : \int_X \Phi(x(t)/\mu)\, dm(t) = \Phi(1)\right\}.$$

The two norms are equivalent; $||x||_\Phi \le c||x||_{N\Phi}$ where c depends only on $\Phi(1)$ (Zygmund [**12**, p. 174]).

For $v \ge 0$, $v \in L^1_{\text{loc}}(X)$ and $X = Q_j^k$ a cube then

$$\begin{aligned}\int_{Q_j^k \cap E_{k+l}} l^2(Q_j^k)\chi_Q v &= |Q_j^k| \int_{Q_j^k} \chi_{Q_j^k \cap E_{k+l}}(y) l^2(Q_j^k)(\chi_Q v)(y) dy/|Q_j^k| \\ &\le c|Q_j^k|\, ||\chi_{Q_j^k \cap E_{k+l}}||_\Psi\end{aligned}$$

by definition of $|| \, ||_\Psi$ since (d) implies

$$c \int_{Q_j^k} \Phi(l^2(Q_j^k)(\chi_Q v)(y)) dy / |Q_j^k| \leq 1.$$

Here the measure on X is $dx/|Q_j^k| = dm$.

Now

$$\begin{aligned} ||\chi_{Q_j^k \cap E_{k+l}}||_\Psi &\leq c ||\chi_{Q_j^k \cap E_{k+l}}||_{N\Psi} \\ &= \inf \left\{ \mu : \int_{Q_j^k} \int^{\chi_{Q_j^k \cap E_{k+l}}(x)/\mu} \Psi(t) \, dt dx / |Q_j^k| \leq \Psi(1) \right\}. \end{aligned}$$

And

$$\begin{aligned} \int_{Q_j^k} \int^{(1/\mu) \cdot \chi_{Q_j^k \cap E_{k+l}(x)}} \Psi(t) \, dt dx / |Q_j^k| &= \int_{Q_j^k \cap E_{k+l}} \int_0^{1/\mu} \Psi(t) \, dt dx / |Q_j^k| \\ &= \frac{|Q_j^k \cap E_{k+l}|}{|Q_j^k|} \Psi(1/\mu); \end{aligned}$$

so for $|| \, ||_{N\Psi}$ one needs

$$\inf \{ \mu : \Psi(1/\mu) \leq c |Q_j^k| / |Q_j^k \cap E_{k+l}| \}.$$

This implies

$$\mu \geq (\Psi^{-1}(c|Q_j^k|/|Q_j^k \cap E_{k+l}|))^{-1}$$

since Ψ is increasing and Ψ^{-1} is also increasing.

Then, taking the inf of such μ,

$$\mu \leq [\Psi^{-1}(c \cdot 2^l)]^{-1} \quad \text{since} \quad |Q_j^k \cap E_{k+l}| \leq 2^{-l} |Q_j^k|. \tag{4}$$

Sublemma. $\varphi(x) = x \log^{2+\varepsilon}(1+x) \Rightarrow \Psi^{-1}(x) \geq c \log^{2+\varepsilon}(1+x)$ *for* $x > 1$.

Proof of Sublemma. $\Phi(x) = x \log^{2+\varepsilon}(1+x)$ and

$$\Phi(x) = \int_0^x \varphi(t) \, dt \Rightarrow$$

$$
\begin{aligned}
\varphi(x) = \Phi'(x) &= \log^{2+\varepsilon}(1+x) + (2+\varepsilon)(x/(1+x))\log^{1+\varepsilon}(1+x) \\
&\Rightarrow \varphi(x) > \log^{2+\varepsilon}(1+x) \\
&\Rightarrow \psi(x) < e^{x^{(1/(2+\varepsilon))}} - 1
\end{aligned}
$$

since $\varphi \circ \psi(x) = x$ and $\varphi > f \Rightarrow \varphi^{-1} < f^{-1}$.

then

$$
\begin{aligned}
\Psi(x) = \int_0^x \psi(t)\,dt &< xe^{x^{(1/(2+\varepsilon))}} - 1 \\
&\leq ce^{2x^{(1/(2+\varepsilon))}} - 1 \\
&\Rightarrow \Psi^{-1}(x) \geq c\log^{2+\varepsilon}(1+x). \qquad \square
\end{aligned}
$$

So by (4)

$$\mu \leq \frac{1}{c\log^{2+\varepsilon}(1+2^l)}.$$

Finally, one obtains:

$$\int_{Q_j^k \cap E_{k+l}} l^2(Q_j^k)\chi_Q v \leq |Q_j^k| \frac{c}{\log^{2+\varepsilon}(1+2^l)}.$$

So, using the same argument as given above in the proof of Theorem 1 for showing (a) $\Rightarrow$ (b$'$), that is, R_k, $D_k = \cup Q_j^k$ and E_{k+l} are all as above, then one has at $(*)$ in the proof of Theorem 1,

$$
\begin{aligned}
&\int_{\mathbf{R}^d} [M_1^d(\chi_Q(v)]^2 \\
&\quad \leq c \sum_{k=-\infty}^{\infty} R_k \sum_j \sum_{l \geq 0} l(Q_j^k) \int_{Q_j^k \cap E_{k+l}} \chi_Q v \\
&\quad = c \sum_k R_k \sum_j \sum_{l \geq 0} \left(\int_{Q_j^k \cap E_{k+l}} \chi_Q v \right)^{1/2} \left(\int_{Q_j^k \cap E_{k+l}} l^2(Q_j^k)\chi_Q v \right)^{1/2} \\
&\quad \leq c \sum_k R_k \sum_j \sum_{l \geq 0} \left(\int_{Q_j^k \cap E_{k+l}} \chi_Q v \right)^{1/2} |Q_j^k|^{1/2} \left(\frac{1}{\log^{2+\varepsilon}(1+2^l)} \right)^{1/2} \\
&\quad = c \sum_{l \geq 0} \frac{1}{\log^{1+(\varepsilon/2)}(1+2^l)} \sum_k R_k \sum_j \left(\int_{Q_j^k \cap E_{k+l}} \chi_Q v \right)^{1/2} |Q_j^k|^{1/2}
\end{aligned}
$$

$$\leq c\sum_{l\geq 0}\frac{1}{\log^{1+(\varepsilon/2)}(1+2^l)}\sum_k R_k\bigg(\sum_j\int_{Q_j^k\cap E_{k+l}}\chi_Q v\bigg)^{1/2}\bigg(\sum_j |Q_j^k|\bigg)^{1/2}$$

$$\leq c\sum_{l\geq 0}\frac{1}{\log^{1+(\varepsilon/2)}(1+2^l)}\sum_k R_k\bigg(\int_{E_{k+l}}\chi_Q v\bigg)^{1/2}|D_k|^{1/2}$$

$$\leq c\sum_{l\geq 0}\frac{1}{\log^{1+(\varepsilon/2)}(1+2^l)}\bigg(\sum_{k=-\infty}^{\infty}R_k^2|D_k|\bigg)^{1/2}\bigg(\int_{\mathbf{R}^d}\chi_Q v\bigg)^{1/2}$$

$$\leq c\bigg(\int_{\mathbf{R}^d}[M_1^d(\chi_Q v)]^2\bigg)^{1/2}\bigg(\int_Q v\bigg)^{1/2}$$

since

$$\sum_{l\geq 0}\frac{1}{\log^{1+(\varepsilon/2)}(1+2^l)}<\infty.$$

So, dividing by $(\int[M_1^d(\chi_Q v)]^2)^{1/2}$, one obtains

$$\int_{\mathbf{R}^d}[M_1^d(\chi_Q v)(x)]^2\,dx\leq c\int_Q v(y)\,dy.$$

Theorem 2 follows from the dyadic case by standard arguments [**6**]. □

ENDNOTES

1. Since this paper was written, several improvements and generalizations on conditions a), b) and c) have been obtained by C. Perez [**8, 9**] and S. Chanillo and E. Sawyer [**4**]. See also the work of J.M. Wilson [**11**].

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