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A REMARK ON L^∞ BOUNDS FOR SOLUTIONS TO QUASILINEAR REACTION-DIFFUSION EQUATIONS

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1. Introduction. The objective of this paper is to obtain an *a priori* L^∞ bound for solutions to certain reaction-diffusion equations. This is accomplished by transforming the system into a single second order parabolic equation with a nonlocal nonlinearity to which may be applied earlier results of the author [**3, 4**]. This approach makes it possible to consider systems which involve nonlocal nonlinearities. We will say that a nonlinearity $\mathcal{F}$ is nonlocal if $\mathcal{F}(\mathbf{u})(\mathbf{x},t)$ depends functionally upon $\mathbf{u}(\cdot,t)$. We believe that our result is new even in case the system has nonlinearities which are local in nature.

The most obvious application of our estimates is to obtain lower bounds for the blow-up time for a system of quasilinear reaction-diffusion equations, irrespective of whether or not the system contains nonlocal terms. However, there are several examples of such systems which do involve nonlocal terms. In both electrophoresis [**1**] as well as models for carrier transport in semiconductors [**5**], one encounters systems of the form

$$\partial u^{(j)}/\partial t = \nabla\cdot(d^{(j)}\nabla u^{(j)} - e^{(j)}u^{(j)}\nabla v) + F^{(j)}(x,t,\mathbf{u},v),$$
$$0 = \nabla v + G(\mathbf{u}),$$

together with boundary conditions. We can, of course, solve the last equation for v in terms of $\mathbf{u}$,

$$v(x,t) = \int_\Omega G(x,\xi)G(\mathbf{u}(\xi,t))\,d\xi,$$

and substitute this into the other equations in order to obtain a system of the form (10). In the case of carrier transport in semiconductors, the function F may contain positive cubic terms (see [**5**, p. 9], while the diffusion coefficients $d^{(j)}$ may depend on v and hence in a nonlocal way depend on $\mathbf{u}$. Prey-predator equations with nonlocal nonlinearities

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have been studied. One such problem involving both nonlocal as well as delayed nonlinearities was studied in [**2**]. In that paper these two complications were assumed to be of such a nature that they could be removed at the expense of increasing the size of the system of equations from two to three. Although we did not look at equations which involve time delays, our result applies to a fairly large class of reaction-diffusion equations and may easily be extended in the obvious way to problems involving constant time delays:

$$\mathbf{u}_t = \nabla \cdot \mathcal{N}(\mathbf{u}, \nabla \mathbf{u}, \mathbf{u}_\tau, \nabla \mathbf{u}_\tau) + \mathcal{F}(\mathbf{u}, \mathbf{u}_\tau),$$

where $\mathbf{u}_\tau(\cdot, t) = \mathbf{u}(\cdot, t - \tau)$.

We will allow nonlinear boundary conditions which are nonlocal and hope that this may prove to be helpful in the study of reaction-diffusion equations which describe processes in which some of the reactions take place on the surface of the container (see [**4**]).

Consider the system of equations

$$\partial u_j / \partial t = \nabla \cdot \mathcal{M}^{(j)}(x, t, \mathbf{u}, D\mathbf{u}) + \mathcal{F}^{(j)}(x, t, \mathbf{u}), \qquad (x, t) \in Q, \tag{1}$$

where $\mathbf{u} = (u_1, u_2, \dots, u_m)$, $D\mathbf{u} = (\partial u_1/\partial x_1, \dots, \partial u_1/\partial x_n, \partial u_2/\partial x_1, \dots, \partial u_m/\partial x_n)$, $Q = \Omega \times (0, T)$, $0 < T \leq \infty$, and Ω is a bounded open set in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$ which possesses a uniquely defined unit outward normal vector field $\nu(x)$. In order to allow for mixed boundary conditions, we suppose that for each $1 \leq j \leq m$ we have $\partial\Omega = \partial\Omega_D^{(j)} \cup \partial\Omega_N^{(j)}$, a disjoint union. Suppose that the following boundary conditions are imposed on $\mathbf{u}$:

$$u_j(x, t) = 0 \quad \text{on } \partial Q_D^{(j)} \tag{2}$$

$$\nu(x) \cdot \mathcal{M}^{(j)}(x, t, \mathbf{u}, D\mathbf{u}) + \mathcal{G}^{(j)}(x, t, \mathbf{u}) = 0 \quad \text{on } \partial Q_N^{(j)}, \tag{3}$$

where $\partial Q_D^{(j)} = \partial\Omega_D^{(j)} \times (0, T)$ and $\partial Q_N^{(j)} = \partial\Omega_N^{(j)} \times (0, T)$.

Let us first transform this to a one-component problem. Let $\xi^{(1)}, \xi^{(2)}, \dots, \xi^{(m)}$ be vectors in $\mathbf{R}^n$ chosen in such a way that the translates $\Omega_j = \xi^{(j)} + \Omega$ of Ω are mutually disjoint. Next we define

$$\begin{aligned}
\Omega^* &= \bigcup_{i=1}^{m} \Omega_i \\
Q^* &= \Omega^* \times (0,T) \\
\partial Q^*_D &= \bigcup_{i=1}^{m} (\partial Q^{(i)}_D + (\xi^{(i)}, 0)) \\
\partial Q^*_N &= \bigcup_{i=1}^{m} (\partial Q^{(i)}_N + (\xi^{(i)}, 0)).
\end{aligned}$$

We define the function u on Q^* as follows

$$u(x,t) = u_i(x - \xi^{(i)}, t) \qquad \text{if } x \in \Omega_i.$$

We define $Du(x,t)$ similarly on Q^*. Conversely, given a function w on Q^* we let the corresponding boldface letter, $\mathbf{w}$ in this case, represent the function from Ω into $\mathbf{R}^m$ given by $\mathbf{w}(x,t) = (w_1(x,t), w_2(x,t), \dots, w_m(x,t))$ where $w_j(x,t) = w(x + \xi^{(j)}, t)$, $x \in \Omega$.

We define

$$\mathcal{M}(x + \xi^{(j)}, t, u, Du) = \mathcal{M}^{(j)}(x, t, \mathbf{u}(x,t), D\mathbf{u}(x,t)), \qquad x \in \Omega.$$

The functions $\mathcal{F}(x,t,u)$ and $\mathcal{G}(x,t,u)$ are defined similarly from the given functions $\mathcal{F}(x,t,\mathbf{u})$ and $\mathcal{G}(x,t,\mathbf{u})$. Problem (1)–(3) is now, at least formally, equivalent to the problem

$$\partial u/\partial t = \nabla \cdot \mathcal{M}(x,t,u,Du) + \mathcal{F}(x,t,u,Du), \qquad (x,t) \in Q^* \tag{4}$$

$$u(x,t) = 0, \qquad (x,t) \in \partial Q^*_D \tag{5}$$

$$\nu^*(x) \cdot \mathcal{M}(x,t,u,Du) + \mathcal{G}(x,t,u) = 0, \qquad (x,t) \in \partial Q^*_N \tag{6}$$

where $\nu^*(x) = \nu(x - \xi^{(j)})$ if $x \in \partial\Omega_j$.

The price of this reduction is that $\mathcal{M}$, $\mathcal{F}$ and $\mathcal{G}$ now depend functionally upon u and Du. When, for each j, the functions $\mathcal{M}^{(j)}$ and $\mathcal{G}^{(j)}$ are independent of the components u_i and Du_i for which $i \neq j$, then we say that the equations are *weakly coupled*. Let us assume that our equations are weakly coupled. Although we assume that the $\mathcal{F}^{(j)}$ do not depend on the derivatives $D\mathbf{u}$, we may allow $\mathcal{F}^{(j)}(x,t,\mathbf{u})$ to depend functionally upon $\mathbf{u}(\cdot,t)$.

2. Results. In this section we state the main hypotheses, the definition of a solution (weak solution) and our results. Let

$$\mathcal{F}_+^{(j)}(x,t,\mathbf{u}) = \max\{0, [\operatorname{sgn} u_j(x,t)]\}\mathcal{F}^{(j)}(x,t,u).$$

Our main hypotheses are as follows.

$$\text{(PBY)} \qquad \mathbf{P}\cdot\mathcal{M}^{(j)}(x,t,s,\mathbf{P}) \ge 0 \qquad \forall\ (x,t,s,\mathbf{P}) \in Q\times\mathbf{R}^1\times\mathbf{R}^n$$

$$\text{(BND)} \qquad \begin{array}{c} s\mathcal{G}^{(j)}(x,t,s) \ge 0 \qquad \forall\ (x,t,s)\in Q\times\mathbf{R}^1 \\ ||\mathcal{F}_+^{(j)}(\cdot,t,\mathbf{u})|| \le g(t)f(||\mathbf{u}(\cdot,t)||) \end{array}$$

where $||\ \ ||$ denotes the $L_\infty(\Omega)$ norm $(||\mathbf{u}(\cdot,t)|| = \max_j ||u_j(\cdot,t)||)$, f is a nondecreasing, nonnegative, Lipschitz continuous function on $[0,\infty)$ and g is a continuous, nonnegative function on $[0,\infty)$ such that $G(t)\equiv\int_0^t g(\tau)\,d\tau<\infty$ for all $0\le t<T$.

$$\text{(MDN)} \qquad \begin{array}{l} \partial\Omega \text{ is a } C^1 \text{ manifold. If } \partial\Omega_D^{(j)} \ne \partial\Omega \text{ and} \\ \partial\Omega_D^{(j)} \ne \varnothing \text{ then } \partial(\partial\Omega_D^{(j)}),\ \text{the boundary of} \\ \partial\Omega_D^{(j)} \text{ in } \partial\Omega,\ \text{is a } C^1 \text{ submanifold of } \partial\Omega. \end{array}$$

Let Q_s denote the set $\Omega\times(0,s)$, then $L_{q,r}(Q_s)$ denotes the Banach space of Lebesgue measurable functions h with finite norm

$$||h||_{q,r} = \left\{\int_0^s \left[\int_\Omega |h(x,t)|^q\,dx\right]^{r/q}\right\}^{1/r}.$$

When h is a function defined on Q_s (respectively, $Q_s^* = \Omega^*\times(0,s)$) such that $h(\cdot,t)\in L_r(\Omega)$ (respectively, $L_r(\Omega^*)$), then we use the corresponding capital letter, in this case H, to denote the function $H(t)=h(\cdot,t)$ from $(0,s)$ into $L_r(\Omega)$ (respectively, $L_r(\Omega^*)$).

Let Ω_s denote the section $\Omega\times\{s\}$. The space of $C^\infty(Q)$ functions whose support does not intersect $\partial Q^{(j)}\cup\Omega_0\cup\Omega_T$ is denoted by $C_{D,j}^\infty(Q)$.

We can similarly define $C_D^\infty(Q^*)$. By $C_D^\infty(Q)$ we will mean the space of C^∞ functions $\mathbf{h}$ from Q into $\mathbf{R}^m$ such that $h_i \in C_{D,i}^\infty(Q)$ for each i.

The term *solution* will denote a distributional solution with a modest amount of regularity:

Definition. A function $\mathbf{u} : Q \to \mathbf{R}^m$ is a solution of (1)–(3) if

(i) $\mathbf{U} \in \mathrm{Lip}\,([0,T), L_\rho(\Omega)^m)$, $\rho \geq 1$.

(ii) For each $0 < s < T$, we have $\mathbf{u} \in L_\infty(Q_s)^m$ and $D\mathbf{u} \in L_{q_2,q_1}(Q_s)^{mn}$, $q_i \in [0,\infty]$, $i = 1, 2$.

(iii) For each j and each $0 < s < T$, we have $\mathcal{M}^{(j)}(x,t,\mathbf{u},D\mathbf{u}) \in L_{p_2,p_1}(Q_s)^n$ and $\mathcal{G}^{(j)}(x,t,\mathbf{u}) \in L_{p_2,p_1}(\partial\Omega_D \times (0,s))$, where $p_i = q_i/(q_i - 1)$ if $q_i > 1$ and $p_i = 1$ if $q_i = \infty$.

(iv) $\mathbf{u}$ is continuous in the interior of Q or $\mathbf{u}$ is a member of the Sobolev space $W_1^1(Q)^m$. If $\rho = 1$, then we also require that $\mathbf{U}$ is a separably valued function into $L_\infty(\Omega)^m$.

(v) $||U_j(t)||$ is continuous from the right for each j.

(vi) For each $\boldsymbol{\psi} \in C_D^\infty(Q)$, we have

(7)

$$\begin{aligned}\iint_{Q_T} \mathbf{u}\cdot\frac{\partial\boldsymbol{\psi}}{\partial t}\,dx\,dt \\ &= \iint_{Q_T} \left\{ \sum_{j=1}^m \mathcal{M}^{(j)}(x,t,\mathbf{u},D\mathbf{u})\cdot\nabla\boldsymbol{\psi}_j - \mathcal{F}(x,t,\mathbf{u})\cdot\boldsymbol{\psi} \right\} dx\,dt \\ &\quad + \sum_{j=1}^m \iint_{\partial Q_N^{(j)}} \mathcal{G}^{(j)}(x,t,\mathbf{u})\psi_j\,dS\,dt.\end{aligned}$$

A solution of problem (4)–(6) is defined entirely analogously: we replace Ω by Ω^*, Q by Q^*, m by 1, and we drop the superscripts ${}^{(j)}$. It is obvious that problems (1)–(3) and (4)–(6) are equivalent.

Before we state the result we need to define a quantity $\mu(\mathcal{F})$ which has to do with the nonlocal character of $\mathcal{F}$. First we define a subset of Q^*:

$$\begin{aligned}Q^*(\rho,h) = \{x \in \Omega^* : |h(x)| > \rho||h||, f(h(x)) > 0\} \\ \times \{t : 0 < t < T, g(t) > 0\}.\end{aligned}$$

Next we define

$$(8) \qquad \mu_\rho(\mathcal{F}) = \sup_{h\in L_\infty(\Omega^*)} \operatorname*{ess\,sup}_{(x,t)\in Q^*(\rho,h)} [g(t)f(h(x))]^{-1}|\mathcal{F}_+(x,t,h)|$$

and

$$\mu(\mathcal{F}) = \lim_{\rho\uparrow 1} \mu_\rho(\mathcal{F}),$$

where $\mathcal{F}_+$ is defined as

$$\mathcal{F}_+(x,t,u) = \max\{0, [\operatorname{sgn}(u(x,t))]\}\mathcal{F}(x,t,u).$$

Lemma. *Suppose that $xf'(\rho x)/f(x)$ exists and is uniformly bounded for $x \geq 0$ and $0 < \rho \leq 1$; then $\mu(\mathcal{F}) \leq 1$.*

Proof. Applying the hypothesis (BND) to equation (8), we see that

$$\begin{aligned}\mu_\rho &\leq \sup_{h\in L_\infty(\Omega^*)} \operatorname*{ess\,sup}_{(x,t)\in Q^*(\rho,h)} [g(t)f(h(x))]^{-1}g(t)f(||h||)\\ &\leq \sup_{||h||} f(||h||)/f(\rho||h||).\end{aligned}$$

This means that

$$\mu(\mathcal{F}) \leq \lim_{\rho\uparrow 1} \sup_{s\geq 0} f(s)/f(\rho s).$$

But we know that there are constants $C > 0$ and $\sigma \in [\rho, 1]$ such that

$$\begin{aligned}1 - f(\rho s)/f(s) &= [f(s) - f(\rho s)]/f(s)\\ &= (1-\rho)sf'(\sigma s)/f(s)\\ &\leq C(1-\rho). \qquad \square\end{aligned}$$

An example of a function which satisfies the hypothesis of the above lemma is a linear combination of positive powers or, more simply, a function of the form $f(s) = A + Bs^r$ with A, B and r positive.

Since problem (4)–(6) satisfies the hypotheses of Theorem 2 in [**3**] and since $||u|| = ||\mathbf{u}||$, we have the following result.

Theorem 1. *Suppose that* $\mathbf{u}$ *is a solution of problem* (1)–(3). *Suppose that the problem is weakly coupled and that hypotheses* (PBY), (BND), *and* (MDN) *are satisfied. Then*

$$\int_{||\mathbf{u}(\cdot,0)||}^{||\mathbf{u}(\cdot,t)||} f(s)^{-1}\,ds \le \mu(\mathcal{F}) \int_0^t g(\tau)\,d\tau \tag{9}$$

for all $t \in [0,T)$. *Moreover, when* $f(s)$ *is a linear combination of positive powers of* s, *then* $\mu(\mathcal{F})$ *may be taken to be* 1.

The requirement that the equations are only weakly coupled excludes some important problems. However, we believe that this restriction may be removed. But, in order to do so, we must generalize the technical Lemma 7 in [**3**] whose proof is already quite lengthy.

The methods used above may be applied to problems of a somewhat different form that were treated in [**4**]. Let us define the elliptic operators

$$\begin{aligned}\mathcal{L}_u^{(j)} = &\sum_{i,k=1}^{n} \partial/\partial x_i\, \mathcal{A}_{i,k}^{(j)}(x,t,\mathbf{u},D\mathbf{u})\partial/\partial x_k \\ &+ \sum_{i=1}^{n} \mathcal{B}_i^{(j)}(x,t,\mathbf{u},D\mathbf{u})\partial/\partial x_i\end{aligned}$$

and the boundary operators

$$\mathcal{C}_u^{(j)} = \sum_{i,k=1}^{n} \nu_i(x)\mathcal{A}_{i,k}^{(j)}(x,t,\mathbf{u},D\mathbf{u})\partial/\partial x_k.$$

We consider the problem

$$\partial u^{(j)}/\partial t = \mathcal{L}_u^{(j)} u + \mathcal{F}(x,t,\mathbf{u}), \qquad (x,t) \in Q, \tag{10}$$

$$u^{(j)}(x,t) = \phi^{(j)}(x,t), \qquad (x,t) \in \partial Q_D^{(j)} \tag{11}$$

$$\mathcal{C}_u^{(j)} u^{(j)}(x,t) + \mathcal{G}^{(j)}(x,t,\mathbf{u}) = 0, \qquad (x,t) \in \partial Q_N^{(j)} \tag{12}$$

$$u^{(j)}(x,0) = u_0^{(j)}(x). \tag{13}$$

We will not demand weak coupling and we will allow the coefficients $\mathcal{A}_{i,k}^{(j)}$ and $\mathcal{B}_i^{(j)}$ as well as $\mathcal{F}$ and $\mathcal{G}$ to depend functionally upon $\mathbf{u}(\cdot,t)$. However, we will restrict ourselves to classical solutions and need the following hypotheses to be satisfied. Let $\mathcal{H}$ denote the space of $C^{2,1}(Q)$ functions w whose derivatives $\partial w/\partial t$ and $\partial^2 w/\partial x_i \partial x_k$ are uniformly Hölder continuous in x with exponent α and whose derivatives $\partial w/\partial t$, $\partial w/\partial x_i$, and $\partial^2 w/\partial x_i \partial x_k$ are uniformly Hölder continuous in t with exponents $\alpha/2$, $(1+\alpha)/2$, and $\alpha/2$, respectively. We assume that the coefficients are sufficiently well behaved so that for each $\mathbf{u} \in \mathcal{H}^m$ the following problem has a classical solution:

$$\text{(XST)}\qquad \begin{gathered} \partial v^{(j)}/\partial t = \mathcal{L}_u^{(j)} v^{(j)}, \qquad (x,t) \in Q, \\ v^{(j)}(x,t) = \phi^{(j)}(x,t), \qquad (x,t) \in \partial Q_D^{(j)}, \\ \mathcal{B}_u^{(j)} v = 0, \qquad (x,t) \in \partial Q_N^{(j)}, \qquad (x,t) \in \partial Q_N^{(j)}, \\ v^{(j)}(x,0) = u_0^{(j)}(x). \end{gathered}$$

We also assume that there exist nondecreasing Lipschitz continuous functions f and $\hat{f}$ on $[0,\infty)$ and functions g and $\hat{g}$ that are continuous nonnegative functions on $[0,T)$ whose integrals $\int_0^T g(t)\,dt$ and $\int_0^T \hat{g}(t)\,dt$ are finite, such that

$$\text{(BND}^*\text{)}\qquad \begin{gathered} ||\mathcal{F}_+^{(j)}(\cdot,t,\mathbf{u})|| \le g(t) f(||\mathbf{u}(\cdot,t)||) \\ s\mathcal{G}^{(j)}(x,t,s) \ge 0 \qquad \forall\ (x,t,s) \in Q \times \mathbf{R}^1 \\ \left\| \sum_{i=1}^n \mathcal{B}_i^{(j)}(x,t,\mathbf{u},D\mathbf{u})^2 \right\|^{1/2} \le \hat{g}(t) \hat{f}(||\mathbf{u}(\cdot,t)||). \end{gathered}$$

We shall also assume uniform parabolicity:

$$\text{(PBY}^*\text{)}\qquad \sum_{i,k=1}^n \mathcal{A}_{ik}^{(j)} p_i p_k \ge \nu_0 \sum_{i=1}^n p_i^2,$$

where ν_0 is a positive constant.

As was done before, we can combine the equations into a single equation on Q^*. The resulting equation satisfies the hypotheses of Theorem 4 in [**4**]. This yields

Theorem 2. *Suppose that* $\mathbf{u}$ *is a solution of problem* (10)–(13). *Suppose that the hypotheses* (PBY*), (BND*), *and* (XST) *are satisfied. Let* $|u_0^{(j)}(x)| \leq M$ *in* Ω *and* $|\phi^{(j)}(x,t)| \leq M$ *in* Q, *then*

$$\int_{2M}^{||\mathbf{u}(\cdot,t)||} f(s)^{-1}\,ds \leq \mu(\mathcal{F}) \int_0^t g(\tau)\,d\tau. \tag{14}$$

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